Sum Rule of Femtoscopic Correlation Function

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in collaboration with

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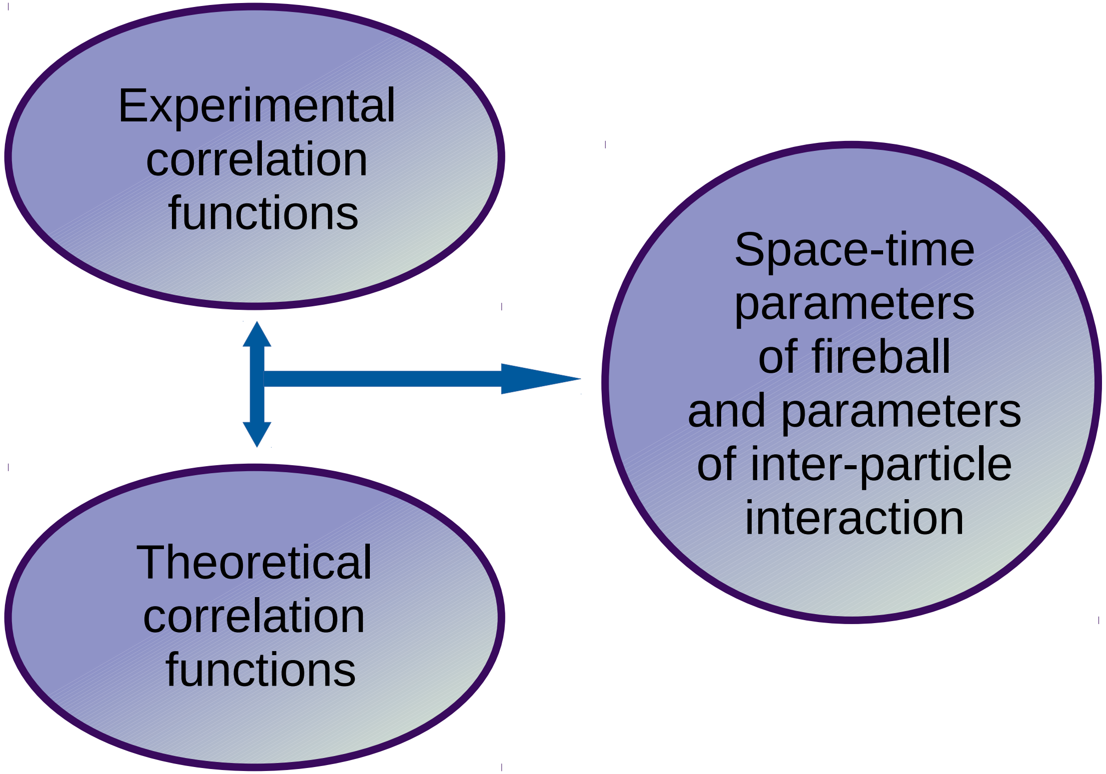
based on arXiv:1908.03178

Motivation


Experimental correlation functions
Theoretical correlation functions
Space-time parameters of fireball and parameters of inter-particle interaction

Correlation function

Definition of the correlation function $R(\mathbf{p}_a, \mathbf{p}_b)$

$$\frac{dP_{ab}}{d^3p_a\, d^3p_b} = R(\mathbf{p}_a, \mathbf{p}_b)\, \frac{dP_a}{d^3p_a}\, \frac{dP_b}{d^3p_b}$$

$\dfrac{dP_a}{d^3p_a}, \dfrac{dP_b}{d^3p_b}$ - probability densities to observe a particle *a* or *b* in a collision final state

$\dfrac{dP_{ab}}{d^3p_a\, d^3p_b}$ - probability density to observe a pair *ab* in a collision final state

Correlation function – nonrelativistic Koonin model

Nonrelativistic model of the correlation function developed by Koonin*

$$R(\mathbf{p}_a, \mathbf{p}_b) = \int d^3 r_a \, d^3 r_b \, D(\mathbf{r}_a) D(\mathbf{r}_b) |\psi(\mathbf{r}_a, \mathbf{r}_b)|^2$$

$D(\mathbf{r}_i)$ - '*effective*' single-particle source function; the probability distribution of emission points of a particle *i*

$$\int d^3 r \, D(\mathbf{r}) = 1$$

$\psi(\mathbf{r}_a, \mathbf{r}_b)$ - the wave function of the two particles *a* and *b*

S. E. Koonin*, Phys. Lett. **B70 (1977) 43

Correlation function – nonrelativistic Koonin model

Change of variables

$$R(\mathbf{p}_a,\mathbf{p}_b)=\int d^3 r_a\, d^3 r_b\, D(\mathbf{r}_a)\, D(\mathbf{r}_b)\, |\psi(\mathbf{r}_a,\mathbf{r}_b)|^2$$

$$\mathbf{R}=\frac{m_a\mathbf{r}_a+m_b\mathbf{r}_b}{M},\quad \mathbf{r}=\mathbf{r}_a-\mathbf{r}_b,$$

$$\mathbf{P}=\mathbf{p}_a+\mathbf{p}_b,\quad \mathbf{q}=\frac{m_a\mathbf{p}_a-m_b\mathbf{p}_b}{M}$$

$$M\equiv m_a+m_b$$

Center-of-mass
and
relative coordinates

$$R(\mathbf{q})=\int d^3 r\, D_r(\mathbf{r})\, |\phi_{\mathbf{q}}(\mathbf{r})|^2$$

$$\psi_{\mathbf{q}}(\mathbf{r}_a,\mathbf{r}_b)=\exp(i\,\mathbf{P}\,\mathbf{R})\,\phi_{\mathbf{q}}(\mathbf{r})$$

$$D_r(\mathbf{r})\equiv\int d^3 R\, D\left(\mathbf{R}+\frac{m_a}{M}\mathbf{r}\right) D\left(\mathbf{R}-\frac{m_b}{M}\mathbf{r}\right)$$

‘*relative*’ source function

Source function

Parametrization as simple as possible – isotropic Gaussian form of single-particle source function

$$D(\mathbf{r}_i) = \frac{1}{(2\pi r_0^2)^{3/2}} \exp\left(-\frac{\mathbf{r}_i^2}{2 r_0^2}\right)$$

gives $\langle \mathbf{r}^2 \rangle = 3 r_0^2$

$$D_r(\mathbf{r}) \equiv \int d^3R\, D\left(\mathbf{R} + \frac{m_a}{M}\mathbf{r}\right) D\left(\mathbf{R} - \frac{m_b}{M}\mathbf{r}\right)$$

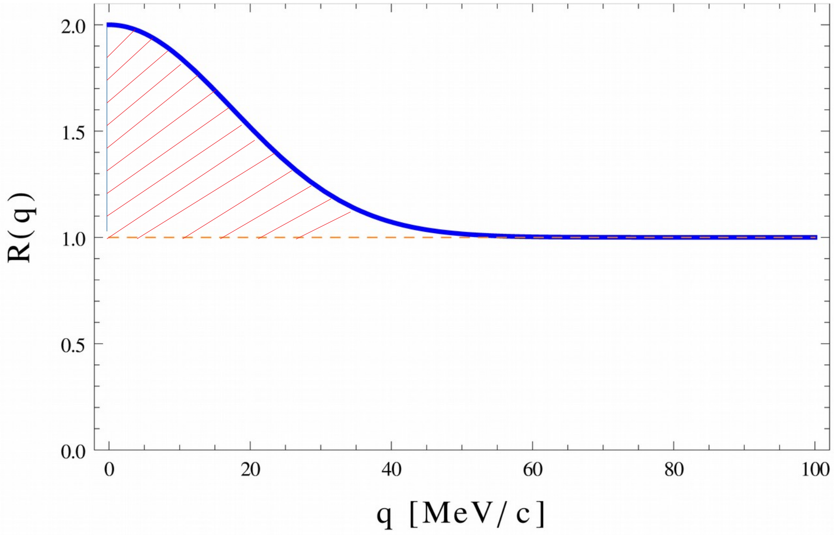
$$D(\mathbf{r}_i) = \frac{1}{(2\pi r_0^2)^{3/2}} \exp\left(-\frac{\mathbf{r}_i^2}{2 r_0^2}\right) \quad \Rightarrow \quad D_r(\mathbf{r}) = \frac{1}{(4\pi r_0^2)^{3/2}} \exp\left(-\frac{\mathbf{r}^2}{4 r_0^2}\right)$$

Gaussian form of the ‘relative’ source,
also spherically symmetric

Point-like ‘relative’ source: $D_r(\mathbf{r}) = \delta^{(3)}(\mathbf{r})$

Sum rule

Sum rule of the correlation function *



The momentum integral of correlation functions – any new information?

$$\lim_{q \to \infty} R(\mathbf{q}) = 1$$

We consider

$$\int d^3q \left(R(\mathbf{q}) - 1 \right) = ?$$

S. Mrówczyński*, Phys. Lett. B **345 (1995) 393

Sum rule

$$\int \frac{d^3q}{(2\pi)^3}(R(\mathbf{q})-1)=\int \frac{d^3q}{(2\pi)^3}\Big(\underbrace{\int d^3r\, D_r(\mathbf{r})|\phi_{\mathbf{q}}(\mathbf{r})|^2}_{R(\mathbf{q})}-1\Big)$$

The source function is normalized

$$\int \frac{d^3q}{(2\pi)^3}(R(\mathbf{q})-1)=\int \frac{d^3q}{(2\pi)^3}\int d^3r\, D_r(\mathbf{r})\left(|\phi_{\mathbf{q}}(\mathbf{r})|^2-1\right)$$

changing the order of integration

$$\int \frac{d^3q}{(2\pi)^3}(R(\mathbf{q})-1)=\int d^3r\, D_r(\mathbf{r})\left[\int \frac{d^3q}{(2\pi)^3}\left(|\phi_{\mathbf{q}}(\mathbf{r})|^2-1\right)\right]$$

Both integrals must exist !

Sum rule

Quantum mechanical completeness condition of quantum states

$$\int \frac{d^3q}{(2\pi)^3}\phi_{\mathbf{q}}(\mathbf{r})\,\phi^*_{\mathbf{q}}(\mathbf{r}')+\sum_\alpha \phi_\alpha(\mathbf{r})\,\phi^*_\alpha(\mathbf{r}')=\delta^{(3)}(\mathbf{r}-\mathbf{r}')\pm\delta^{(3)}(\mathbf{r}+\mathbf{r}')$$

Integral representation of Dirac delta distribution

$$\delta^{(3)}(\mathbf{r}-\mathbf{r}')=\int\frac{d^3q}{(2\pi)^3}e^{i\,\mathbf{q}(\mathbf{r}-\mathbf{r}')}$$

$$\int \frac{d^3q}{(2\pi)^3}\phi_{\mathbf{q}}(\mathbf{r})\,\phi^*_{\mathbf{q}}(\mathbf{r}')-\delta^{(3)}(\mathbf{r}-\mathbf{r}')+\sum_\alpha \phi_\alpha(\mathbf{r})\,\phi^*_\alpha(\mathbf{r}')=\pm\delta^{(3)}(\mathbf{r}+\mathbf{r}')$$

Proper symmetry in the case of identical particles

Limit $\mathbf{r}'\rightarrow\mathbf{r}$

$$\boxed{\int \frac{d^3q}{(2\pi)^3}\left(|\phi_{\mathbf{q}}(\mathbf{r})|^2-1\right)=\pm\delta^{(3)}(2\mathbf{r})-\sum_\alpha|\phi_\alpha(\mathbf{r})|^2}$$

Sum rule

$$\int \frac{d^3q}{(2\pi)^3}\left(|\phi_{\mathbf{q}}(\mathbf{r})|^2-1\right)=\underbrace{\pm\delta^{(3)}(2\mathbf{r})-\sum_\alpha|\phi_\alpha(\mathbf{r})|^2}$$

Completeness condition of quantum states

$$\int \frac{d^3q}{(2\pi)^3}(R(\mathbf{q})-1)=\int d^3r\,D_r(\mathbf{r})\left[\int \frac{d^3q}{(2\pi)^3}\left(|\phi_{\mathbf{q}}(\mathbf{r})|^2-1\right)\right]$$

Integral of interest

Sum Rule:

$$\boxed{\int d^3q\,(R(\mathbf{q})-1)=\pm\pi^3 D_r(0)-\sum_\alpha A_\alpha}$$

$A_\alpha=(2\pi)^3\int d^3r\,D_r(\mathbf{r})|\phi_\alpha(\mathbf{r})|^2$ - the formation rate of a bound state

$$\frac{dP_\alpha}{d^3P}=A_\alpha\frac{dP_a}{d^3p_a}\frac{dP_b}{d^3p_b}$$

Test of the sum rule

$$\int d^3q\left(R(\mathbf{q})-1\right)=\pm\pi^3 D_r(0)-\sum_\alpha A_\alpha$$

Sum rule predictions

$$\int d^3q\left(R(\mathbf{q})-1\right)=\pi^3 D_r(0)$$ - identical bosons

$$\int d^3q\left(R(\mathbf{q})-1\right)=-\sum_\alpha A_\alpha$$ - nonidentical particles forming bound states

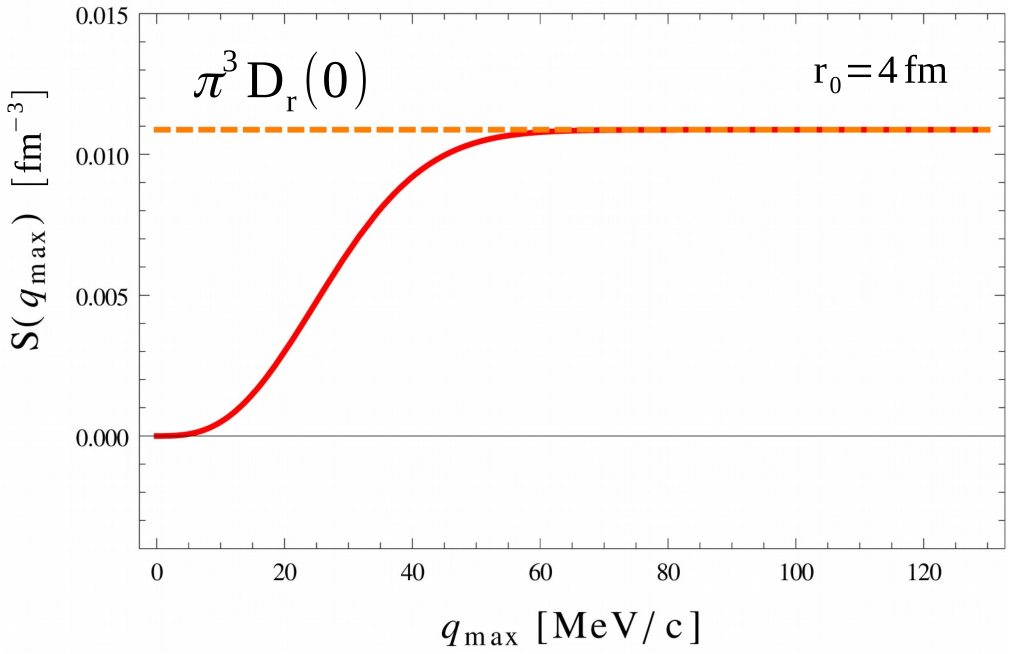
$$\int d^3q\left(R(\mathbf{q})-1\right)=0$$ - nonidentical particles which do not form a bound state

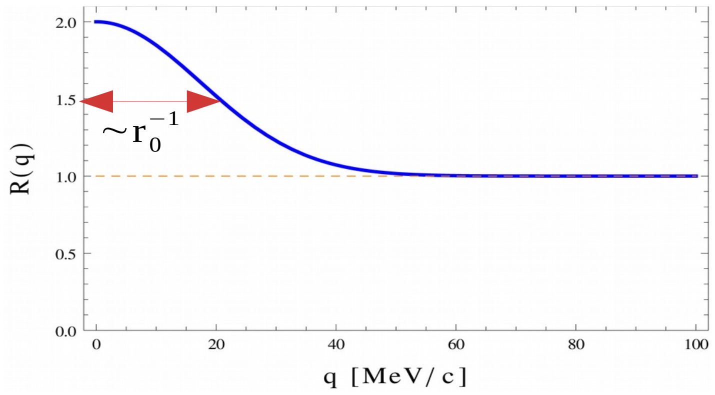
Function showing how fast the integral saturates

$$S(q_{max})\equiv 4\pi\int_0^{q_{max}} dq\, q^2\left(R(q)-1\right)$$

Test of the sum rule

Free identical bosons Expected result: $\int d^3q\left(R(\mathbf{q})-1\right)=\pi^3 D_r(0)$

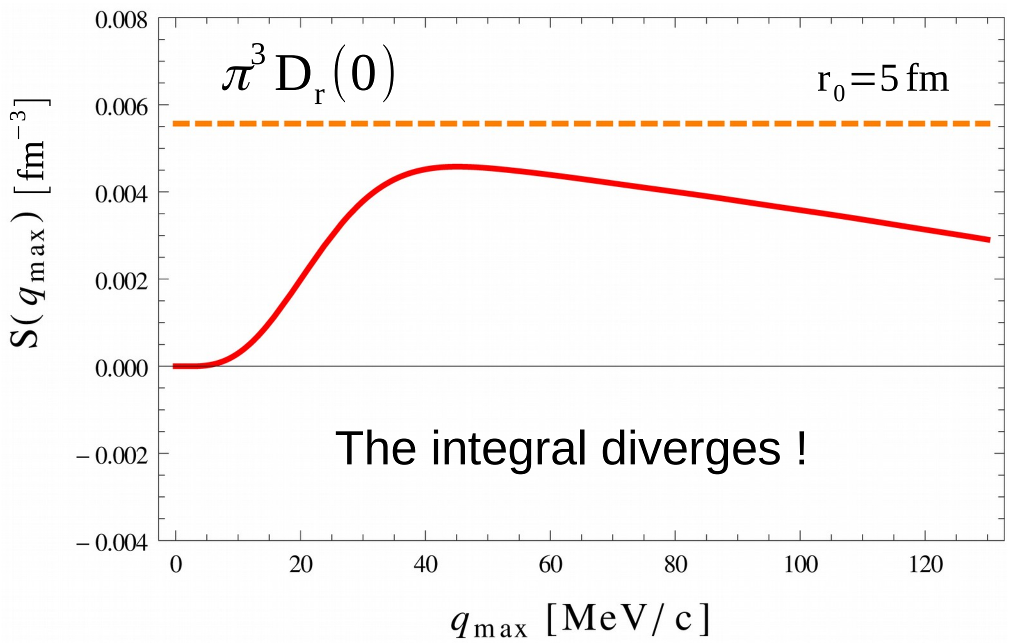


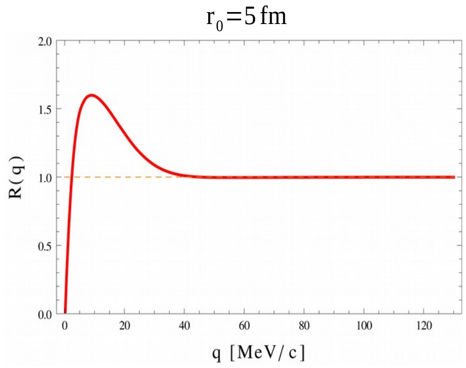


Sum rule works!

Test of the sum rule

Identical pions $\pi^+\pi^+,\ \pi^-\pi^-$ Expected result: $\int d^3q\,(R(\mathbf{q})-1)=\pi^3 D_r(0)$





$$\int \frac{d^3q}{(2\pi)^3}(R(\mathbf{q})-1)=\int \frac{d^3q}{(2\pi)^3}\int d^3r\, D_r(\mathbf{r})\left(|\phi_{\mathbf{q}}(\mathbf{r})|^2-1\right)$$

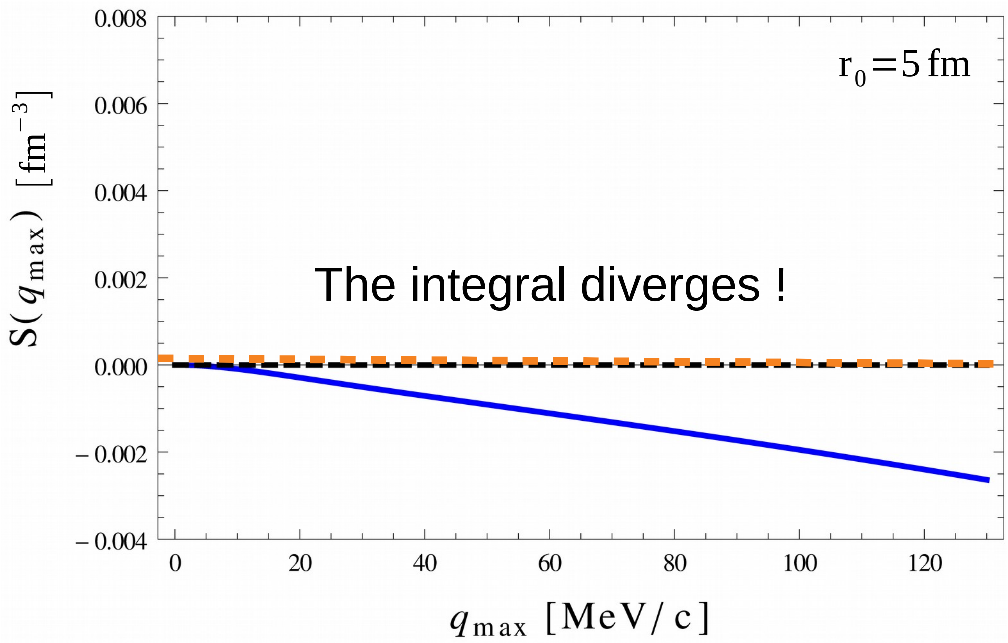
$$\int \frac{d^3q}{(2\pi)^3}(R(\mathbf{q})-1)=\int d^3r\, D_r(\mathbf{r})\left[\int \frac{d^3q}{(2\pi)^3}\left(|\phi_{\mathbf{q}}(\mathbf{r})|^2-1\right)\right]$$

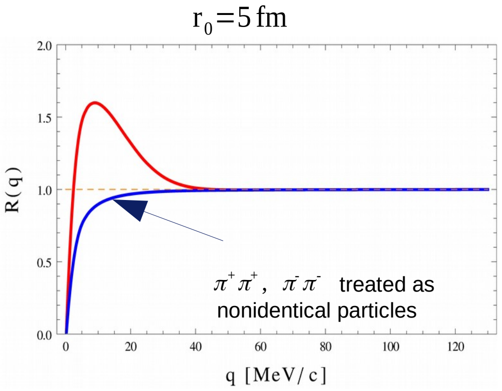
Sum rule does not work

Test of the sum rule

Pions $\pi^+\pi^+$, $\pi^-\pi^-$ treated as nonidentical particles

Expected result: $\int d^3q\,(R(\mathbf{q})-1)=0$





$$\int \frac{d^3q}{(2\pi)^3}(R(\mathbf{q})-1)=\int \frac{d^3q}{(2\pi)^3}\int d^3r\,D_r(\mathbf{r})\left(|\phi_{\mathbf{q}}(\mathbf{r})|^2-1\right)$$

$$\int \frac{d^3q}{(2\pi)^3}(R(\mathbf{q})-1)=\int d^3r\,D_r(\mathbf{r})\left[\int \frac{d^3q}{(2\pi)^3}\left(|\phi_{\mathbf{q}}(\mathbf{r})|^2-1\right)\right]$$

Sum rule does not work

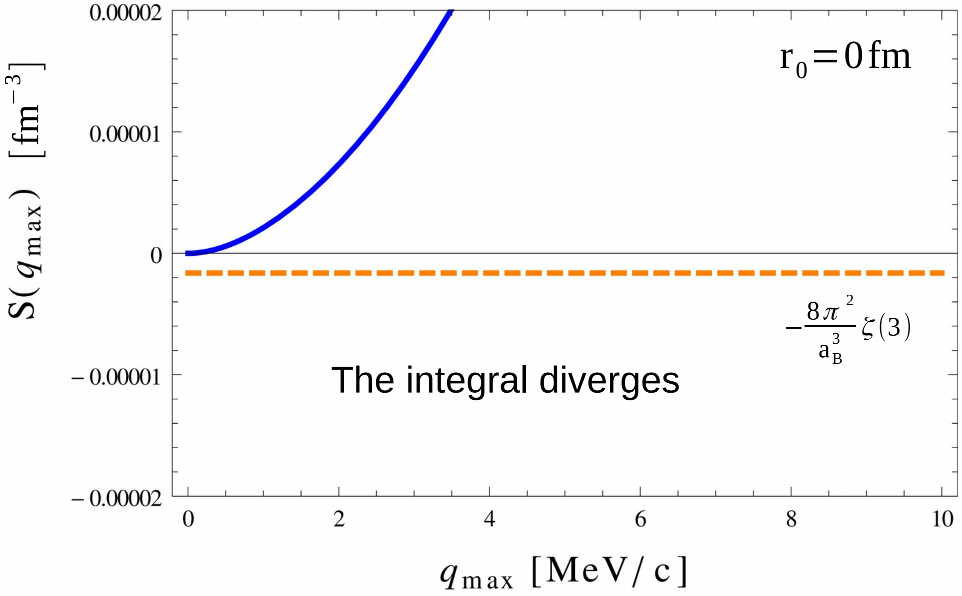
Test of the sum rule

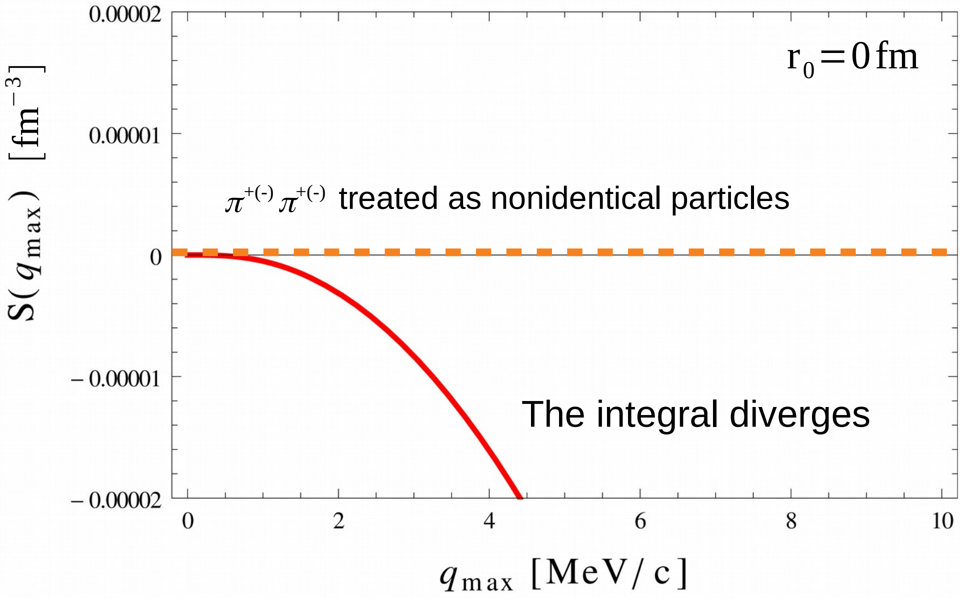
Pions $\pi^+ \pi^-$ $\qquad D_r(\mathbf{r}) = \delta^{(3)}(\mathbf{r})$ - point-like source

$$\int d^3q \left(R(\mathbf{q}) - 1\right) = -\sum_{n=1}^{\infty} A_{nlm} = -\frac{8\pi^2}{a_B^3}\zeta(3)$$

Expected results:

$$\int d^3q \left(R(\mathbf{q}) - 1\right) = 0$$





Sum rule does not work

Improved sum rule

The sum rule can be improved if we consider sum or difference of two correlation functions:

$\mathrm{R}(\mathbf{q})$ - correlation function of interest

$\tilde{\mathrm{R}}(\mathbf{q})$ - appropriately chosen correlation function - "*regulator*" which cancels out ultraviolet divergence

We should choose the "*regulator*" function in such a way that the difference or the sum of two correlations functions tends to zero faster than q^{-3} as q grows:

$$S(q_{max}) \equiv 4\pi \int_0^{q_{max}} dq\, q^2 (R(q) - 1)$$

If we choose as a "*regulator*" the correlation function for pair of nonidentical particles which do not form any bound states, that is:

$$\tilde{\mathrm{R}}(\mathbf{q}): \quad \int \frac{d^3 q}{(2\pi)^3} \tilde{\phi}_{\mathbf{q}}(\mathbf{r}) \tilde{\phi}_{\mathbf{q}}^*(\mathbf{r}') = \delta^{(3)}(\mathbf{r} - \mathbf{r}') \quad \Rightarrow \quad \int \frac{d^3 q}{(2\pi)^3} \left(|\tilde{\phi}_{\mathbf{q}}(\mathbf{r})|^2 - 1 \right) = 0$$

- the "regulator" does not change any information carried by the sum rule.

Improved sum rule

‘Regulator’ ensures existence of the integral – it allows to change the order of integration

Difference:

$$\int \frac{d^3q}{(2\pi)^3}\left[(R(\mathbf{q})-1)-(\widetilde{R}(\mathbf{q})-1)\right]=\int d^3r\left[D_r(\mathbf{r})\int \frac{d^3q}{(2\pi)^3}\left(|\phi_{\mathbf{q}}(\mathbf{r})|^2-1\right)-\widetilde{D}_r(\mathbf{r})\int \frac{d^3q}{(2\pi)^3}\left(|\widetilde{\phi}_{\mathbf{q}}(\mathbf{r})|^2-1\right)\right]$$

Sum:

$$\int \frac{d^3q}{(2\pi)^3}\left[(R(\mathbf{q})-1)+(\widetilde{R}(\mathbf{q})-1)\right]=\int d^3r\left[D_r(\mathbf{r})\int \frac{d^3q}{(2\pi)^3}\left(|\phi_{\mathbf{q}}(\mathbf{r})|^2-1\right)+\widetilde{D}_r(\mathbf{r})\int \frac{d^3q}{(2\pi)^3}\left(|\widetilde{\phi}_{\mathbf{q}}(\mathbf{r})|^2-1\right)\right]$$

Assumption: $D_r(\mathbf{r})=\widetilde{D}_r(\mathbf{r})$

Difference:

$$\int d^3q\left(R(\mathbf{q})-\widetilde{R}(\mathbf{q})\right)=\pm\pi^3 D_r(0)-\sum_\alpha A_\alpha$$

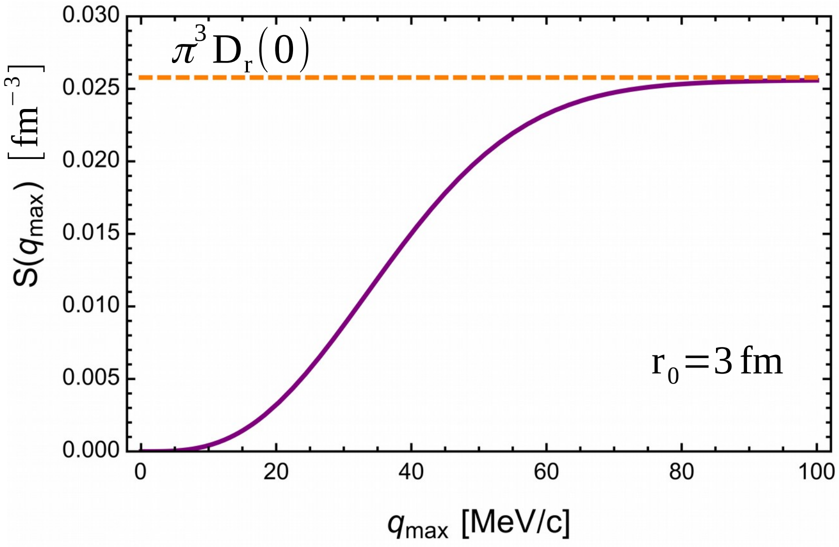
Sum:

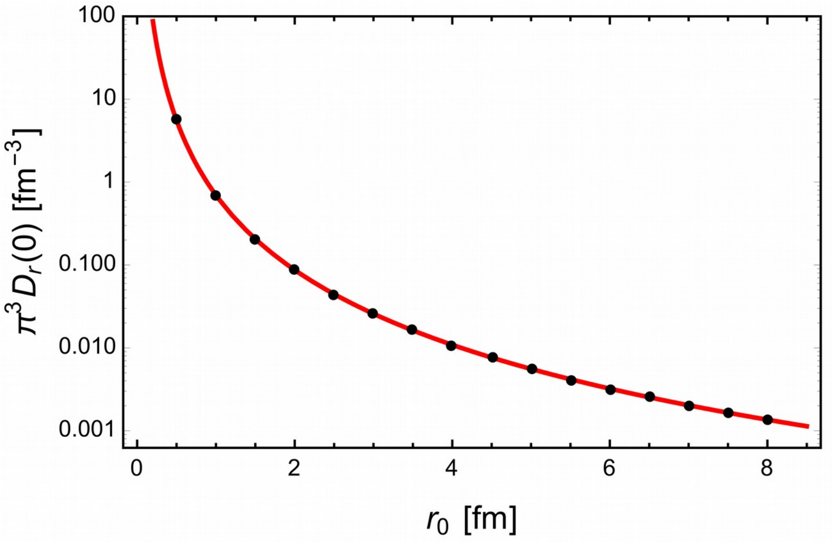
$$\int d^3q\left(R(\mathbf{q})+\widetilde{R}(\mathbf{q})-2\right)=\pm\pi^3 D_r(0)-\sum_\alpha A_\alpha$$

Test of improved sum rule

Identical pions $\pi^+\pi^+,\ \pi^-\pi^-$ Expected result: $\int d^3q\left(R(\mathbf{q})-\widetilde{R}(\mathbf{q})\right)=\pi^3 D_r(0)$

$\widetilde{R}(\mathbf{q})$ - is the correlation function of the pions treated as nonidentical particles





New sum rule works perfectly well !

Test of improved sum rule

Pions $\pi^+ \pi^-$

Point-like source $\boxed{D_r(\mathbf{r}) = \delta^{(3)}(\mathbf{r})}$

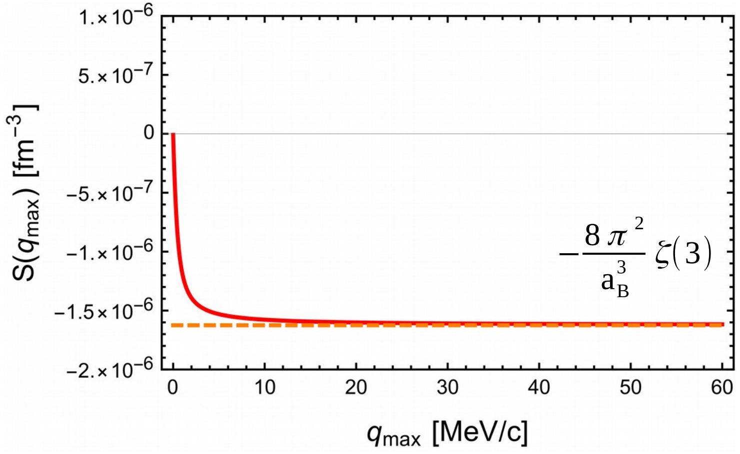
Expected result:

$$\int d^3q \left(G^{+-}(\mathbf{q}) + G^{++}_{--}(\mathbf{q}) - 2 \right) = -\frac{8\pi^2}{a_B^3}\zeta(3)$$

$\widetilde{R}(\mathbf{q}) = G^{++}_{--}(\mathbf{q})$ - the correlation function for point-like source and pions treated as nonidentical particles

Problem: integrand expanded in powers of $\left[\frac{2\pi}{(a_B q)}\right]$ reveals divergent term.
It has to be removed from integrand.

$$\int d^3q \left(G^{+-}(\mathbf{q}) + G^{++}_{--}(\mathbf{q}) - 2 - \frac{2\pi^2}{3 a_B^2 q^2} \right) = -\frac{8\pi^2}{a_B^3}\zeta(3)$$



New sum rule works perfectly well !

Improved sum rule as a tool to test models

Neutron – proton (singlet and triplet spin states) $\int d^3q\left(R^t(\mathbf{q})-R^s(\mathbf{q})\right)=-A_D$

Triplet state – deuteron formation

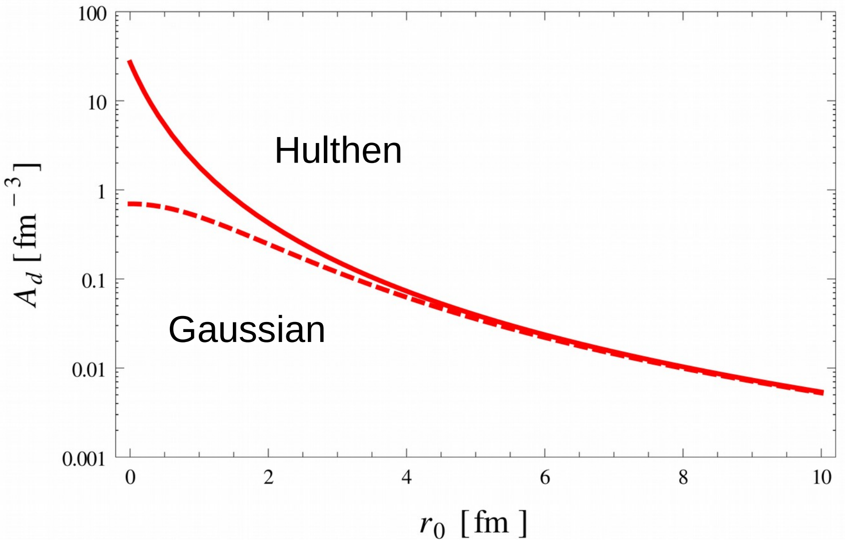
Gaussian wave function of deuteron

$$|\phi_D(\mathbf{r})|^2=\frac{1}{(4\pi r_D^2)^{3/2}}\exp\left(-\frac{r^2}{4r_D^2}\right) \longrightarrow A_D=\frac{\pi^{3/2}}{(r_0^2+r_D^2)^{3/2}}, \qquad r_D=2\,\mathrm{fm}$$

Hulthen wave function

$$\phi_D(\mathbf{r})=\left(\frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2}\right)^{1/2}\frac{e^{-\alpha r}-e^{-\beta r}}{2}$$

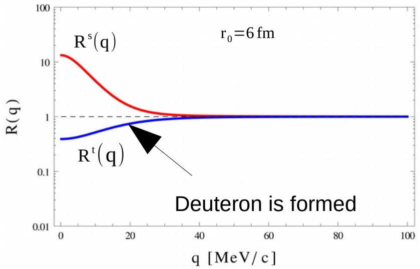
$$\alpha=0.23\,\mathrm{fm}^{-1},\quad \beta=1.61\,\mathrm{fm}^{-1}$$



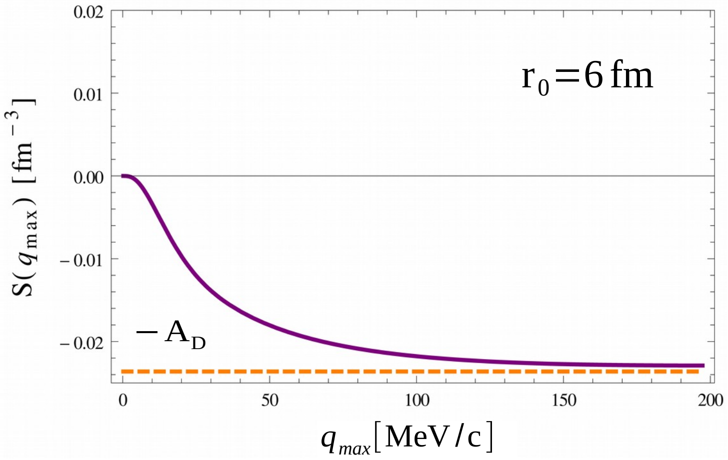
Improved sum rule as a tool to test models

Neutron – proton

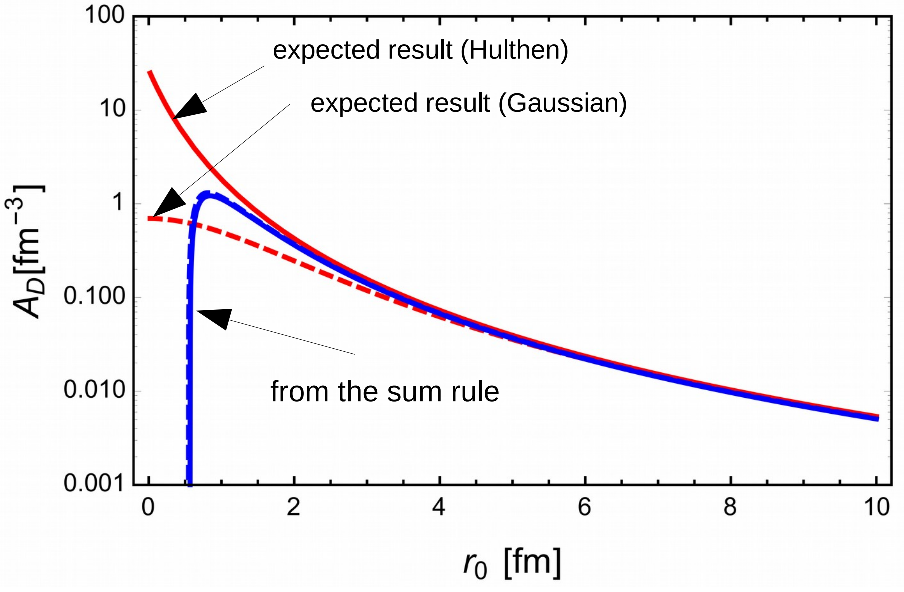
Expected result: $$\int d^3q \left(R^t(q) - R^s(q)\right) = -A_D$$



$$\tilde{R}(\mathbf{q}) = R^s(\mathbf{q})$$



The improved sum rule allows us to check the various models



Conclusions

The original sum rule works only for non-interacting identical particles.

The improved sum rule is shown to work for interacting particles as well.

The sum rule can be used to test an accuracy and range of applicability of the approximate models.