Lattice QCD and the photon emission rate of the quark-gluon plasma

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Outline

Part 1: numerics

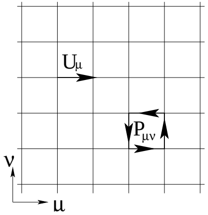
- Probing the photon rate: a calculation in lattice QCD with dynamical up and down quarks.

Part 2: theory aspects

- Interpretation of the vector spectral functions for spacelike momenta;
- Dispersion relation of momentum-space Euclidean correlators at fixed, vanishing photon virtuality.

Work done in collaboration with Marco Cè, Tim Harris, HM, Aman Steinberg, Arianna Toniato; see 1710.07050 (LAT2017) and 1807.00781 (EPJA).

Lattice QCD and vector correlators



Gluons: $U_\mu(x) = e^{iag_0 A_\mu(x)} \in SU(3)$
'link variables'

Quarks: $\psi(x)$ 'on site', Grassmann

Gauge-invariance exactly preserved.

Imaginary-time path-integral representation of QFT (Matsubara formalism).

Imaginary-time **vector correlators** ($\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} = 2\text{diag}(1,-1,-1,-1)$),

$$G^{\mu\nu}(x_0, \boldsymbol{k}) = \int d^3x \; e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \, \text{Tr}\Big\{ \frac{e^{-\beta H}}{Z(\beta)} j^\mu(x)\, j^\nu(0) \Big\}, \qquad j^\mu = \sum_f Q_f \, \bar\psi_f \gamma^\mu \psi_f$$

Spectral representation (u is a real four-vector):

$$u_\mu G^{\mu\nu} u_\nu(x_0, \boldsymbol{k}) = \int_0^\infty \frac{d\omega}{2\pi} \underbrace{\frac{(u_\mu \rho^{\mu\nu} u_\nu)(\omega, \boldsymbol{k})}{\sinh(\beta\omega/2)}}_{\geq 0} \cosh[\omega(\beta/2 - x_0)].$$

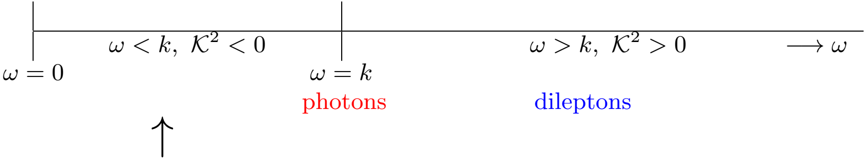
Physical significance of the spectral function for $\mathcal{K}^2 \geq 0$

- Rate of dilepton production per unit volume plasma:

$$d\Gamma_{\ell^+\ell^-}(\mathcal{K}) = \alpha^2 \, \frac{d^4\mathcal{K}}{6\pi^3\mathcal{K}^2} \, \frac{-\rho^{\mu}{}_{\mu}(\mathcal{K})}{e^{\beta\mathcal{K}^0} - 1}$$

- Rate of photon production per unit volume plasma:

$$d\Gamma_{\gamma}(\boldsymbol{k}) = \alpha \, \frac{d^3k}{4\pi^2\, k} \, \frac{-\rho^{\mu}{}_{\mu}(k, k)}{e^{\beta k} - 1}.$$



What about $\mathcal{K}^2 < 0$?
More on this later.

Alternative expression for the photon rate

- current conservation: $\omega^2\rho^{00}(\omega,k) = k^i k^j \rho^{ij}(\omega,k)$.
- $\Rightarrow (\hat{k}^i\hat{k}^j\rho^{ij} - \rho^{00})/\omega$ has the same sign as $\mathcal{K}^2 \equiv \omega^2 - k^2$, and vanishes at $\omega = k$ (photon kinematics).

Therefore, introduce ($k \equiv |\boldsymbol{k}|, \quad \hat{k}^i = k^i/k$)

$$\rho(\omega,k,\lambda) = (\delta^{ij} - \hat{k}^i\hat{k}^j)\rho^{ij} + \lambda\,(\hat{k}^i\hat{k}^j\rho^{ij} - \rho^{00}).$$

In particular,

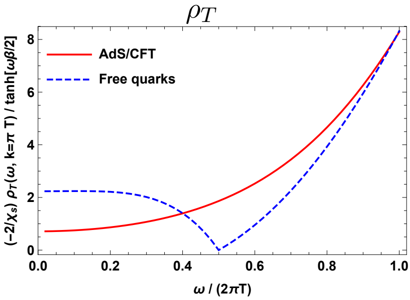
$$\rho(\omega,k,\lambda) = \begin{cases} -\rho^{\mu}{}_{\mu}(\omega,k) = 2\rho_T + \rho_L & \lambda = 1 \\ (\delta^{ij} - 3\hat{k}^i\hat{k}^j)\rho^{ij} + 2\rho^{00} = 2(\rho_T - \rho_L) & \lambda = -2. \end{cases}$$

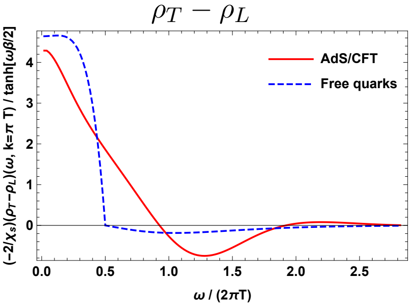
Photon rate can be written ($\forall\lambda$)

$$d\Gamma_\gamma(\boldsymbol{k}) = \alpha\,\frac{d^3k}{4\pi^2\,k}\,\frac{\rho(k,k,\lambda)}{e^{\beta k}-1}.$$

Choosing a favourable λ: weak and strong coupling spectral fcts

Spatial momentum $k = \pi T$:

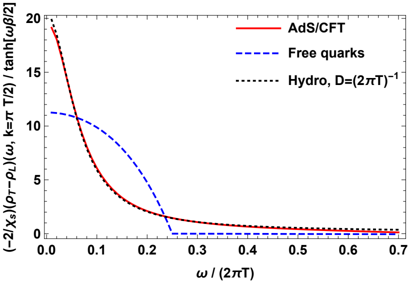




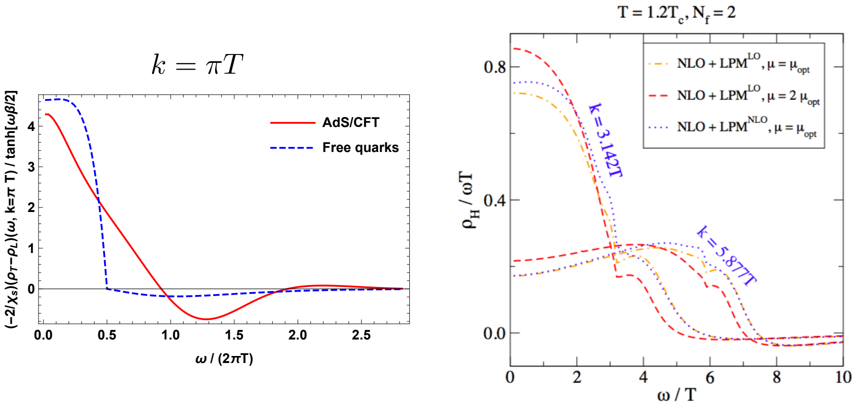
Spatial momentum $k = \pi T/2$:
At strong coupling, hydro works:

$$-2(\rho_T - \rho_L)(\omega, k)/\omega \approx \frac{4\chi_s\, Dk^2}{\omega^2 + (Dk^2)^2},$$

Refs: hep-th/0607237 and 1310.0164.



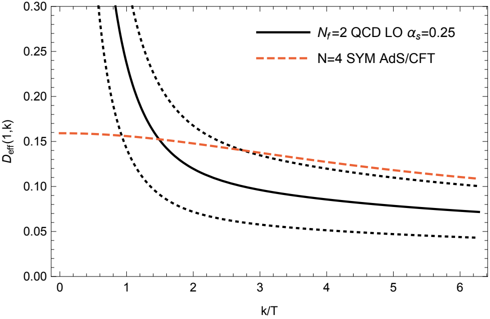
New NLO weak-coupling result Jackson & Laine 1910.09567



- at finite coupling, the kink on the light cone gets smoothened out, resulting in an O($\alpha_s \log 1/\alpha_s$) photon rate.

Summary: properties of $\rho(\omega,k) \equiv \rho(\omega,k,\lambda=-2) = 2(\rho_T - \rho_L)$

- non-negative for $\omega \leq k$
- $\rho(\omega,k) \overset{\omega\to\infty}{\sim} k^2/\omega^4$
- sum rule: $\int_0^\infty \mathrm{d}\omega\, \omega\rho(\omega,\mathrm{k}) = 0$ (so $\rho(\omega,k)$ must go negative somewhere for $\omega > k$)
- effective diffusion coefficient $D_{\rm eff}(1,k) \equiv \frac{\rho(k,k)}{4\chi_s k} \propto$ photon rate.

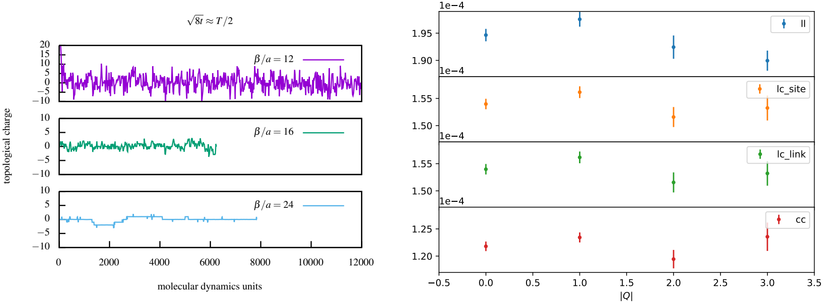


Results from Arnold, Moore, Yaffe hep-ph/0111107 (JHEP); AdS/CFT from hep-ph/0607237.

Lattice set-up with $N_\mathrm{f} = 2$ $\mathrm{O}(a)$-improved Wilson fermions

T (MeV)	T/T_c	β_LAT	β/a	L/a	$m_{\overline{\mathrm{MS}}(2\,\mathrm{GeV})}$ (MeV)	N_meas
250	1.2	5.3	12	48	12	8256
"	"	5.5	16	64	"	4880
"	"	5.69	20	80	"	25000
"	"	5.83	24	96	"	9600

- enables continuum limit at $T = 250$ MeV



- only weak dependence of observable on topological charge
- impact of long autocorrelation time on vector correlator under control.

Continuum limit 1/3

There are four independent discretizations of the $\lambda = -2$ isovector vector correlator

$$G^{\lambda=-2}(x_0, \boldsymbol{k}) = \left(\delta^{ij} - \frac{3k^i k^j}{k^2}\right) G^{ij}(x_0, \boldsymbol{k}) + 2G^{00}(x_0, \boldsymbol{k})$$

where $G^{\mu\nu}(x_0, \boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \langle j^\mu(x) j^\nu(0)\rangle$ using both the local or exactly-conserved lattice vector current.

Project to all spatial momenta, on and off-axis, with $k\beta \leq 2\pi$.

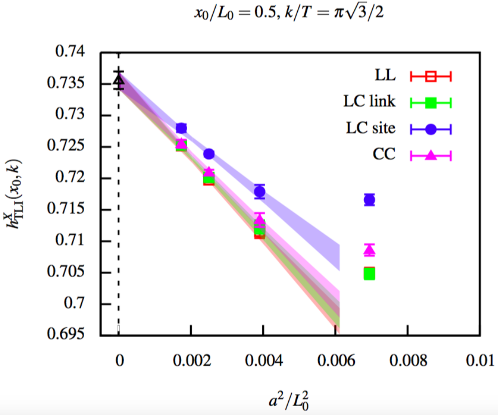
In the chirally-symmetric phase, the matrix-elements of the $\mathrm{O}(a)$-improvement counterterms are suppressed, so we perform a continuum limit in a^2/β^2.

Static susceptibility: $\chi_s \equiv \int d^4x \langle j^0(x) j^0(0)\rangle = 0.880(9)_{\rm stat}(8)_{\rm syst} \cdot T^2$ (this quantity is unity for free massless quarks).

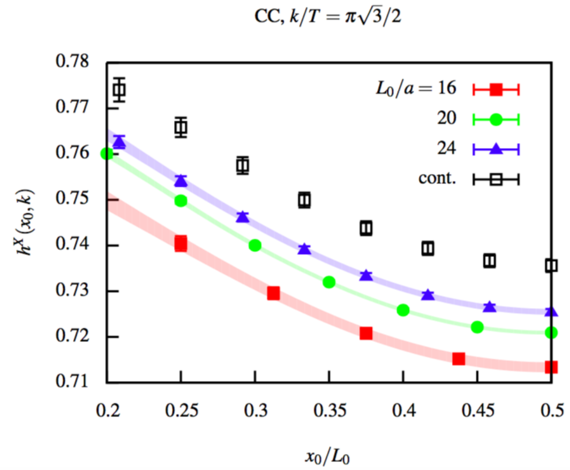
Continuum limit 2/3

Tree-level improvement: $$G(x_0,\boldsymbol{k}) \to \frac{G_{\rm cont.t.l.}(x_0,\boldsymbol{k})}{G_{\rm lat.t.l}(x_0,\boldsymbol{k})} G(x_0,\boldsymbol{k})$$

A piecewise spline interpolation is used before taking the combined continuum limit of the four discretizations of $G(x_0,\boldsymbol{k})/\chi_s$. For $x_0 = \beta/2$:



Continuum limit 3/3 using tree-level improved at $k = \pi T$



- Coarsest ensemble $N_t = 12$ is not included in the continuum extrapolation.
- In the subsequent analysis, we use the continuum-extrapolated correlator at $x_0 \geq \beta/4$.

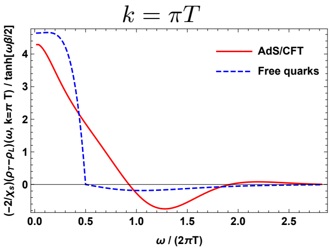
Can the lattice distinguish a weak- from a strong-coupling $\rho(\omega, k)$?

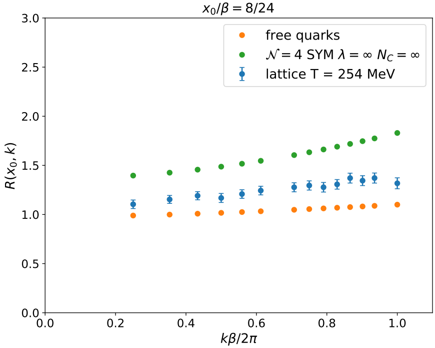
In the "transverse minus longitudinal" channel, consider the ratio

$$R(x_0, k) \equiv \frac{16\pi}{(\beta - 2x_0)^2 k^2} \left[\frac{G(x_0, k)}{G(\beta/2, k)} - 1\right]$$

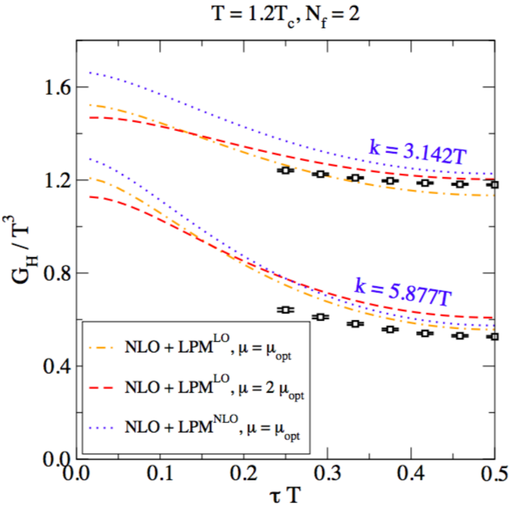
$$= \frac{16\pi}{(\beta - 2x_0)^2 k^2} \frac{\int_0^\infty d\omega\ \rho(\omega, k)(\cosh[\omega(\beta/2 - x_0)] - 1)/\sinh(\omega\beta/2)}{\int_0^\infty d\omega\ \rho(\omega, k)/\sinh(\omega\beta/2)}.$$

This observable differs by a factor ~ 1.5 between the extreme cases of AdS/CFT and non-interacting quarks. Lattice data lies in between.





Comparison with recent NLO calculation ($T = 250\,\text{MeV}$)



- difference between continuum-extrapolated lattice data and NLO calculation is small. Figure from 1910.09567.

Padé fit ansatz for the spectral function

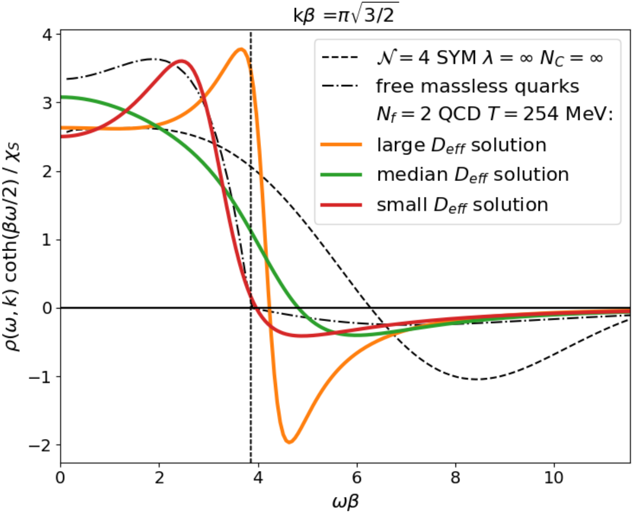
$$\frac{\rho(\omega,k)}{\tanh[\omega\beta/2]} = \frac{A(1+B\omega^2)}{[(\omega-\omega_0)^2+b^2][(\omega+\omega_0)^2+b^2][\omega^2+a^2]},$$

- $\rho(\omega,k)\sim 1/\omega^4$ at large ω (consistent with OPE);
- sum rule $\Rightarrow B = B(a,b,\omega_0)$;
- assume piece-wise linear dependence of (a,b,ω_0) on k^2, i.e. locally $a(k)=a_0+a_2k^2$ etc.
- scan in the non-linear parameters $(a_0,a_2,b_0,b_2,\omega_0^{(0)},\omega_0^{(2)})$, A chose to minimize χ^2
- accept all solutions that satisfy:
  1. $\rho(\omega,k)\geq 0$ for $\omega\leq k$;
  2. p-value above 32% (using the full, regularized covariance matrix);
  3. "there can be no arbitrarily long relaxation times":
     $\min(a,b) > \min(D_{\rm AdS/CFT}k^2, D_{\rm pert}^{-1})$

$D_{\rm AdS/CFT} = \frac{1}{2\pi T}$, $\qquad D_{\rm pert}^{-1} = \mathrm{O}(\alpha_s^2)T = 0.46T$, $\qquad \alpha_s = 0.25$.

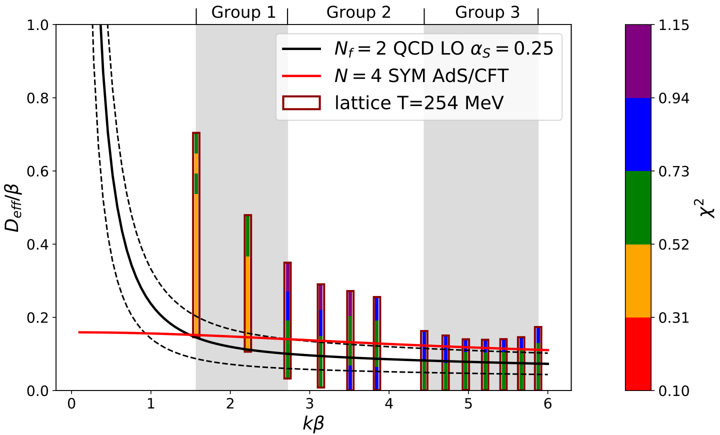
↑ Arnold, Moore, Yaffe hep-ph/0302165

Typical spectral functions resulting from the Padé fit



- All three describe the lattice data, fullfill the positivity requirement and do not have singularities too close to the real axis.

Result at $T = 250\,\text{MeV}$



- Near-final result. Used covariance matrix C with modest amount of regularization.
- Results very much in line with weak-coupling prediction; lattice data presently unable to exclude large values of D at momenta $k \lesssim 2.5T$.

Interpretation of the spectral function for spacelike momenta

Cross-section per unit volume for a lepton scattering on the medium through the exchange of a space-like photon ($\ell^{\mu\nu} \equiv 2(p^\mu p'^\nu + p^\nu p'^\mu - g^{\mu\nu}(p\cdot p'))$):

$$\frac{d^2\sigma}{L^3 dp^{0\prime} d\Omega} = \frac{e^4 (p^{0\prime}/p^0)}{32\pi^3 \mathcal{K}^4} \ell_{\mu\nu} \frac{\rho^{\mu\nu}(k^0, \boldsymbol{k})}{1-e^{-\beta k^0}},$$

with p and p' respectively the initial and final lepton momenta and $k = p - p'$.

More generally: vector spectral functions measure the ability of the medium to convert the energy stored in external electromagnetic fields into heat:

- Couple the plasma to a harmonic external vector potential $\boldsymbol{A}(t,\boldsymbol{x}) = \mathrm{Re}(\boldsymbol{A}_{\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)})$, via Hamiltonian $\Delta H = -e\int d^3x\, \boldsymbol{j}\cdot\boldsymbol{A}$.
- If energy of external electromagnetic fields: $E_{\rm e.m.} = \frac{1}{2}\int d^3x\, (\boldsymbol{E}^2 + \boldsymbol{B}^2)$,

$$\boldsymbol{A}_{\boldsymbol{k}} \perp \boldsymbol{k} : \quad \frac{-1}{E_{\rm e.m.}}\frac{dE_{\rm e.m.}}{dt} = \alpha \frac{2\pi\omega\,(\delta^{ij} - \hat{k}^i\hat{k}^j)\rho^{ij}(\omega,\boldsymbol{k})}{\omega^2 + k^2},$$

$$\boldsymbol{A}_{\boldsymbol{k}} \parallel \boldsymbol{k} : \quad \frac{-1}{E_{\rm e.m.}}\frac{dE_{\rm e.m.}}{dt} = \alpha \frac{4\pi \hat{k}^i\hat{k}^j \rho^{ij}(\omega,\boldsymbol{k})}{\omega}$$

Dispersion relation for a Euclidean correlator at zero virtuality

- Let $\sigma(\omega) \equiv \rho_T(\omega, |\boldsymbol{k}| = \omega)$ be the relevant spectral function proportional to the photon emission rate;
- let $H_E(\omega_n) \equiv G_E(\omega_n, k = i\omega_n)$ the momentum-space Euclidean correlator with Matsubara frequency ω_n and imaginary spatial momentum $k = i\omega_n$;
- once-subtracted dispersion relation: $(\sigma(\omega) \sim \omega^{1/2}$ at weak coupling)

$$H_E(\omega_n) - H_E(\omega_r) = \int_0^\infty \frac{d\omega}{\pi}\, \omega\, \sigma(\omega) \Big[\frac{1}{\omega^2 + \omega_n^2} - \frac{1}{\omega^2 + \omega_r^2}\Big], \quad n, r \neq 0.$$

Recall photon rate:

$$\frac{d\Gamma_\gamma}{d\omega} = \frac{\alpha}{\pi} \frac{1}{e^{\beta\omega} - 1} \cdot (\omega\, \sigma(\omega)).$$

[HM, 1807.00781.]

Conclusion

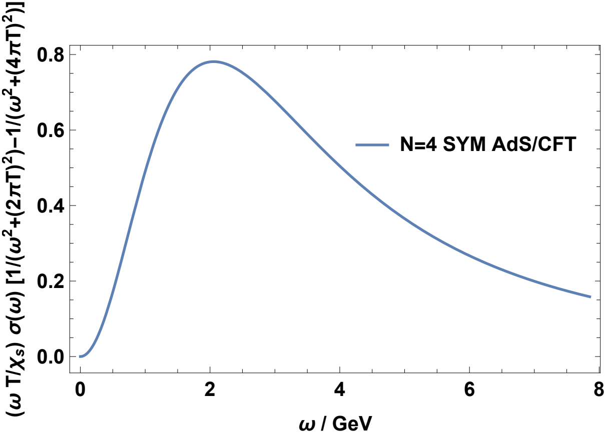
- Photon rate: first lattice calculation in dynamical QCD with continuum limit.
- The transverse-minus-longitudinal combination cancels a large ultraviolet and admits a super-convergent sum rule.
- Euclidean correlators show only small differences with weak-coupling prediction; results for $d\Gamma_\gamma(k \geq \pi T)$ compatible with weak-coupling prediction.
- Dispersion relation at fixed photon virtuality $q^2 = 0$ can be used to probe exclusively the photon rate (rather than the full (ω, k) dependence).

List of references

Lattice papers on the photon rate:

- Karsch, Laermann, Petreczky, Stickan, Wetzorke 2002; S. Gupta 2004; Aarts, Allton, Foley, Hands, Kim 2007: quenched calculations, $k = 0$.
- hep-lat/0610061 (LAT06): Aarts, Allton, Foley, Hands: quenched, $k \neq 0$
- 1012.4963 (PRD): Ding, Francis, Kaczmarek, Karsch, Laermann, Soeldner, quenched calculation with continuum limit, $k = 0$.
- 1212.4200 (JHEP): Brandt, Francis, HM, Wittig: $N_f = 2$, $N_t = 16$, $k = 0$, $m_\pi = 270$, $T = 250$MeV.
- 1307.6763 (PRL), 1412.6411 (JHEP): Aarts, Allton, Amato,Giudice, Hands, Skullerud: $N_f = 2 + 1$, $k = 0$, anisotropic, fixed-scale temperature scan, $m_\pi = 384$ MeV
- 1512.07249 (PRD): Brandt, Francis, Jäger, HM, $N_f = 2$, $k = 0$, $N_t = 12 \to 24$, $m_\pi = 270$, fixed-scale scan across the phase transition.
- 1604.07544 (PRD): Ghiglieri, Kaczmarek, Laine, F. Meyer: quenched calculation with continuum limit, $k \neq 0$.
- 1310.0164, Laine; 1910.09567, Jackson, Laine: spectral functions at NLO.
- here: $N_f = 2$ calculation with continuum limit at $T = 250$MeV, $k \neq 0$.

What the dispersive integrand might look like for $T = 250\,$MeV



- the lattice observable $[H_E(2\pi T) - H_E(4\pi T)]$ is mostly sensitive to the rate of emission of multi-GeV photons.

A sum rule for $\rho \equiv \rho_{\lambda=-2}$

i. Lorentz invariance and transversity $\Rightarrow \tilde{G}_{\rm E}(\omega_n, k) = 0$ in vacuum and UV finite at $T > 0$

ii. OPE: from power-counting one expects $\tilde{G}_{\rm E}(\omega_n, k) \sim \frac{\langle \mathcal{O}_4 \rangle}{\omega_n^2}$

Furthermore, charge conservation demands $\tilde{G}_{\rm E}(\omega_n, k) \to 0$ as $k \to 0$ and $\omega_n \neq 0$, so actually

$$\tilde{G}_{\rm E}(\omega_n, k) \sim \frac{k^2 \langle \mathcal{O}_4 \rangle}{\omega_n^4}$$

iii. From the dispersive representation:

$$\tilde{G}_{\rm E}(\omega_n, k) = \int_0^\infty \frac{{\rm d}\omega}{\pi} \omega \frac{\rho(\omega, k)}{\omega^2 + \omega_n^2} \stackrel{\omega_n \to \infty}{\longrightarrow} \frac{1}{\pi \omega_n^2} \int_0^\infty {\rm d}\omega\, \omega\, \rho(\omega, {\rm k})$$

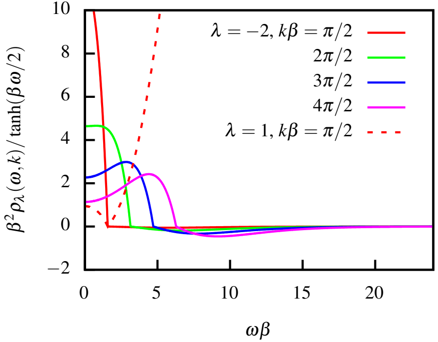
The two expressions are only compatible if the super-convergent sum rule

$$\boxed{\int_0^\infty {\rm d}\omega\, \omega \rho(\omega, {\rm k}) = 0}$$

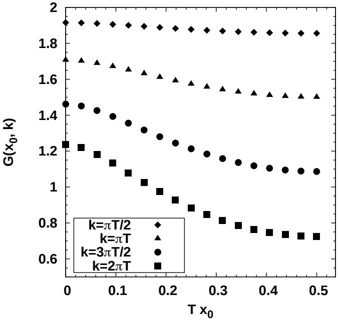
holds.

Choosing a favourable λ: non-interacting fermions

Spectral function



Euclidean correlator with $\lambda = -2$



- We choose $\lambda = -2$ from now on: UV-finite correlator even at $x_0 = 0$.
- for $k = \mathrm{O}(\pi T)$, $\rho(k, k, \lambda) = \mathrm{O}(\alpha_s \log \alpha_s)$ in perturbation theory.

Sketch of the (standard) derivation of the dispersion relation

$G_R(\omega, k) = i(\delta_{il} - \frac{k_i k_l}{k^2}) \int d^4x \, e^{i\mathcal{K}\cdot x}\theta(x^0) \left\langle [\mathrm{j}^i(x), \mathrm{j}^l(0)] \right\rangle$. But

$$[j^\mu(x), j^\nu(0)] = 0 \quad \text{for} \quad x^2 < 0,$$

$\Rightarrow$ the retarded correlator $H_R(\omega) \equiv G_R(\omega, k = \omega)$ at lightlike momentum is analytic for $\mathrm{Im}\,(\omega) > 0$. Similarly, the advanced correlator $H_A(\omega)$ is analytic for $\mathrm{Im}\,(\omega) < 0$.

Define the function $H(\omega) = \begin{cases} H_R(\omega) & \mathrm{Im}\,(\omega) > 0 \\ H_A(\omega) & \mathrm{Im}\,(\omega) < 0 \end{cases}$.

It is analytic everywhere, except for a discontinuity on the real axis:

$$H(\omega + i\epsilon) - H(\omega - i\epsilon) = H_R(\omega) - H_A(\omega) = i\sigma(\omega),$$

Write a Cauchy contour-integral representation (using two half-circles) of $H(\omega)$ just above the real axis, where it coincides with $H_R(\omega)$:

$$H_R(\omega) = H_R(\omega_r) + \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, \sigma(\omega') \Big[\frac{1}{\omega' - \omega - i\epsilon} - \frac{1}{\omega' - \omega_r - i\epsilon} \Big].$$

The dispersion relation for the Euclidean correlator follows from the observation $G_E(\omega_n, k^2) = G_R(i\omega_n, k^2), \qquad n > 0.$

Representation through non-static screening masses

$$\tilde{G}_E(\omega_r, x_3) = -2\int_0^\beta dx_0\; e^{i\omega_r x_0} \int dx_1 dx_2\; \langle J_1(x) J_1(0)\rangle = \sum_n |A_n^{(r)}|^2 e^{-E_n^{(r)}|x_3|}$$

$$\Rightarrow \underbrace{H_E(\omega_r)}_{=\mathrm{O}(g^2)} \equiv \int_{-\infty}^{\infty} dx_3\; \tilde{G}_E(\omega_r, x_3)\; e^{\omega_r x_3} = 2\omega_r^2 \sum_{n=0}^{\infty} \underbrace{|A_n|^2}_{=\mathrm{O}(g^4)} \underbrace{\frac{1}{E_n^{(r)}\,(E_n^{(r)\,2} - \omega_r^2)}}_{=\mathrm{O}(g^{-2})}.$$

This helps explain the connection observed in [Brandt et al, 1404.2404] between non-static screening masses and the LPM-resummation contributions to the photon emission rate [Aurenche et al, hep-ph/0211036].

In lattice regularization, Lorentz symmetry is absent $\Rightarrow H_E(\omega_r)$ does not vanish in vacuum as it does in the continuum. Explicit subtraction of the *in vacuo* $H_E(\omega_r)$ from the thermal $H_E(\omega_r)$ is necessary.