Entropy Growth and Equilibration in Strongly Coupled Gauge Theories

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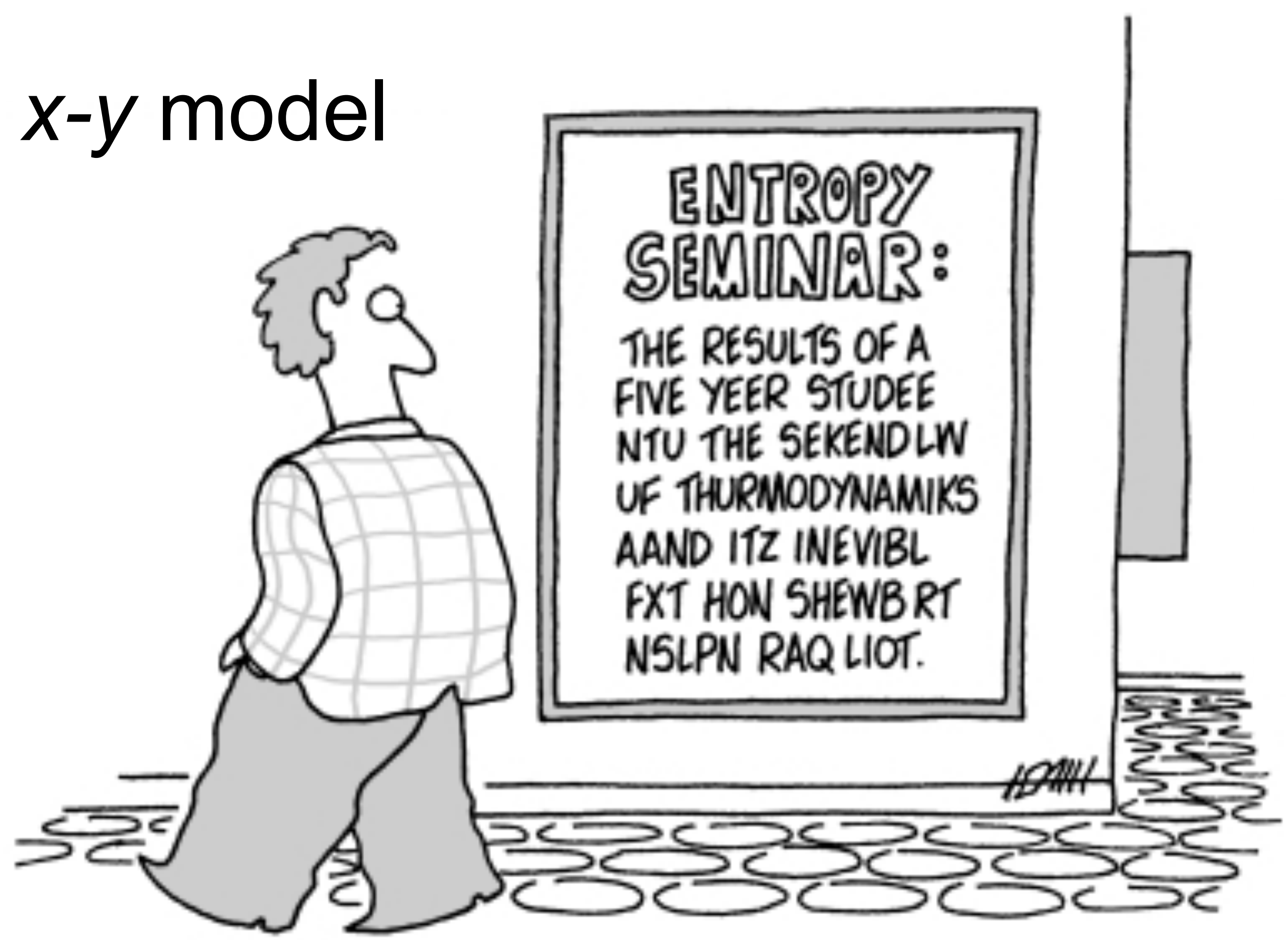
Overview

- Concepts of entropy
  - von Neumann entropy
  - Coarse grained entropy
  - Relevant entropy
  - Entanglement entropy
  - Husimi-Wehrl entropy
- Examples: Quantum quench, *x-y* model
- Lattice gauge theory
- Holographic thermalization



Thermalization

Thermalization means that a system loses all information about its history.

This can happen in two ways:

1. The system exchanges information with its environment (heat bath). This is true thermalization. The thermal state of the system is characterized by a density matrix, which only depends on the conserved quantum numbers (energy, particle number, charge, etc.). The entropy of the system is a measure of its information loss to the environment. In this case, the quantum state of the system becomes *entangled* with the quantum state of its environment.
2. The state of the system evolves by itself into a complicated superposition of components that cannot be distinguished by any practical measurements. This is apparent thermalization, implied by the *coarse graining* inherent in physical observations. A single eigenstate of the system can appear thermal (*eigenstate thermalization*). The physical mechanism by which a system can evolve into such complex states under its own dynamics is called *quantum chaos*.

Density matrix

The state of a system (ensemble) is specified by the density matrix:

$$\rho = \rho^{\dagger}; \quad \mathrm{tr}(\rho) = 1; \quad \mathrm{tr}(\rho^2) \le 1; \quad \langle \Psi | \rho | \Psi \rangle \ge 0 \ \ \forall | \Psi \rangle$$

The density matrix evolves according to the von Neumann equation:

$$i\hbar \frac{\partial}{\partial t} \rho = [H, \rho] \qquad \rightarrow \qquad \rho(t) = e^{-iHt/\hbar} \rho(0) e^{iHt/\hbar}$$

The unitary time evolution implies that the von Neumann entropy

$$S_{\mathrm{vN}} = \mathrm{tr}(\rho \ln \rho)$$

does not change with time: Information about the quantum system is never lost.

However, not all information about the quantum system may be recoverable by an observer, in principle or in practice: “coarse graining” or “entanglement”.

Nakajima-Zwanzig theory

For a highly complex system (many degrees of freedom) usually only simple, slowly varying observables (few-body, low resolution, etc.) can be measured.

Split the density matrix into a relevant part ρ_R that determines the value of the observable A and an irrelevant part ρ_I that has no influence on the value of A:

$$\rho = \rho_R + \rho_I \quad \text{with} \quad \langle A \rangle = \mathrm{tr}(\rho A) = \mathrm{tr}(\rho_R A); \quad \mathrm{tr}(\rho_I A) = 0$$

Define a projection operator P such that: $\rho_R = P\rho$

Then:

$$\frac{\partial}{\partial t}\rho_R = -PL\rho_R(t) - iPL\,e^{-i(1-P)L}\rho_I(0) - \int_0^t d\tau\, G(\tau)\rho_R(t-\tau)$$

where $L = \frac{1}{\hbar}[H,\circ]$ $\quad G(\tau) = PL\,e^{-i(1-P)L\tau}(1-P)LP$ (memory kernel)

[For a review, see e.g.: J. Rau, BM, *Physics Reports* 272 (1996) 1]

Time scales

An important question is which observables A should be considered to define the relevant part ρ_R of the density matrix. These should be experimentally measurable quantities, which implies that they should vary only on observable time scales: they must be slowly varying observables.

In many cases the memory kernel, which describes the feedback from the irrelevant degrees of freedom, decays much faster than the characteristic time scale on which the value of the observables change. The evolution equation for ρ_R then becomes effectively Markoff.

Any analysis of the problem of entropy creation and thermalization in the Nakajima-Zwanzig formalism thus starts from an analysis of time scales.

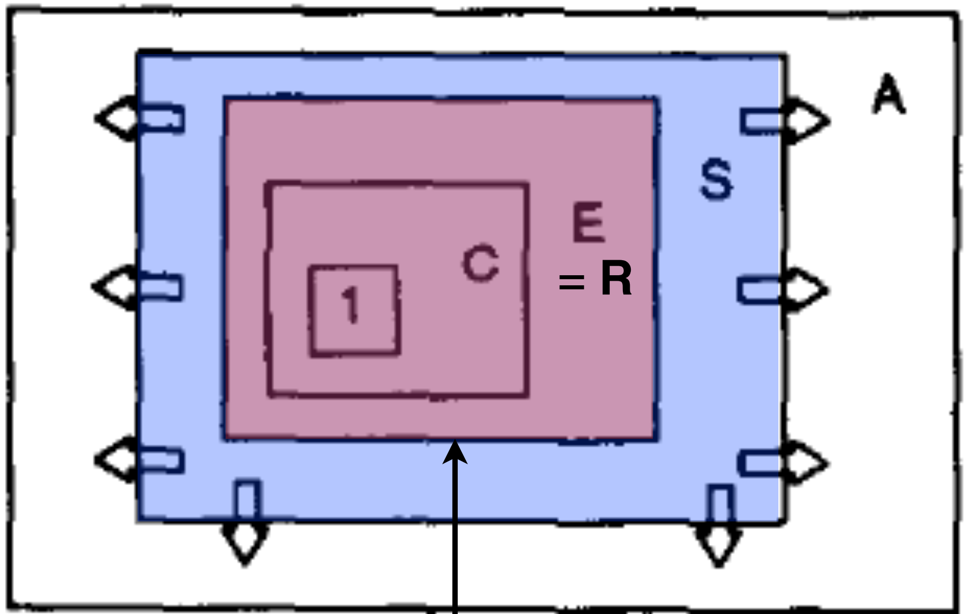
Note: The projector P ensuring $\mathrm{tr}(P\rho A) = \mathrm{tr}(\rho A)$ is called the Kawasaki-Gunton projector; the resulting evolution equation for ρ_R is called the Robertson equation. Because ρ is time dependent, P depends on time.
An alternative formulation is due to Mori, who defined the projector such that $\mathrm{tr}(P_M\rho_{eq}A) = \mathrm{tr}(\rho_{eq}A)$, which makes P_M time independent.

Relevant entropy

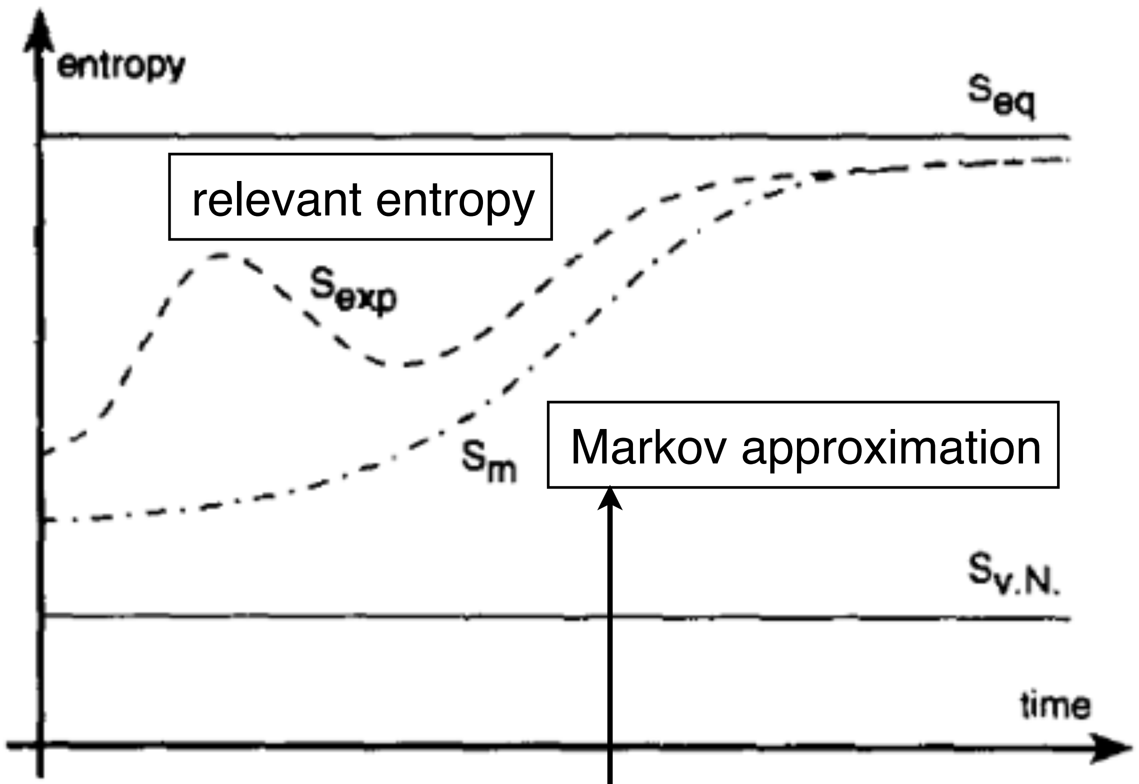
The relevant entropy $S_R = \mathrm{tr}(\rho_R \ln \rho_R)$ generally increases with time (but not necessarily monotonously), because information gets transferred into irrelevant degrees of freedom. Special case: 1-P = projector on the environment.

The relevant entropy is “in the eye of the beholder”.

C = conserved observables
E = experimentally relevant observables
S = “slowly varying” observables
A = all observables



“Level of description” of the system



Good if large separation of time scales

Husimi coarse graining

A minimal coarse-graining of a quantum system is achieved by projecting its density matrix on a coherent state (Husimi ["*Fushimi*"] 1940):

$$H(x,p) = \langle z|\rho|z\rangle \quad \text{with} \quad z = \frac{x}{\sqrt{\hbar\Delta^{-1}}} + \frac{ip}{\sqrt{\hbar\Delta}} \quad \Rightarrow \quad \rho_H = \sum_z |z\rangle H(z) \langle z|$$

The Husimi phase-space density is positive semi-definite and can be used to define a coarse grained entropy (Wehrl, 1978):

$$S_H = -\mathrm{Tr}\left[\rho_H \ln \rho_H\right] = -\int \frac{dx\,dp}{2\pi\hbar} H(x,p) \ln H(x,p)$$

As opposed to the von Neumann entropy S = -Tr(ρlnρ), the Husimi-Wehrl entropy is not conserved by unitary evolution. Its value depends on Δ, but its growth rate at large times is independent of the smearing Δ (Kunihiro *et al*. [KMOS], 2008). Far off equilibrium it is equal to the *Kolmogorov-Sinaï* (KS) entropy growth rate:

$$\frac{dS_H}{dt} \xrightarrow{t\to\infty} \sum_{\alpha}^{\lambda_\alpha > 0} \lambda_\alpha = \dot{S}_{\mathrm{KS}}$$

Husimi II

Husimi density can be understood as smearing of the Wigner function with a Gaussian minimum-uncertainty wave packet:

$$H_\Delta(p,x;t) \equiv \int \frac{dp'\,dx'}{\pi\hbar} \exp\left(-\frac{1}{\hbar\Delta}(p-p')^2 - \frac{\Delta}{\hbar}(x-x')^2\right) W(p',x';t)$$

Special case of Gaussian smearing with $\sigma_p\sigma_x = \hbar/2$:

$$H_{\sigma_p\sigma_x}(p,x;t) = \int \frac{dp'dx'}{2\pi\sqrt{\sigma_p\sigma_x}} \exp\left(-\frac{(p-p')^2}{2\sigma_p^2} - \frac{(x-x')^2}{2\sigma_x^2}\right) W(p',x';t)$$

Formally, the Husimi transformation of the density matrix is of the form:

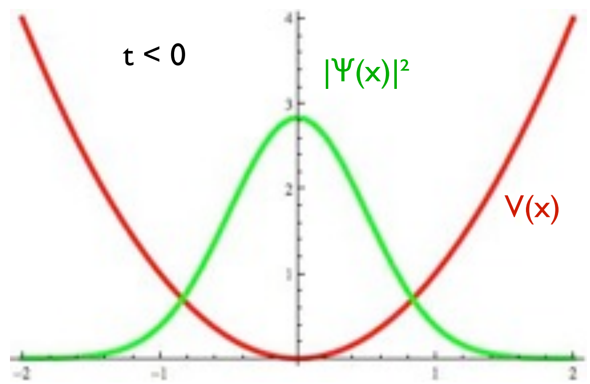
$$\rho_H = \Gamma_H \rho = \Gamma\left(\sigma_p,\sigma_x\right)\rho$$

with $\sigma_p^2 = \hbar\Delta/2$, $\sigma_x^2 = \hbar/2\Delta$. Note that Γ is not quite a projection operator:

$$\Gamma\left(\sigma_p,\sigma_x\right)^2 = \Gamma\left(\sqrt{2}\sigma_p,\sqrt{2}\sigma_x\right)$$

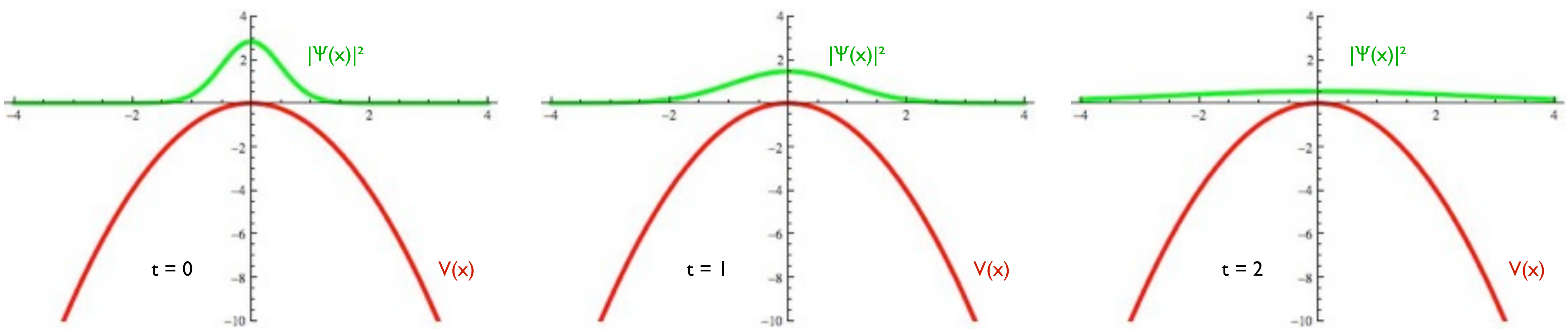
Quantum quench

The decay of an unstable vacuum state is a common problem, e.g., in cosmology and in condensed matter physics. Paradigm case: inverted oscillator.



$$\hat{H}(t) = \frac{p^2}{2} + \frac{m(t)^2}{2} x^2$$

with $$m(t)^2 = \omega^2 \theta(-t) - \lambda^2 \theta(t)$$

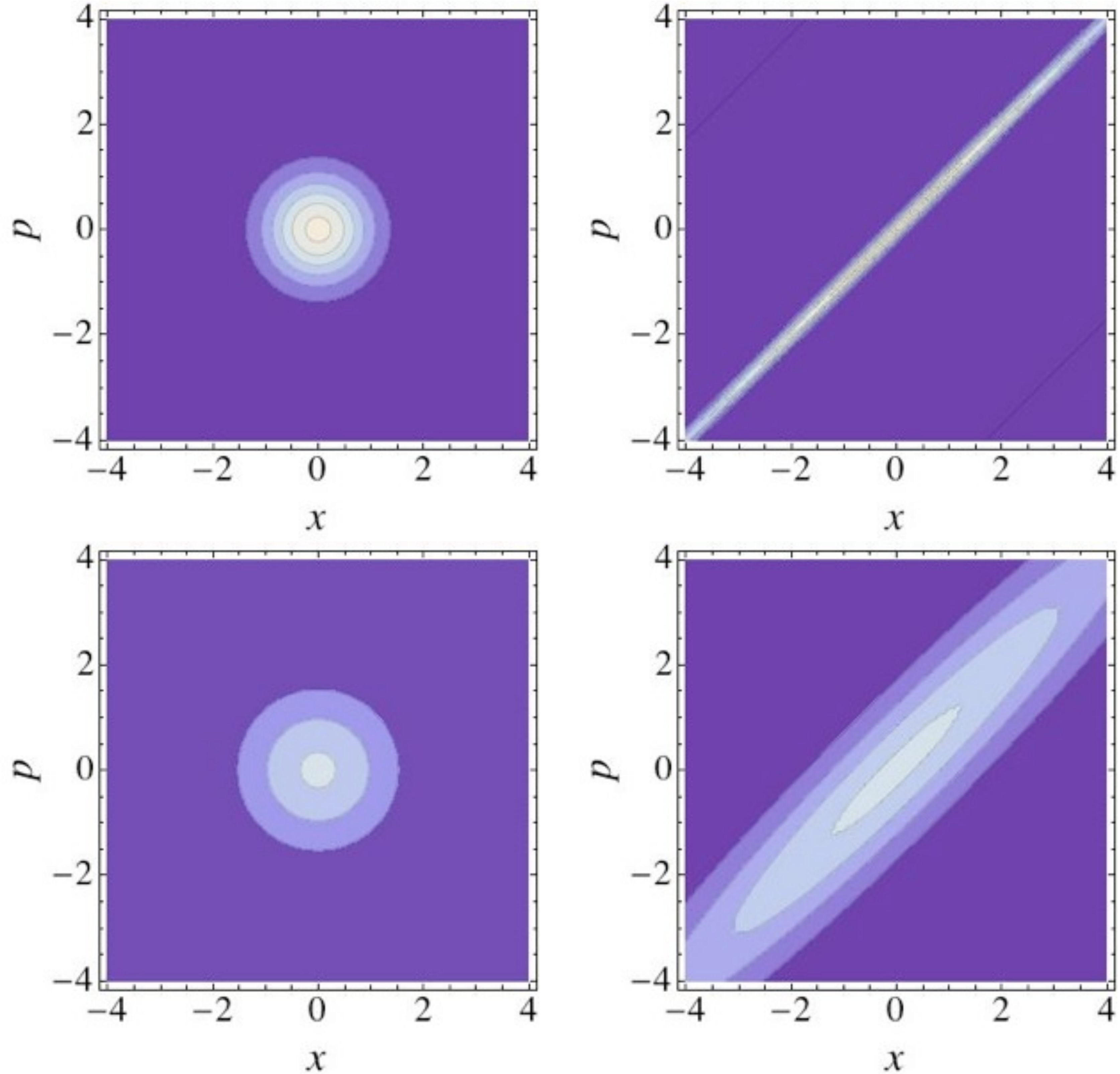


Wigner function: $$W(q,p;t) = \int du\, e^{-ipu} \langle q + \frac{1}{2}u|\, \hat{\rho}(t)\, |q - \frac{1}{2}u\rangle$$

Wigner vs. Husimi

Wigner function

t = 0



t = 2

Husimi function

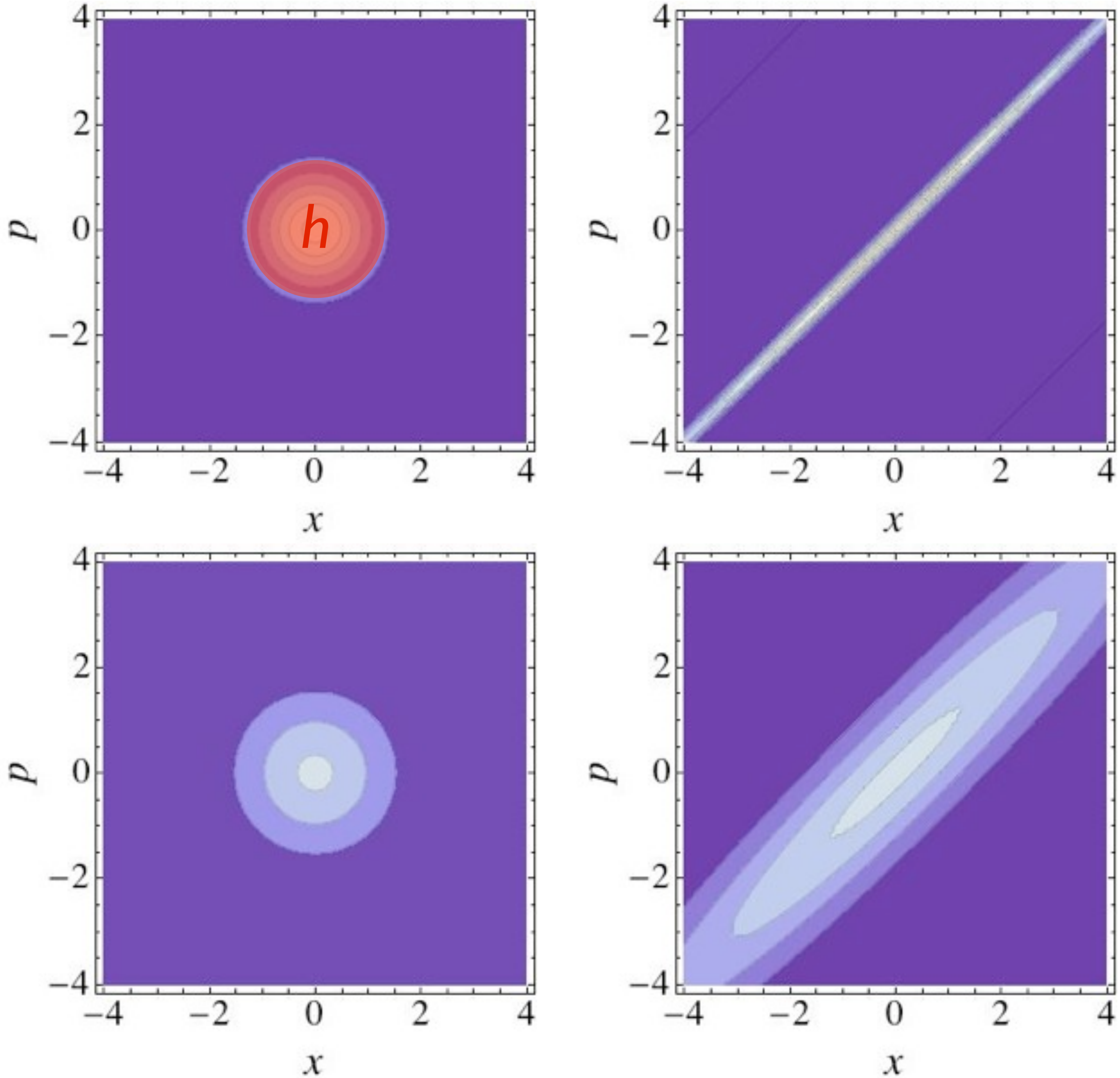
t = 0

t = 2

Wigner vs. Husimi

Wigner function

$t = 0$ | $t = 2$



Husimi function

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Wigner vs. Husimi

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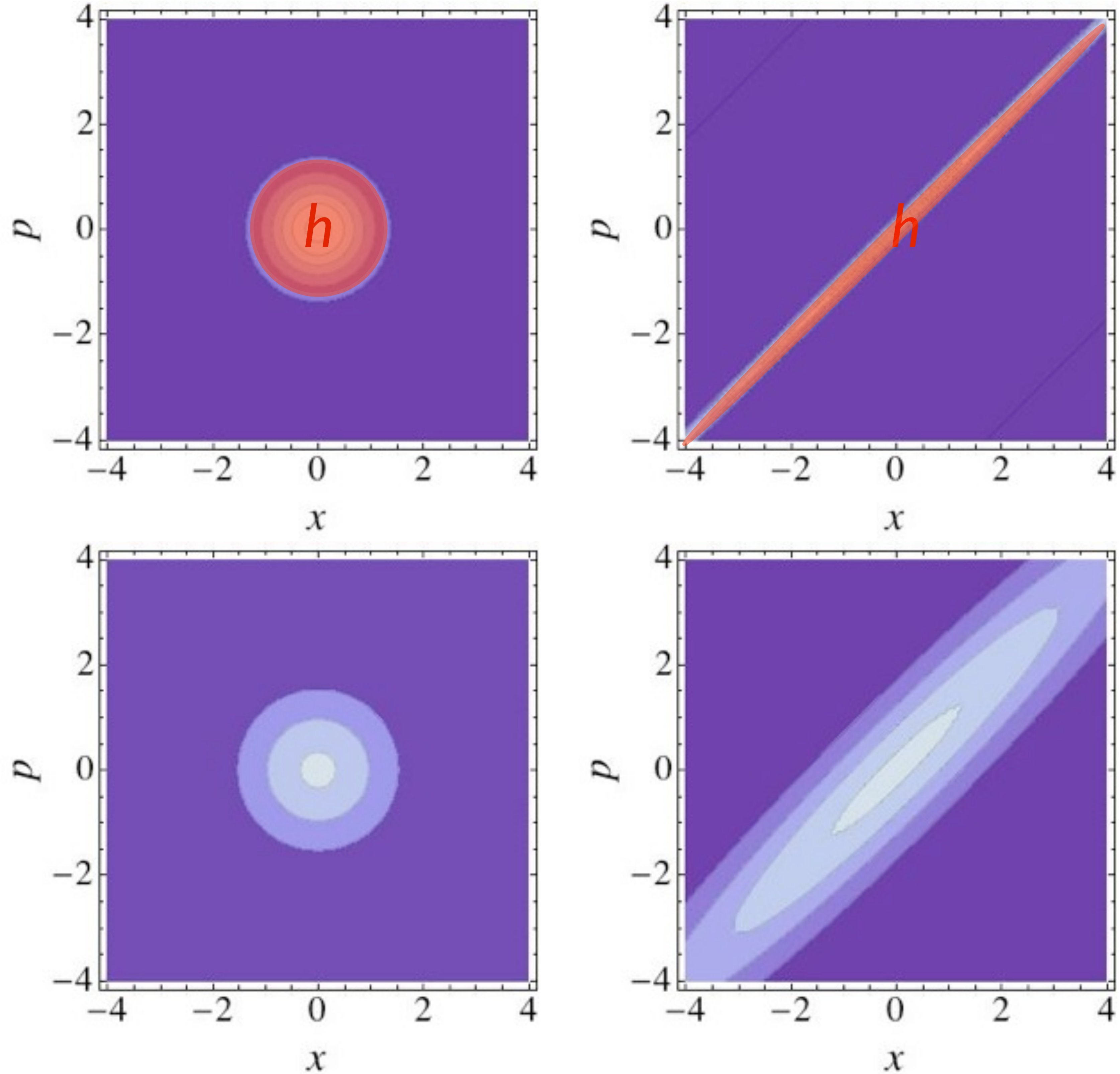

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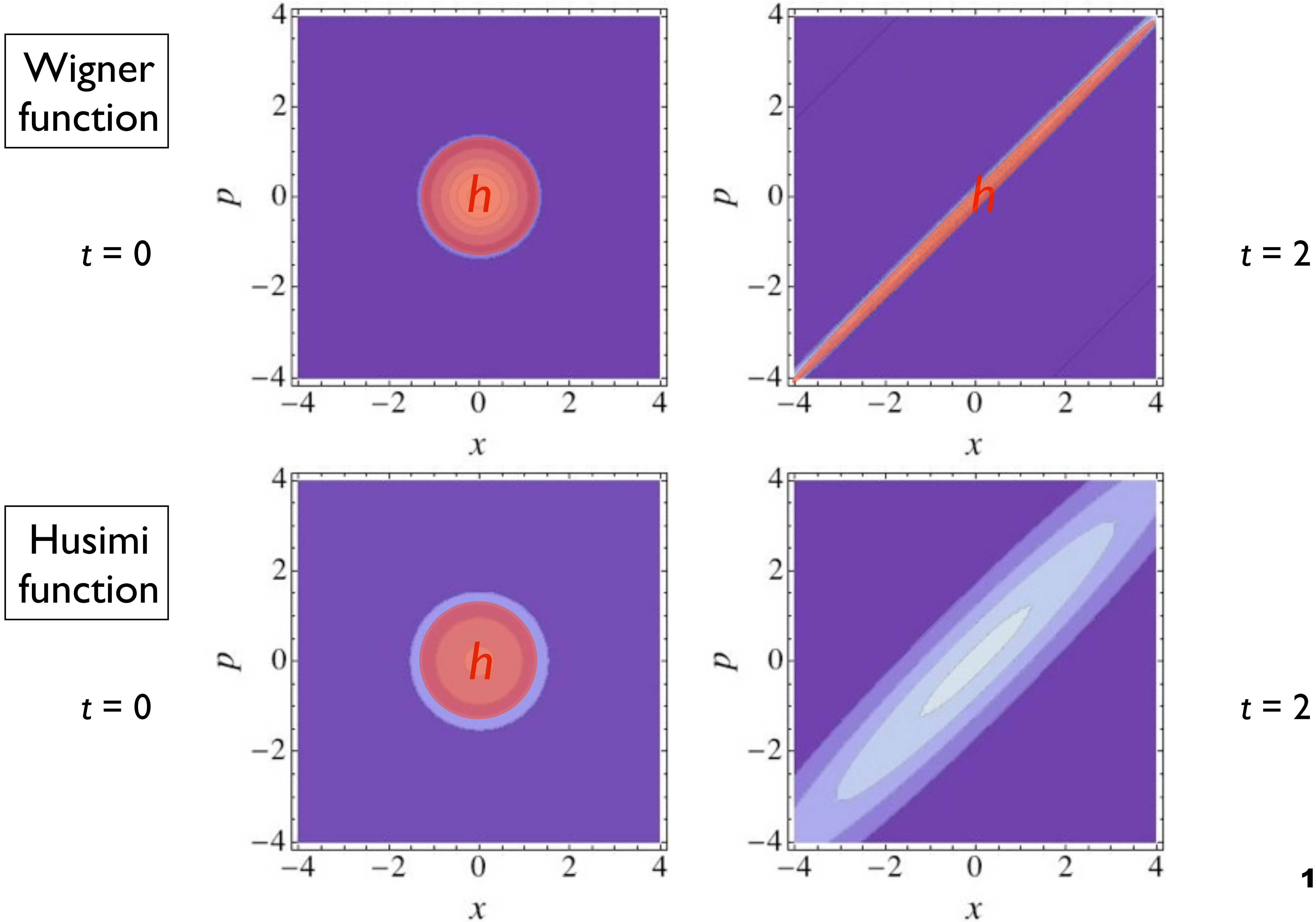

Wigner vs. Husimi

Wigner function

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$t = 2$

Husimi function

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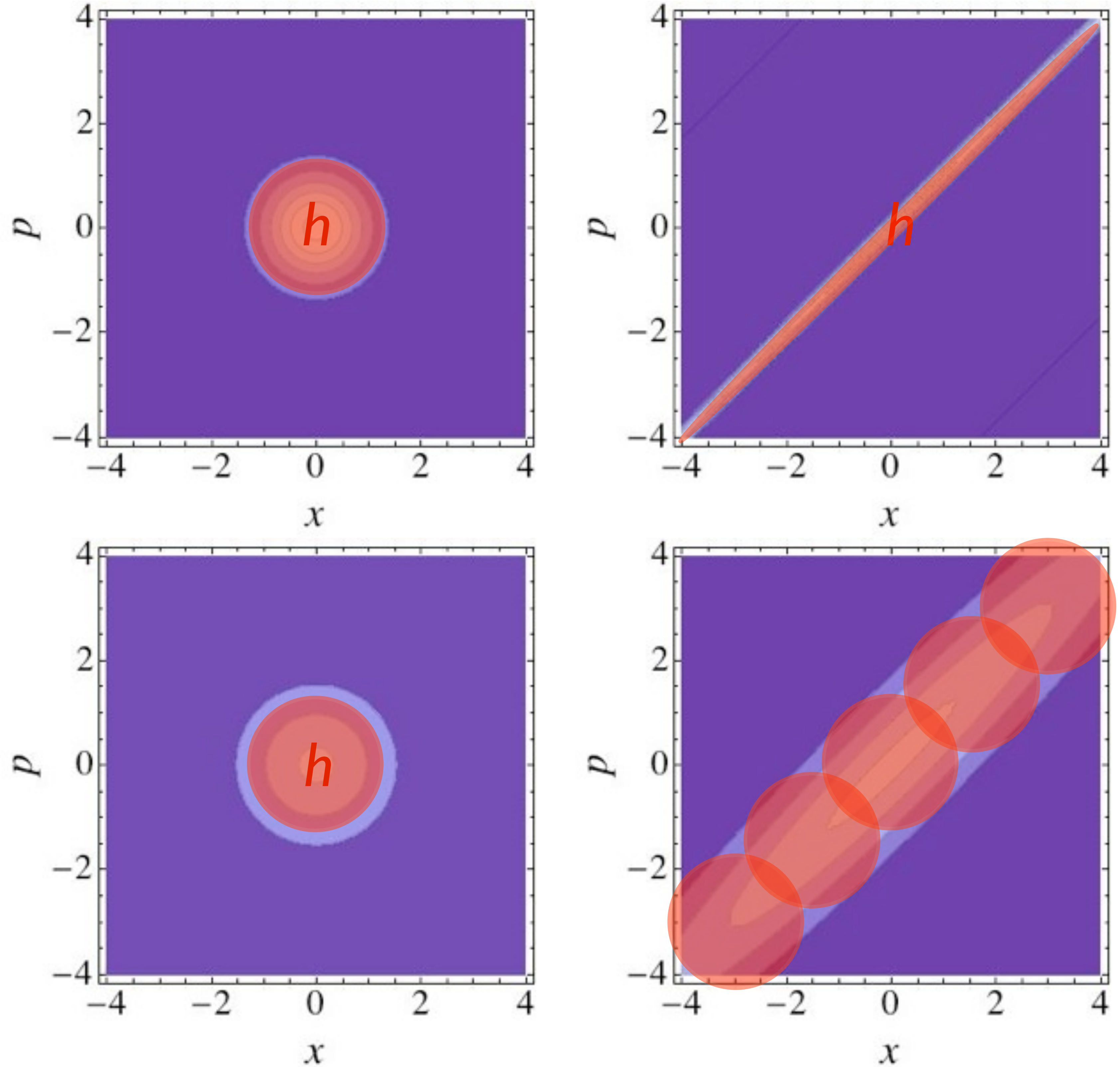


YMQM (x-y) model

A simple example of a non-trivial chaotic quantum system is given by the infrared limit of SU(2) gauge theory (*Yang-Mills Quantum Mechanics*):

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{3} p_i^2 + \frac{g^2}{4}\sum_{i\neq k}^{3} x_i^2 x_k^2$$

Further simplification: $x_1 = x,\ x_2 = y,\ x_3 = 0$ (x-y *model*): $\mathcal{H} = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{g^2}{2}x^2y^2$

Solve equation of motion for Husimi density $H(x,y,p_x,p_y,t)$ using superposition of Gaussians with time-dependent positions and widths:

$$H(\xi_i,t) = \sum_{\alpha}\exp\left[-\sum_{ij} c_{ij}(t)\left(\xi_i - \xi_i^{(\alpha)}(t)\right)\left(\xi_j - \xi_j^{(\alpha)}(t)\right)\right] \quad \text{with} \quad \xi_i = (x,y,p_x,p_y)$$

Evolution conserves the coarse grained Hamiltonian

$$\mathcal{H}_H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{g^2}{2}x^2y^2 - \frac{g^2\hbar}{4\Delta}\left(x^2 + y^2\right) + \frac{g^2\hbar^2}{8\Delta^2} - \frac{1}{2}\hbar\Delta$$

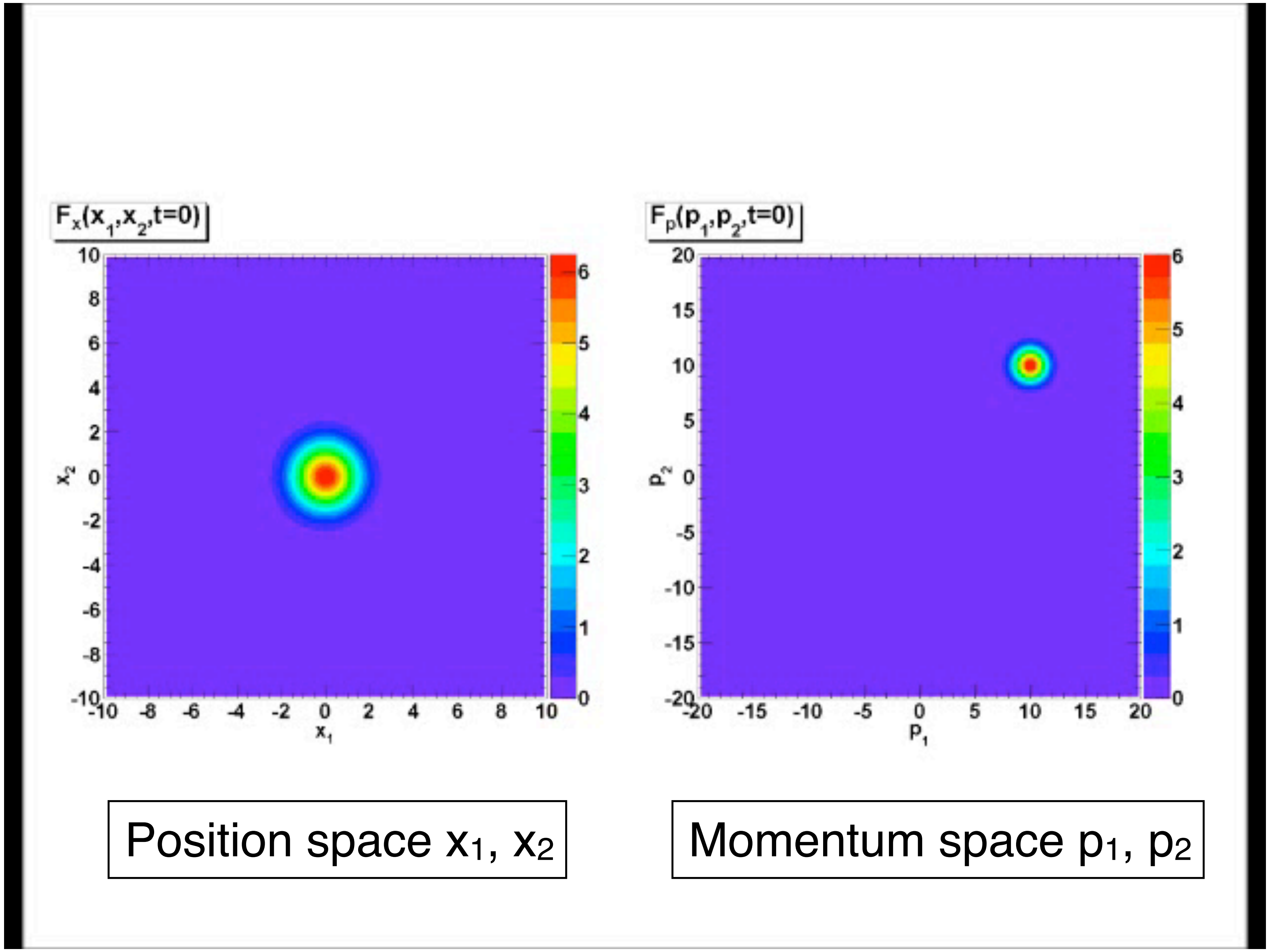
YM-QM - the movie

Position space x_1, x_2

Momentum space p_1, p_2

Hung-Ming Tsai & BM, arXiv:1011.3508

YM-QM - the movie



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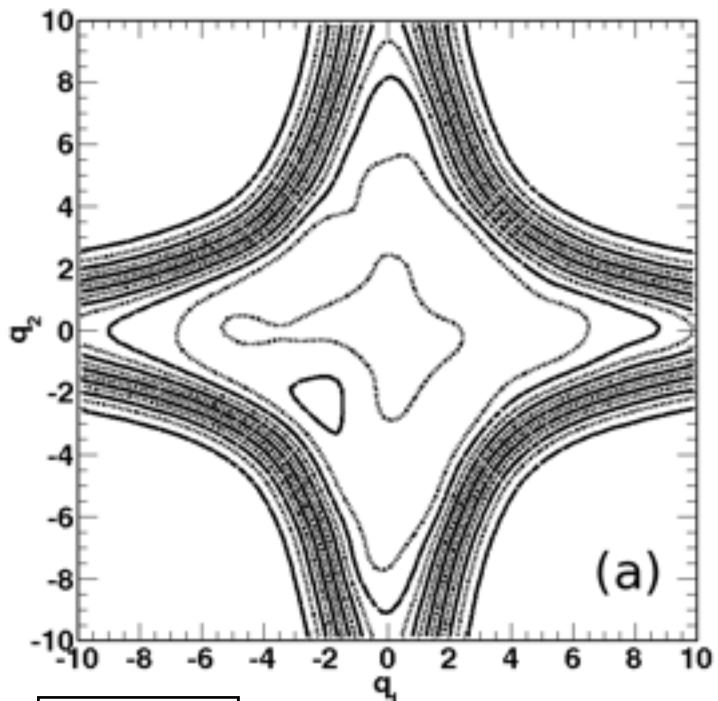
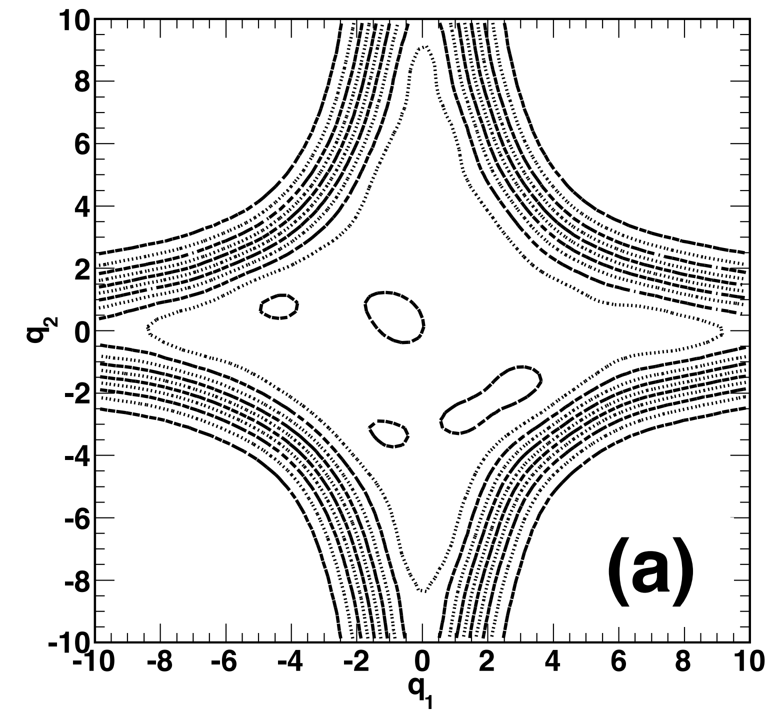
YM-QM equilibration



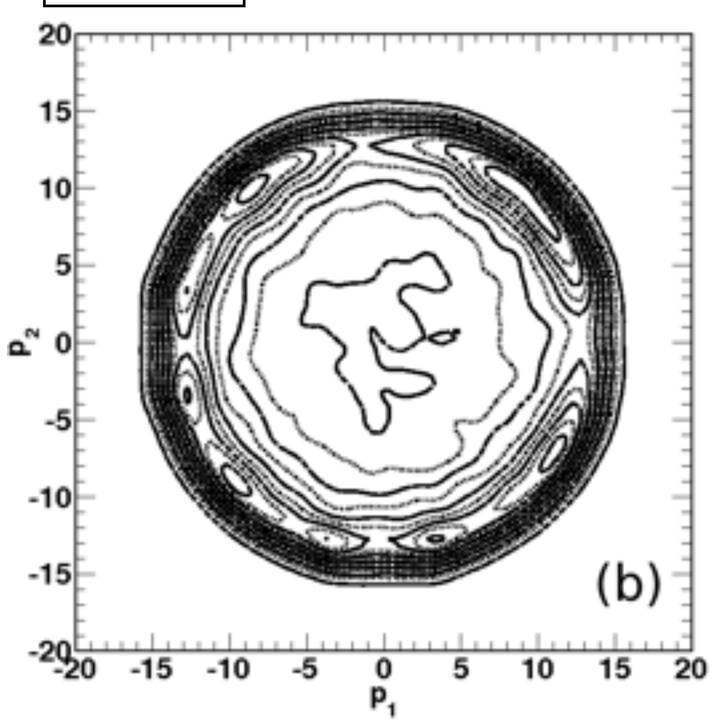
(x,y)



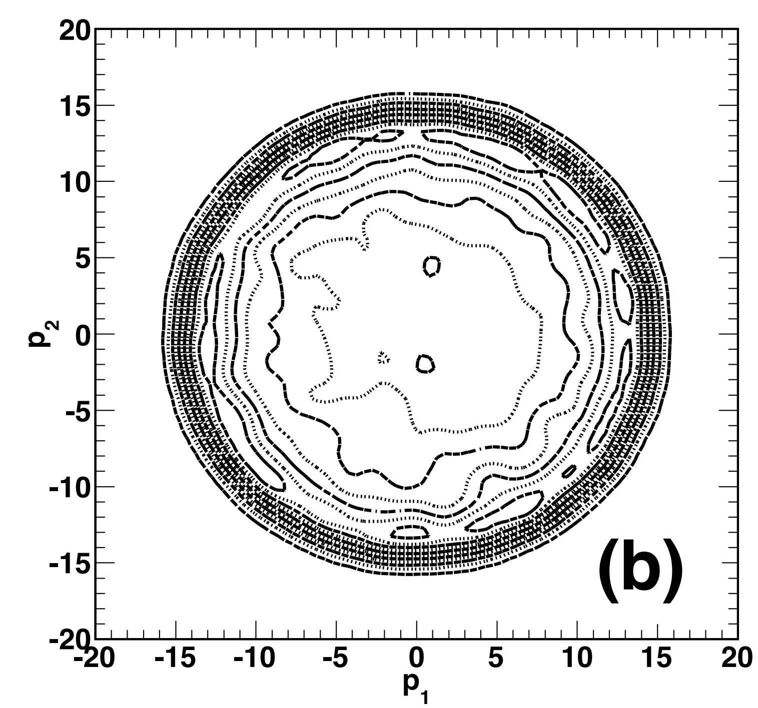
$t = 10$

80,000 test functions

microcanonical equil.

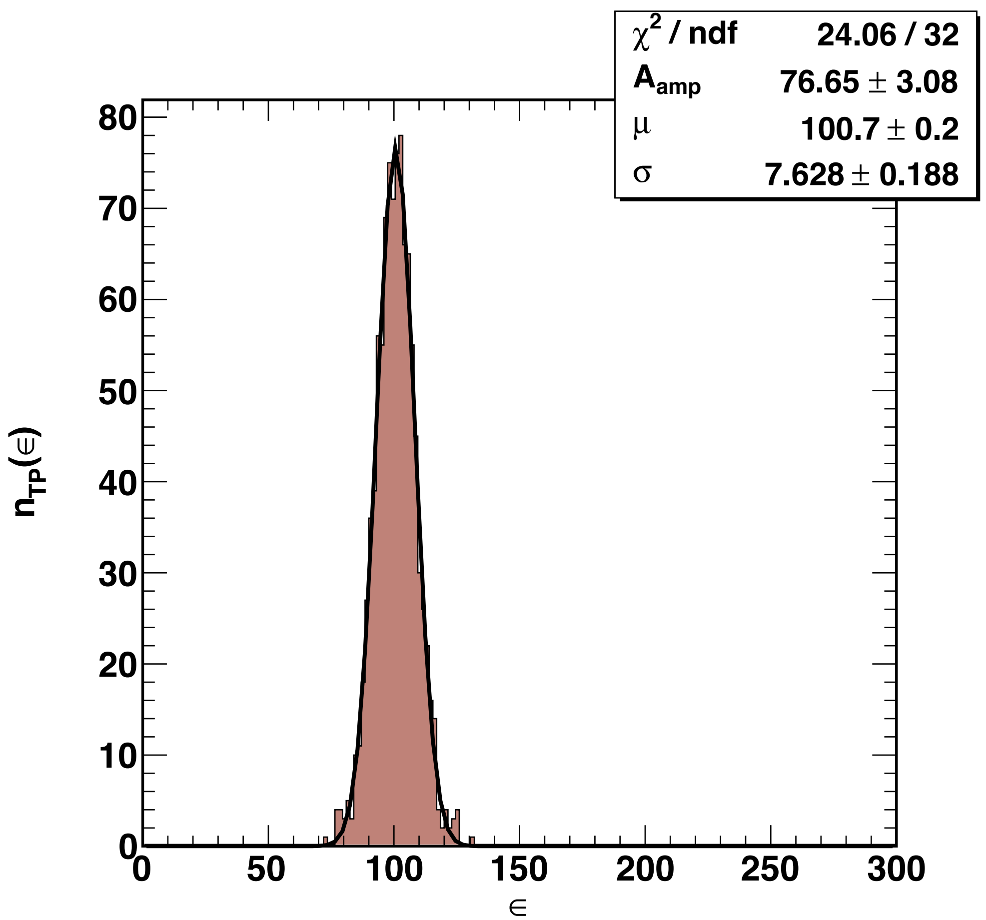


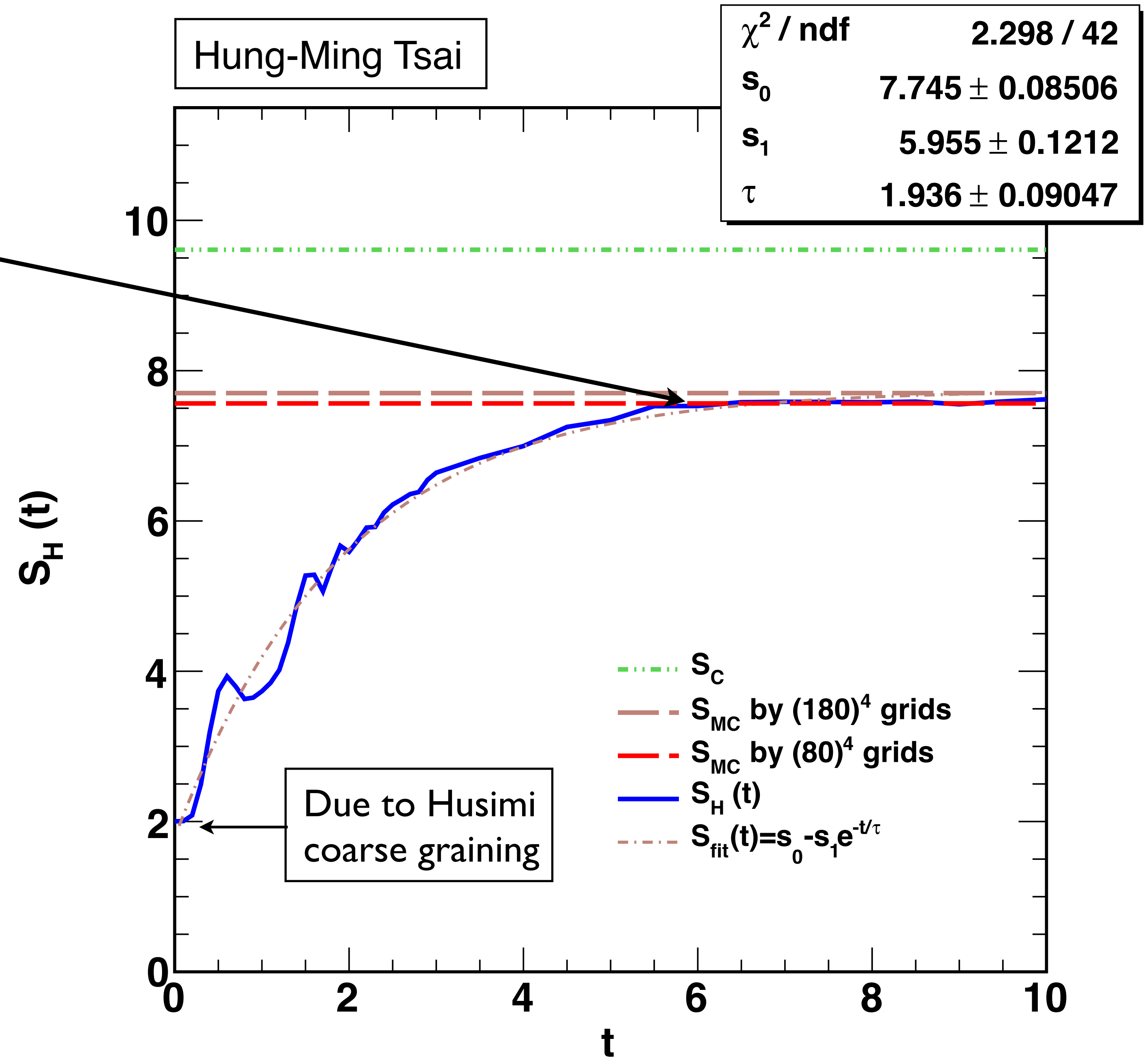
(p_x,p_y)



YM-QM equilibration - II

Initial Husimi density has a narrow energy distribution peaked around $E \approx 100$. Final Wehrl entropy agrees with micro-canonical equilibrium !





Reversals of fortune

An isolated finite-dimensional system with a compact phase space will eventually return to its initial configuration (*Poincaré recurrence*). Any thermalization or equilibration is thus only *apparent*. In realistic systems the Poincaré recurrence time is usually too large to observe.

"If we have everlasting life, what about *entropy*?"

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A simpler question:

Can the microcanonical equilibration visible in the Husimi distribution be undone by running the evolution backwards?

How *real* is the Wehrl entropy?

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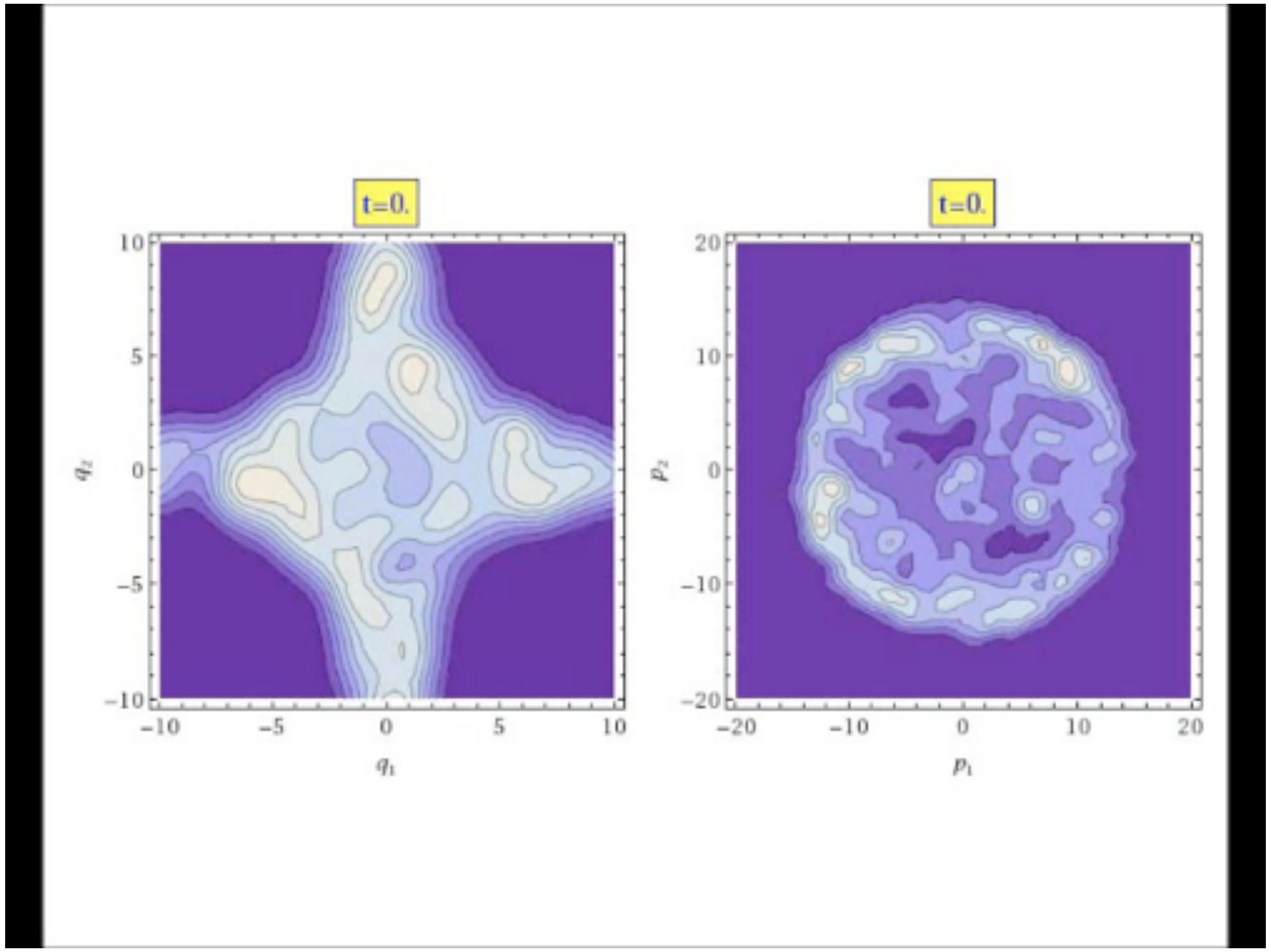

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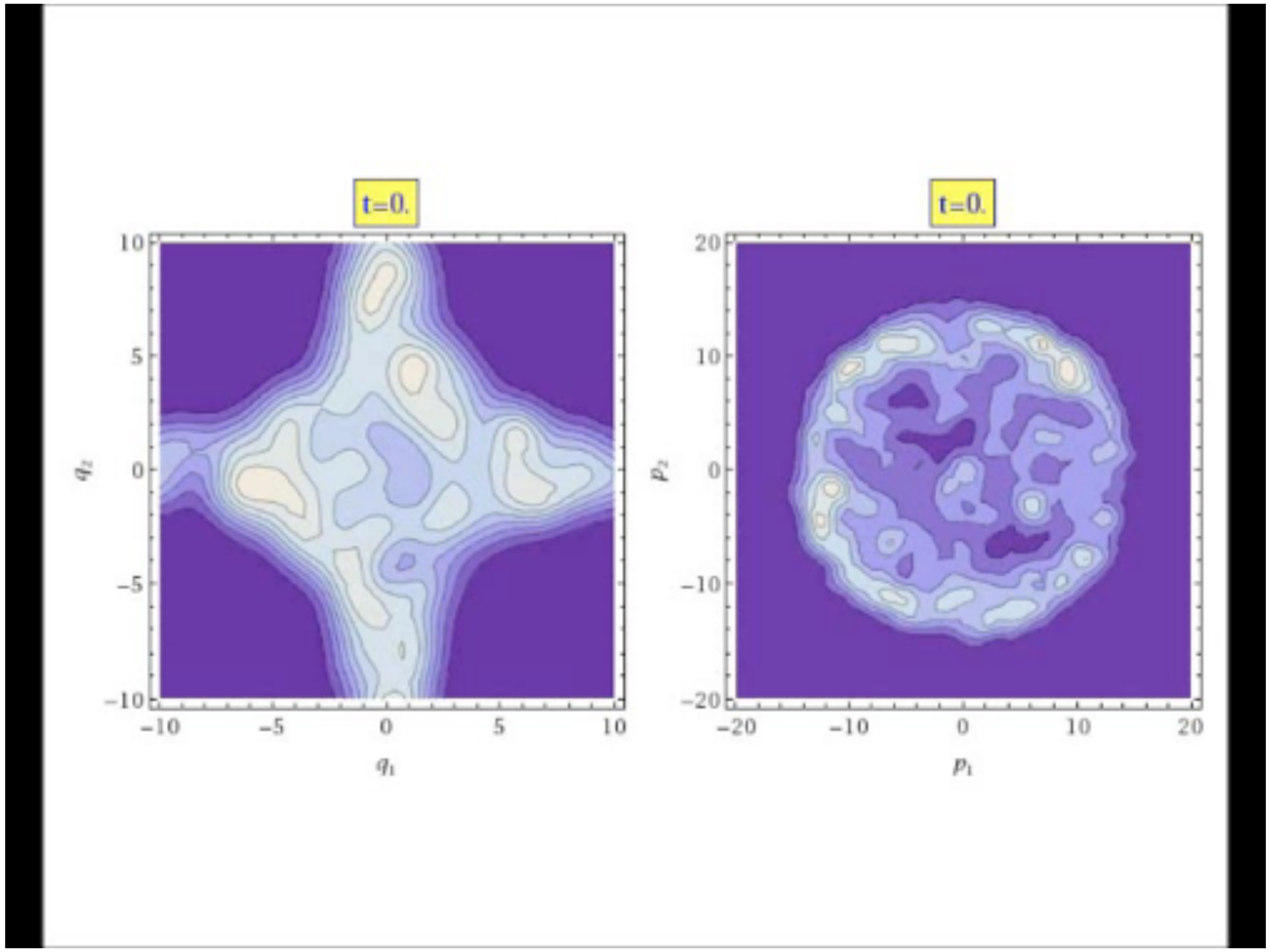

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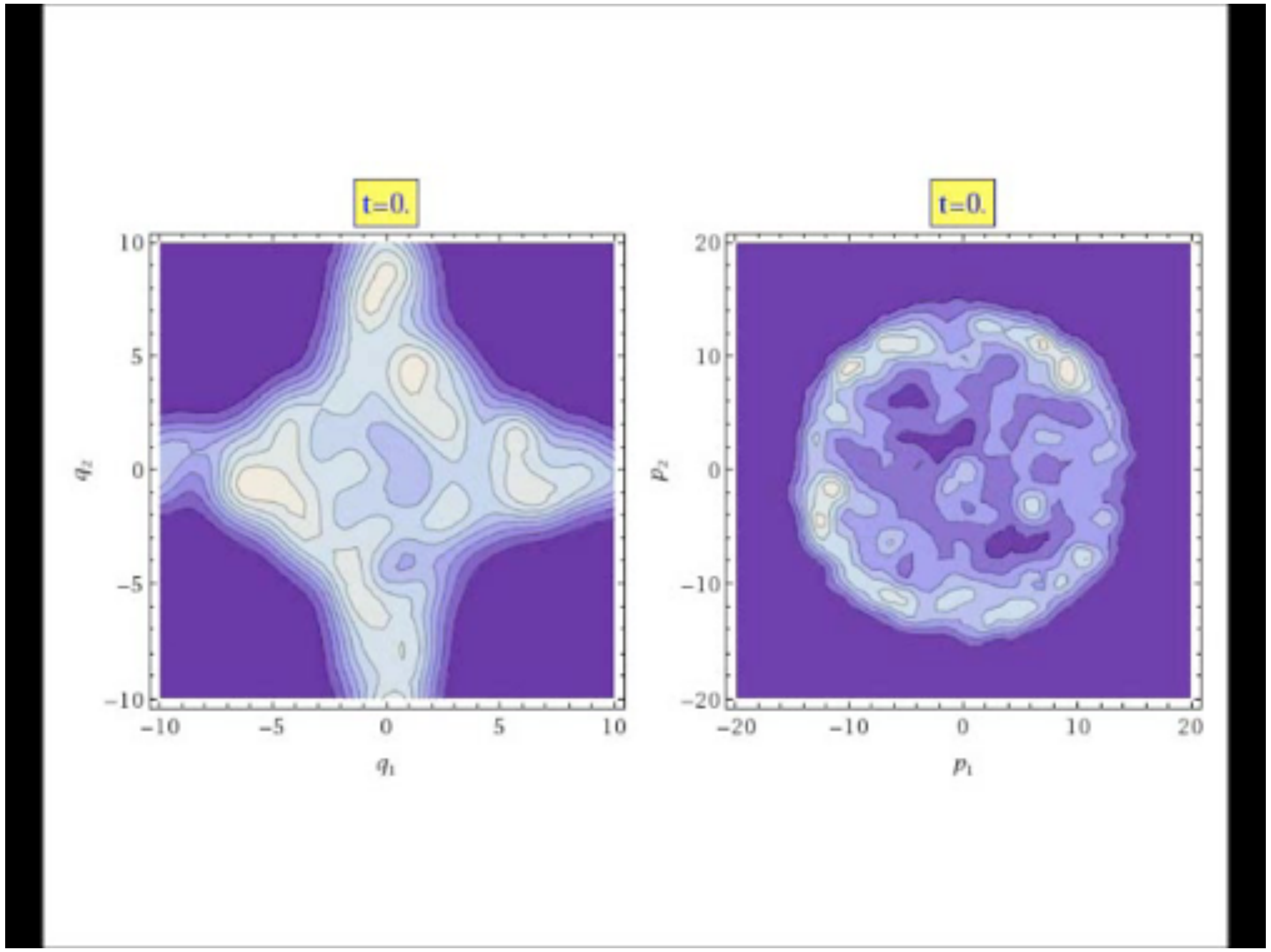

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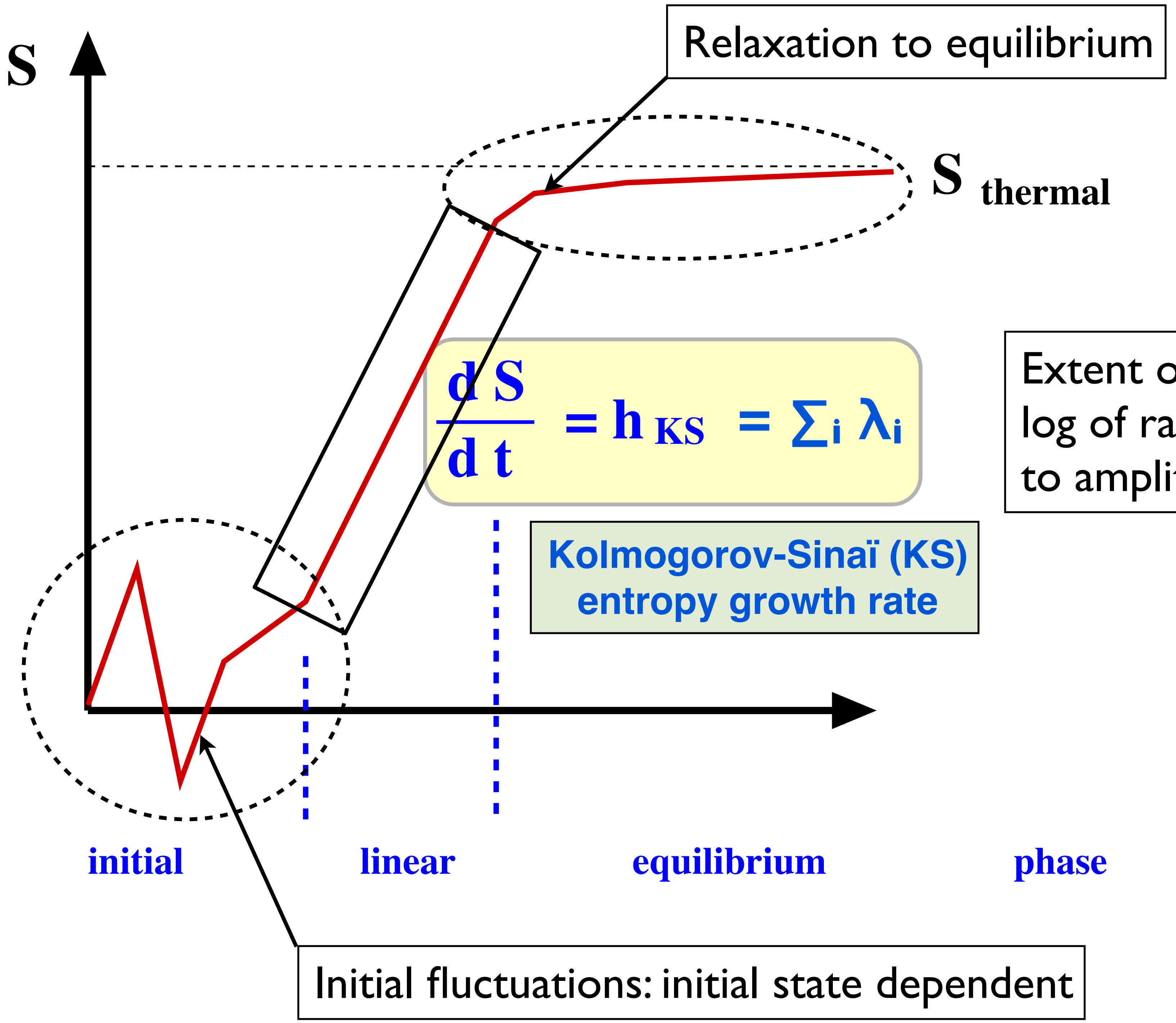
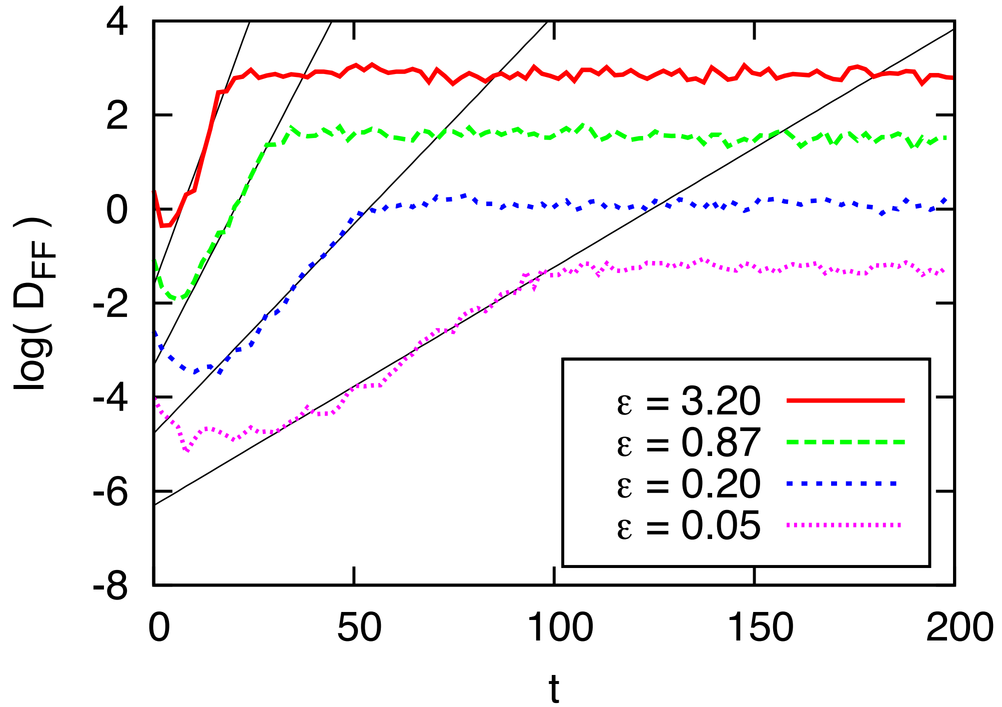
A small perturbation before time reversal will destroy the recurrence of the initial state.

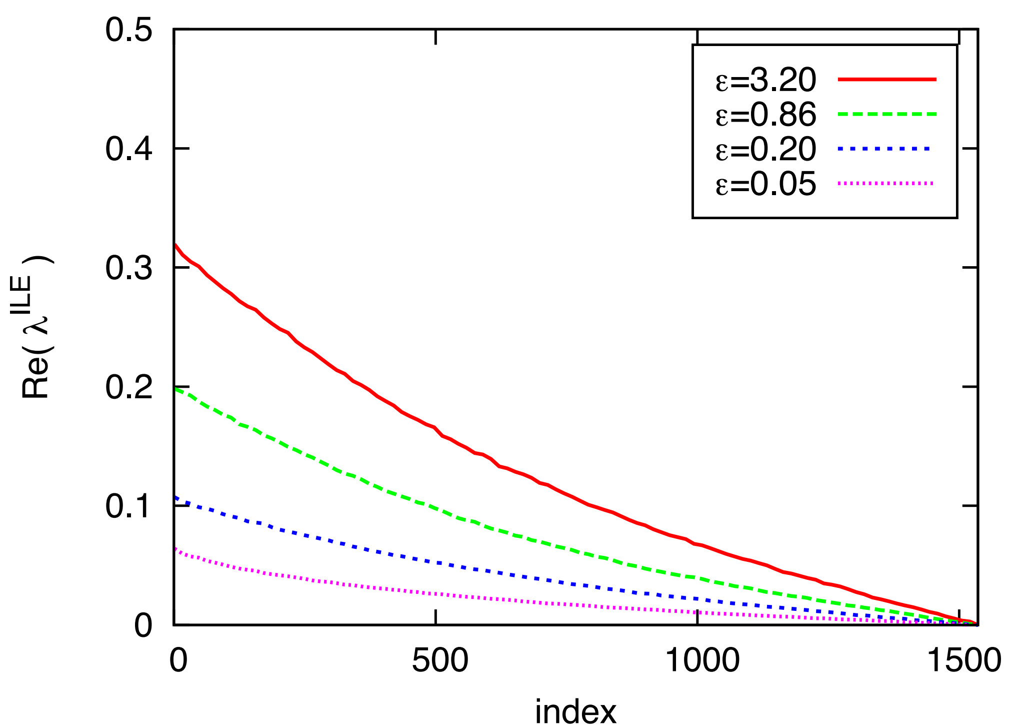
General picture



Extent of linear region depends on log of ratio of thermal fluctuations to amplitude of initial fluctuations.

Classical lattice SU(3)





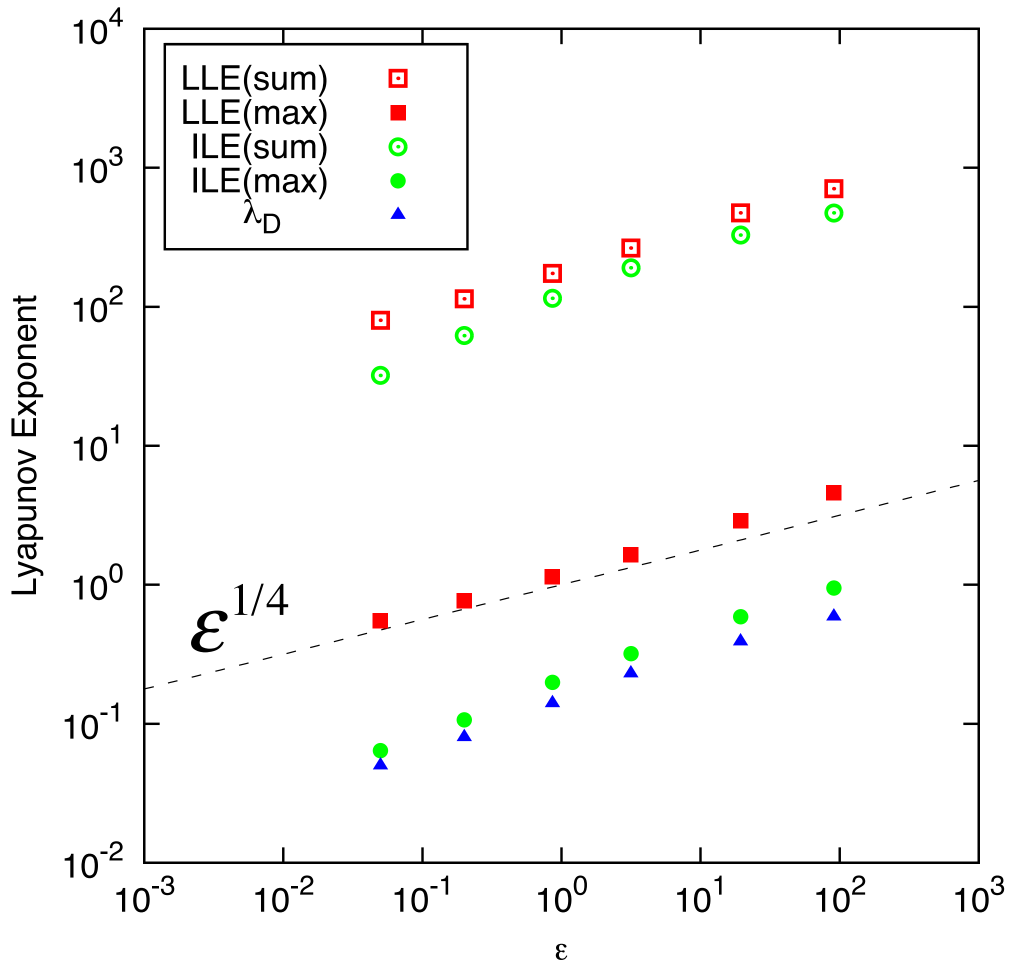
T. Kunihiro, BM, A. Ohnishi, A. Schäfer, T. Takahashi & A. Yamamoto, PRD 82 (2010) 114015

LLE = Local Lyapunov exponents:
= Eigenvalues of the Hesse matrix

ILE = Intermediate Lyapunov exponents:
= Growth rate of distance between neighboring gauge field config's

GLE = Global Lyapunov exponents:
= Asymptotic divergence rate of neighboring gauge field config's
= Standard definition of LE's

Equilibration time



Lattice: $\varepsilon_{\rm cl}(T) = \frac{\varepsilon^{\rm L}}{a^4 g^2} = 2(N_c^2-1) C_{\rm L} \frac{T}{a^3}$

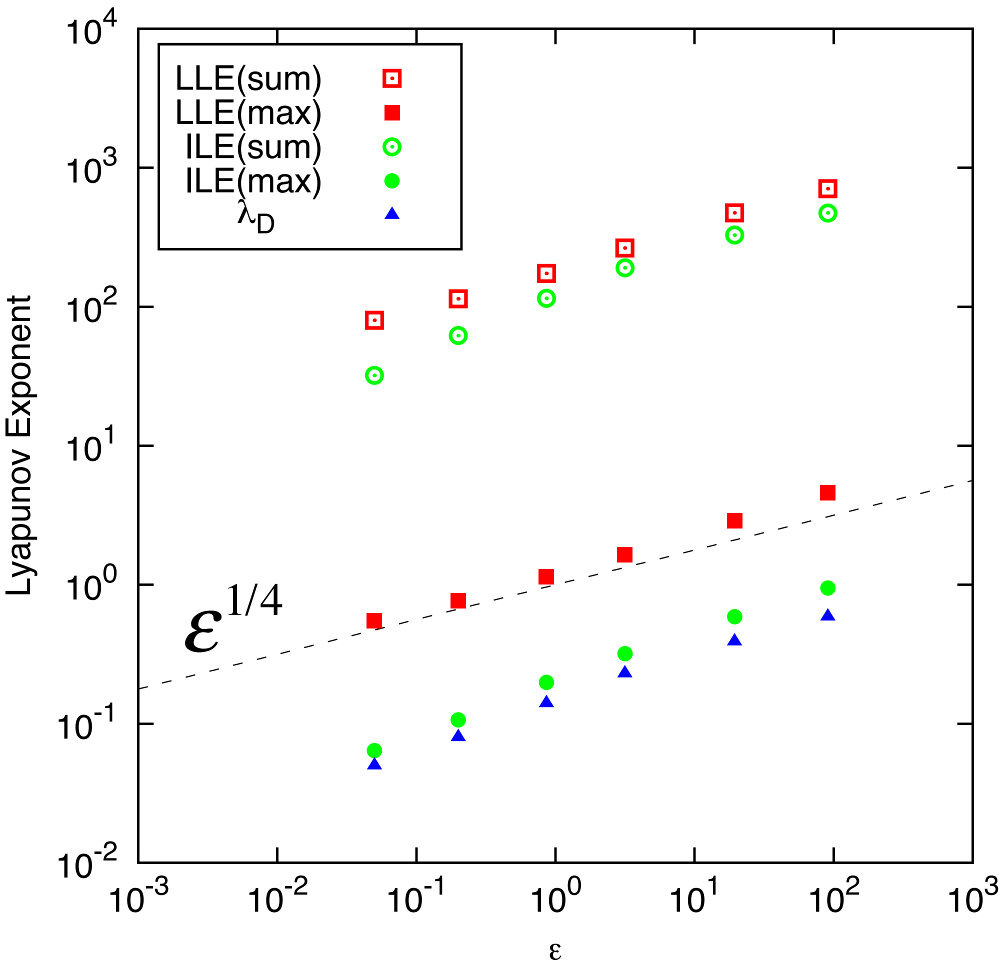
Continuum: $\varepsilon(T) = 2(N_c^2-1)\frac{\pi^2}{30}T^4$

Correspondence: $a \geq \frac{\theta}{T}$ $\theta \approx 1.45$

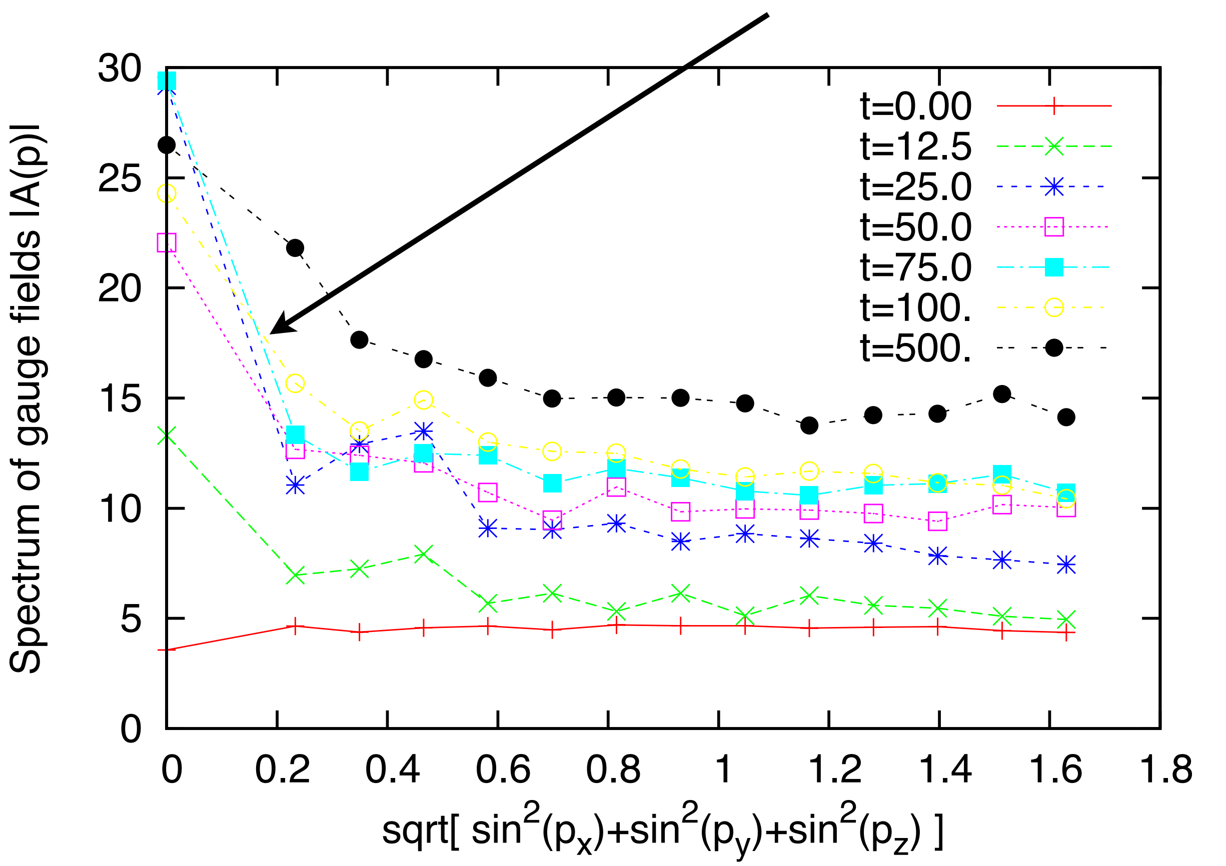
$s_{\rm KS} = \lambda^{\rm ILE}_{\rm sum}/L^3 \simeq 2 \times \varepsilon^{1/4} = c_{KS} \times \varepsilon^{1/4}$

$\tau_{\rm eq} = 2(N_c^2-1)\frac{4\, C_{\rm L}\theta}{3\, c_{\rm KS} T}(\varepsilon^{\rm L})^{-1/4} \simeq \frac{5}{T}$

Equilibration time



Enhancement of low-p modes



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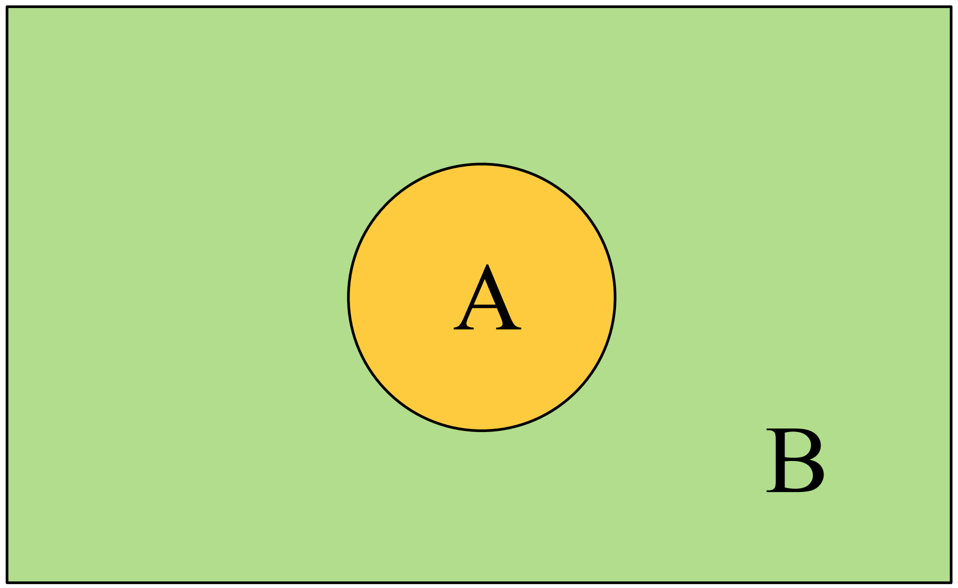
Entanglement entropy - I



Consider a vacuum QFT in a large box.

An observer restricted to subvolume *A* will experience a reduced density matrix

$$\rho_A = \mathrm{Tr}_B(\rho) = \mathrm{Tr}_B\left(|0\rangle\langle 0|\right)$$

Special case of Nakajima-Zwanzig projection!

The *entanglement entropy* between *A* and *B* is defined as $S_A = -\mathrm{Tr}_A\left(\rho_A \ln \rho_A\right)$

It measures the loss of information to the observer from not knowing exactly what the state of the field in the subvolume *A* is, if she does not know the state in *B.*

S_A is a useful measure of how entangled the wave function of the ground state $|0\rangle$ is between *A* and *B*. Naïvely, one would expect that any mode component in *A* with wave number *k* "knows" about the presence of *B* if it is located within distance $\hbar/k$ of the boundary.

Entanglement entropy - II

Therefore one expects (Srednicki, 1993): $S_A \sim \int (\partial A) \sum_{k}^{k_{\max}} \frac{\hbar}{|k|} \sim \kappa \|\partial A\| k_{\max}^2$

The entanglement entropy is thus proportional to the *surface area* of *A*. If one chooses $k_{\max} \sim M_{\mathrm{Pl}}$, S_A becomes the Bekenstein entropy of a black hole with surface area $\|\partial A\|$. Black hole entropy is thus a form of entanglement entropy.

Interactions introduce finite corrections to the UV divergent entanglement entropy. These provide a measure of the range of quantum correlations in the ground state wave function.

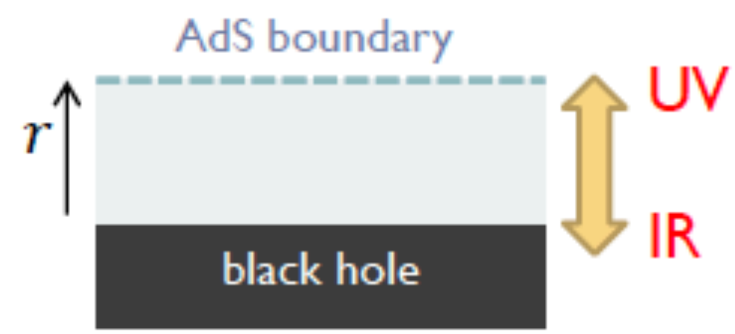
Another variant is when the QFT is not considered in the vacuum state, but at finite temperature *T*. The entanglement entropy then receives a contribution proportional to Vol(*A*), which is precisely the thermal equilibrium entropy.

AdS/CFT dictionary

- Want to study strongly coupled phenomena in QCD
- Toy model: $\mathcal{N} = 4\ SU(N_c)$ SYM

$\mathcal{N} = 4$ SYM	holo. dual ⟷ AdS/CFT	String theory in AdS_5
Strong coupling	⟷	Weak coupling
Vacuum	⟷	Empty AdS_5
Thermal state (equilibrated plasma)	⟷	AdS BH
Thermalization	⟷	BH formation
HI collision	⟷	Energy injection

$$\left(\frac{R_{AdS}}{l_s}\right)^4 = g_{YM}^2 N_c \equiv \lambda$$



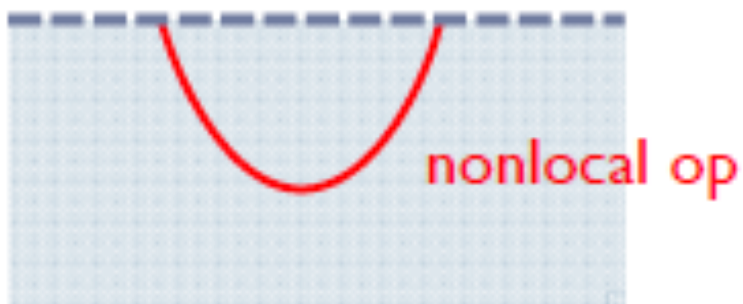
Questions to answer

- What is the measure of thermalization on the boundary?
  - Local operators are not sufficient

$$\langle T_{\mu\nu} \rangle \text{ etc.}$$

  - Nonlocal operators are more sensitive

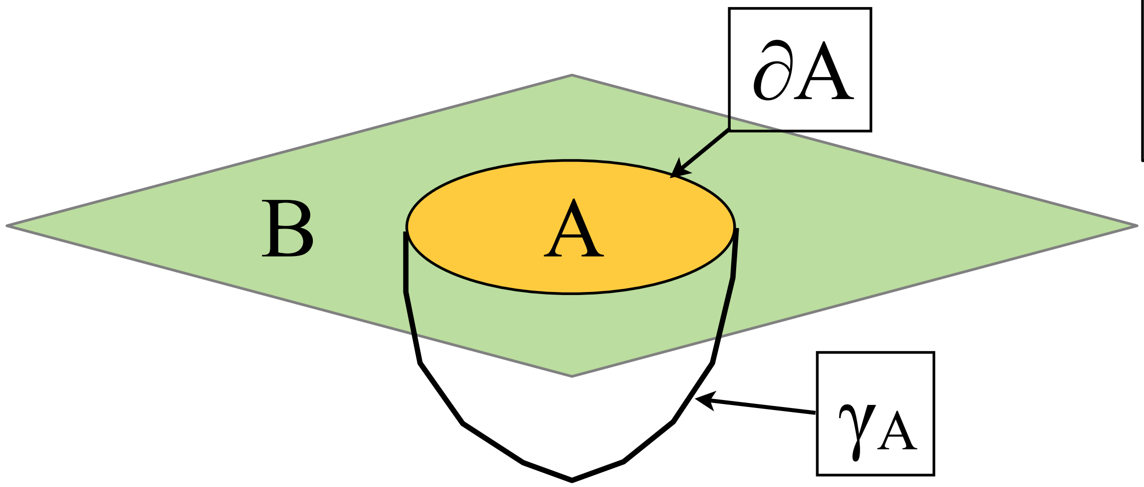
$$\langle O(x) O(x') \rangle \text{ etc.}$$



- What is the thermalization time?
  - When observables reach their thermal values
  - Entropy is the “gold standard”

Entanglement entropy in AdS/CFT

For a (d+1)-dimensional QFT with a holographic gravity dual, S_A can be calculated in the dual theory from the area of the extremal surface γ_A in the bulk, with has the same boundary ∂A as A: $\partial(\gamma_A) = \partial A$.



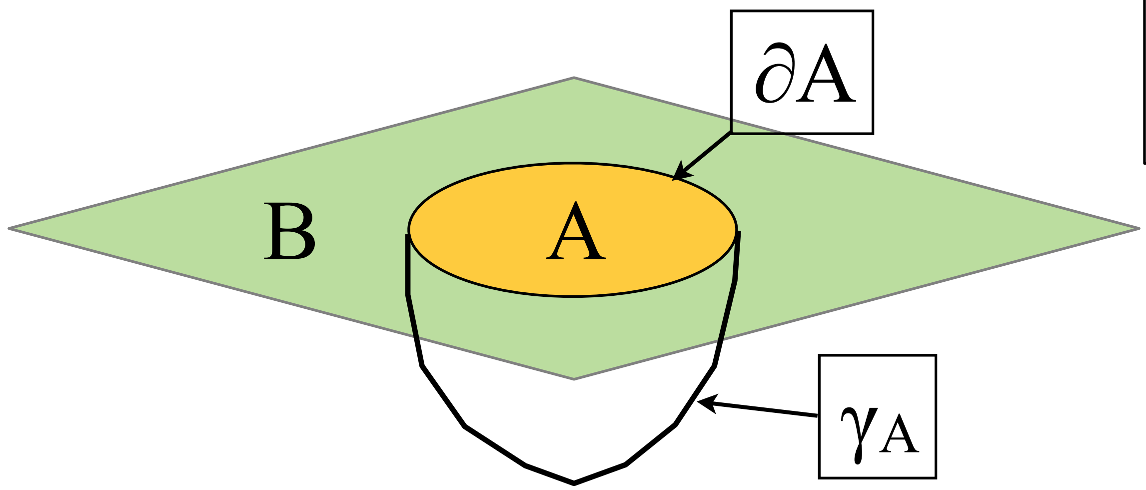
$$S_A = \frac{\|\gamma_A\|}{4G_N^{(d+2)}}$$

see review by:
Nishioka, Ryu,
Takayanagi,
arXiv:0905.0932

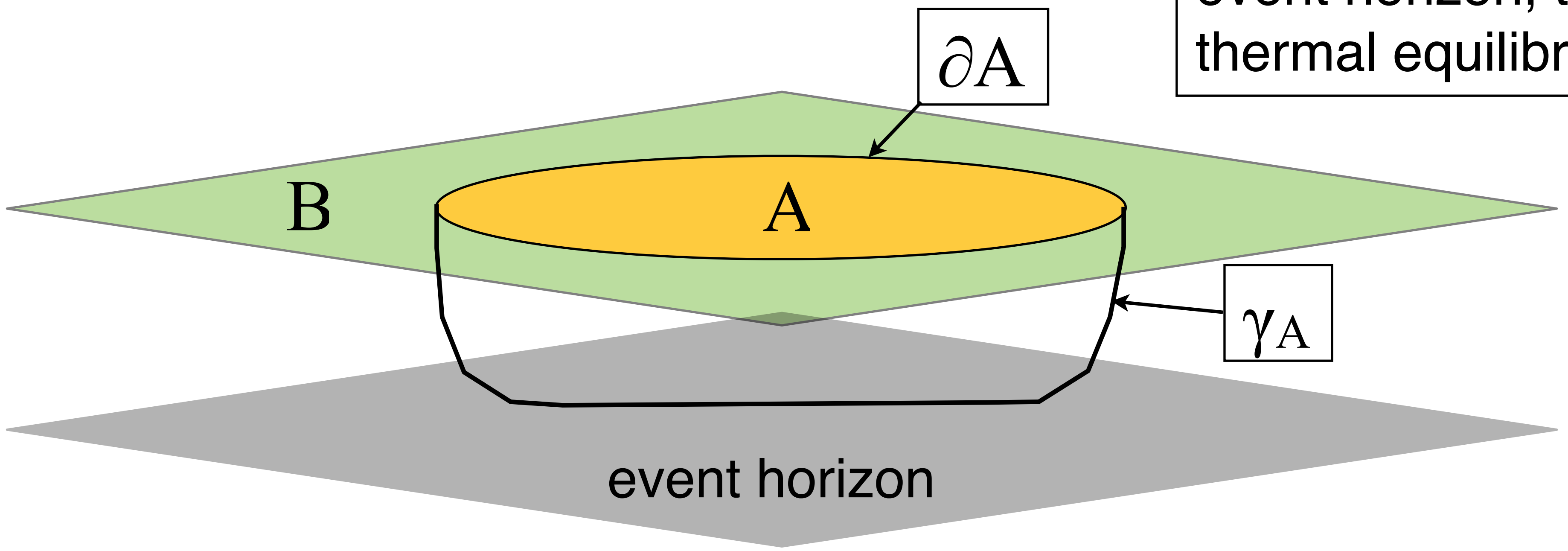
Entanglement entropy in AdS/CFT

For a (d+1)-dimensional QFT with a holographic gravity dual, S_A can be calculated in the dual theory from the area of the extremal surface γ_A in the bulk, with has the same boundary ∂A as A: $\partial(\gamma_A) = \partial A$.

$$S_A = \frac{\|\gamma_A\|}{4G_N^{(d+2)}}$$



At finite temperature, a BH is present, and the surface γ_A picks up a part of the event horizon, thus accounting for the thermal equilibrium entropy of A.



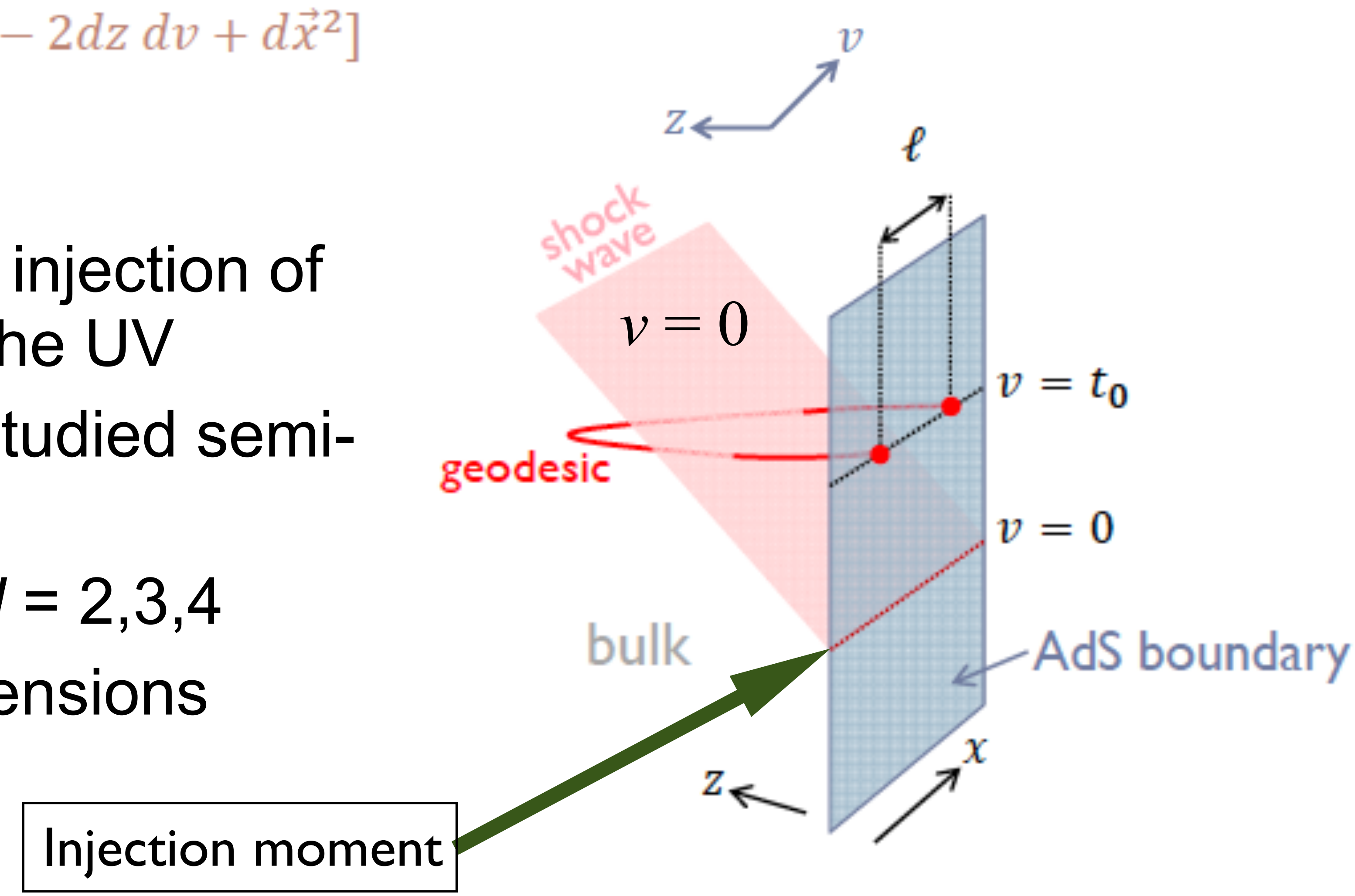
see review by: Nishioka, Ryu, Takayanagi, arXiv:0905.0932

Vaidya-AdS geometry

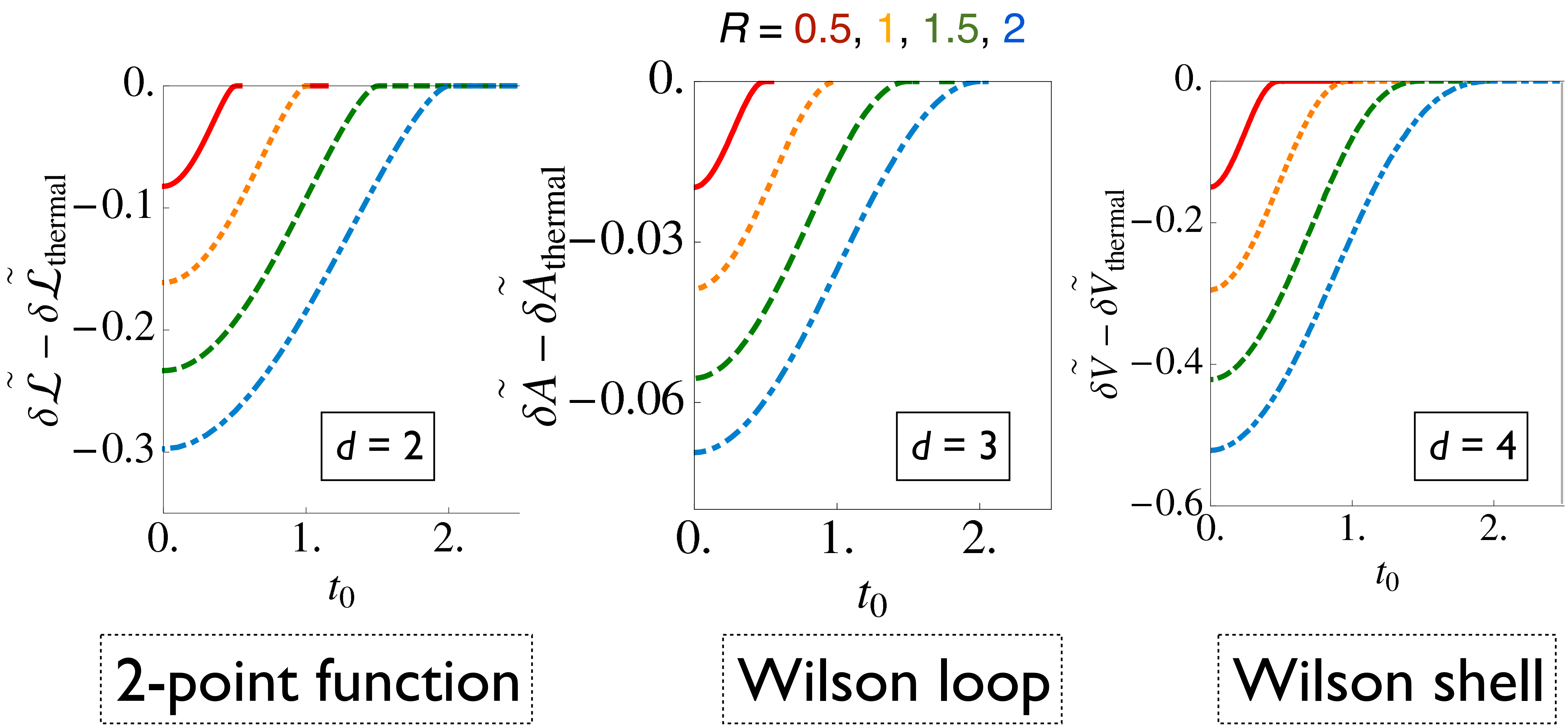
- Light-like (null) infalling energy shell in AdS (shock wave in bulk)
  - *Vaidya-AdS space-time* (analytical)

$$ds^2 = \frac{1}{z^2}[-(1 - m(v)z^d)dv^2 - 2dz\,dv + d\vec{x}^2]$$

  - $z = 0$: UV $\quad z = \infty$: IR
  - Homogeneous, sudden injection of entropy-free energy in the UV
  - Thin-shell limit can be studied semi-analytically
  - We studied AdS_{d+1} for d = 2,3,4
  - ⇔ Field theory in d dimensions

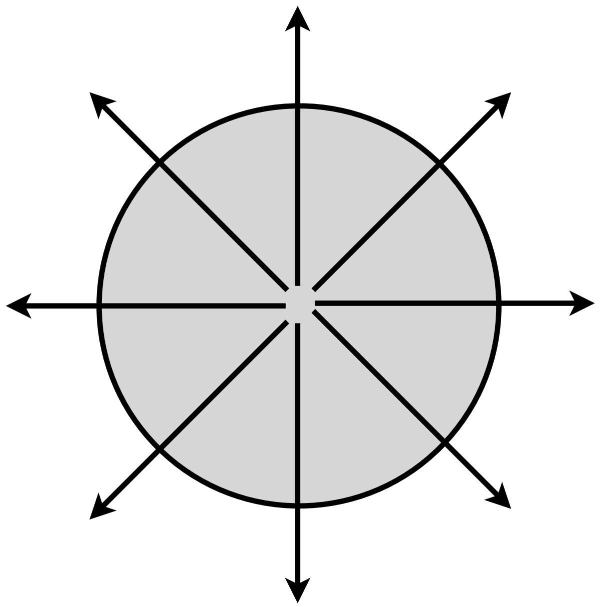


Entanglement entropy



For details: *V. Balasubramanian, et al.*, PRL **106**, 191601 (2011); PRD **84**, 026010

Information escape time



Puzzling question:
What transports information at the speed of light ??

Information takes $c\tau = R/2$ to escape from circular loop

(Very crude) phenomenology for QGP:

$\tau_{crit} \sim 0.5\ \hbar/T \approx 0.3$ **fm/c for** $T = 300 - 400$ **MeV**