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Equilibration in the Osaka / Stavanger approach & the lattice $Q\bar{Q}$ potential

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References:
T. Miura, Y. Akamatsu, M. Asakawa, A.R. arXiv:1908.06293
P. Petreczky, A.R., J. Weber, NPA982 (2019) 735
D. Lafferty, A.R., arXiv:1906.00035

Motivation: Osaka/Stavanger approach

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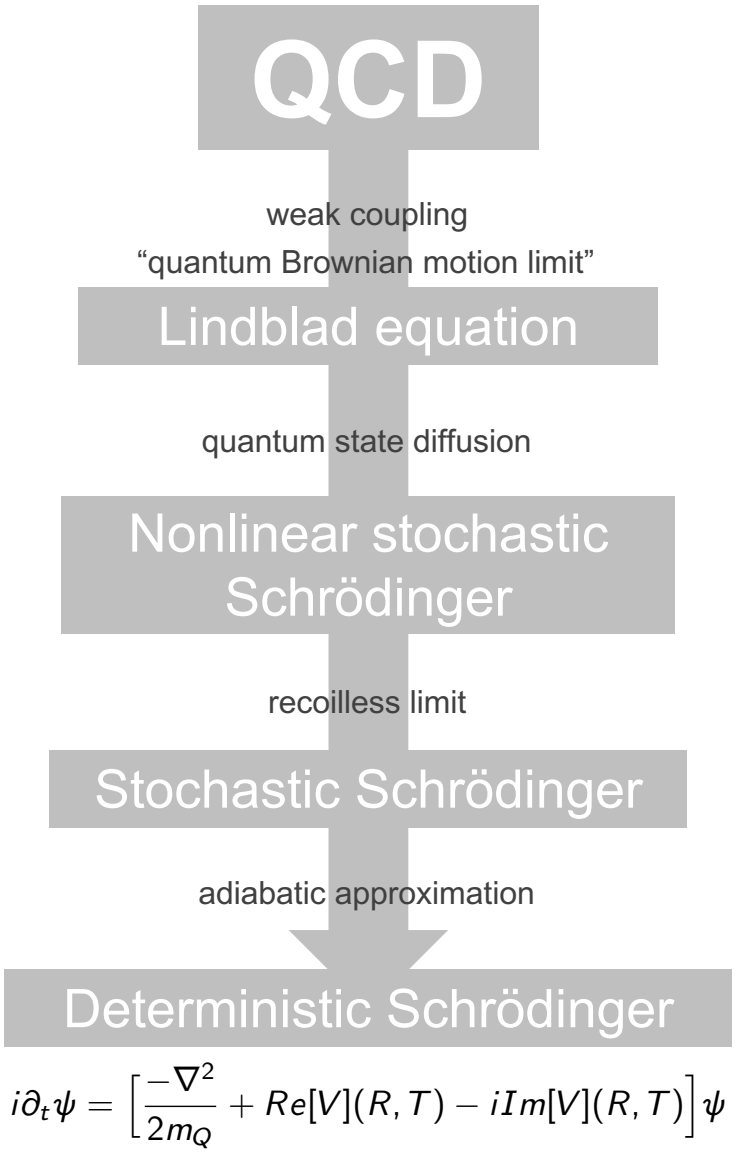
- Main goal: derive a **potential based** e.o.m. for quarkonium wavefunctions from QCD

$$i\partial_t\psi(t) = F[\psi, Re[V], Im[V], \ldots]$$

  - **conceptual**: connect imaginary part of the EFT potential to microscopic quarkonium dynamics
  - **technical**: range of validity of phenomenological models
  - **practical**: provide path for incremental improvement of current approaches

- Based on **open-quantum-systems** picture (no semiclassical approximation)



Reaching equilibrium (proof-of-principle)

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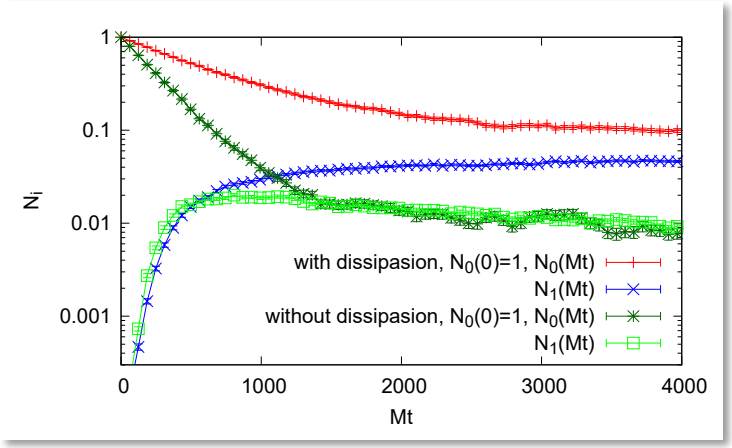
- Dissipation is essential for thermalization: linear vs. nonlinear stoch. Schroedinger



$$H = \frac{p^2}{m_Q} - \frac{\alpha_S}{\sqrt{x^2 + x_c^2}} e^{-m_D|x|}$$

$$D(x) = \gamma \exp\left[-x^2/\ell_{\text{corr}}^2\right]$$

$$\alpha = \frac{g^2}{4\pi}, \quad m_D = gT\sqrt{\frac{N_c}{3} + \frac{N_f}{6}} \simeq \frac{2}{\ell_{\text{corr}}}$$

$$\gamma = D(0) = \frac{g^2 T}{4\pi}$$

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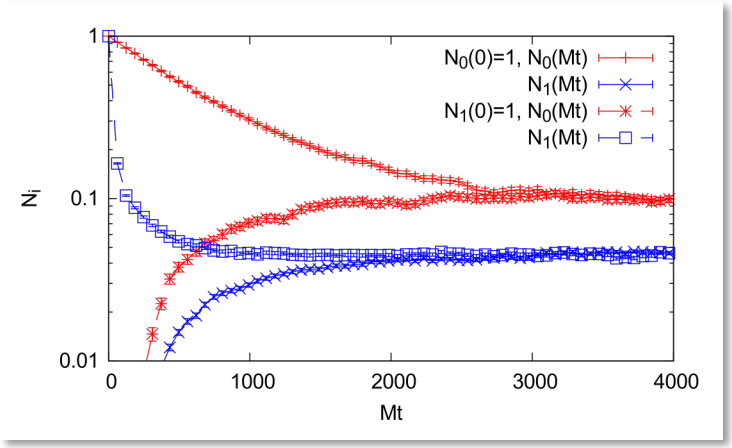

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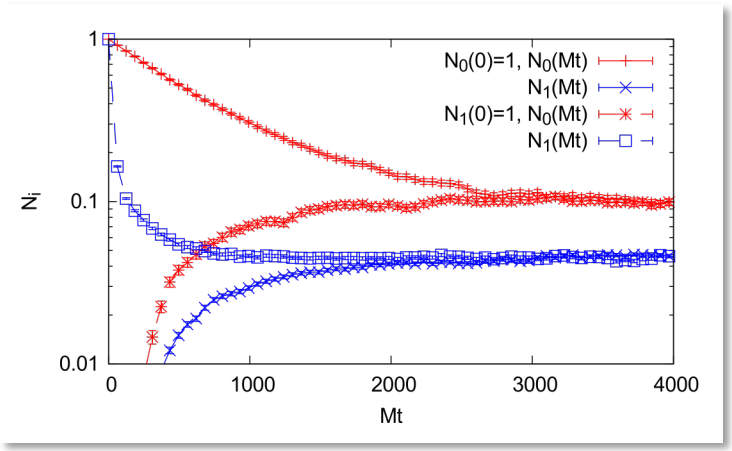

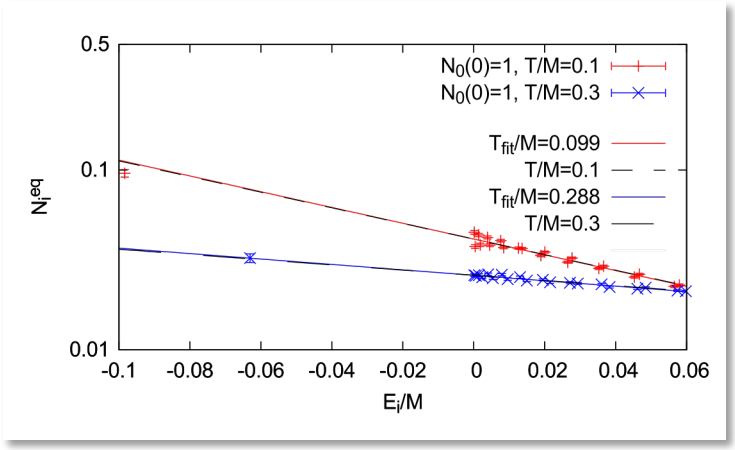

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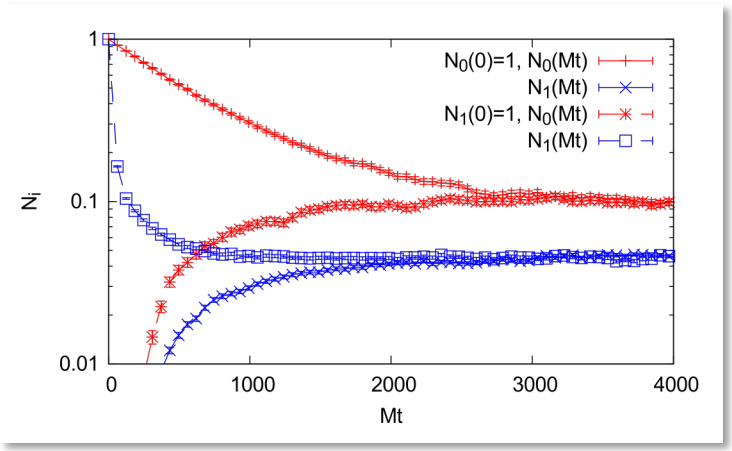

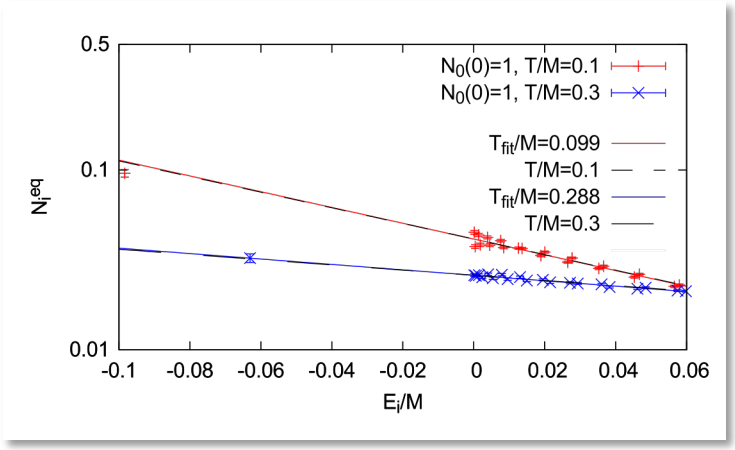

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Position basis master equation

$$\begin{aligned}\partial_t \rho(R,r) = &\left[\frac{2i}{M}\vec{\nabla}_R \cdot \vec{\nabla}_r - i\left\{V\left(R+\frac{r}{2}\right) - V\left(R-\frac{r}{2}\right)\right\}\right]\rho(R,r) \\ &+ \left[2F_1\left(\frac{r}{2}\right) - 2F_1(0) + F_1\left(R+\frac{r}{2}\right) + F_1\left(R-\frac{r}{2}\right) - 2F_1(R)\right]\rho(R,r) \\ &+ \left[\begin{array}{l}\left\{4\vec{F}_2\left(\frac{r}{2}\right) + 2\vec{F}_2\left(R+\frac{r}{2}\right) - 2\vec{F}_2\left(R-\frac{r}{2}\right)\right\}\cdot\vec{\nabla}_r \\ + \left\{\vec{F}_2\left(R+\frac{r}{2}\right) + \vec{F}_2\left(R-\frac{r}{2}\right) - 2\vec{F}_2(R)\right\}\cdot\vec{\nabla}_R\end{array}\right]\rho(R,r).\end{aligned}$$

$$F_1(x) \equiv D(x) + \frac{\vec{\nabla}^2 D(x)}{4MT} \simeq D(x), \quad \vec{F}_2(x) \equiv \frac{\vec{\nabla} D(x)}{4MT}$$

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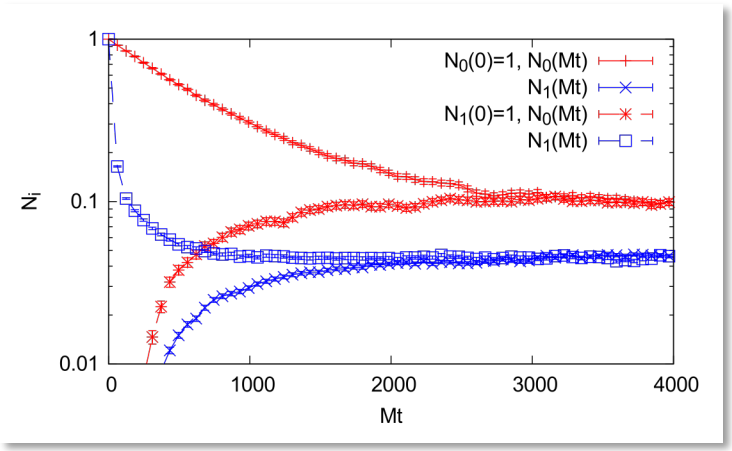

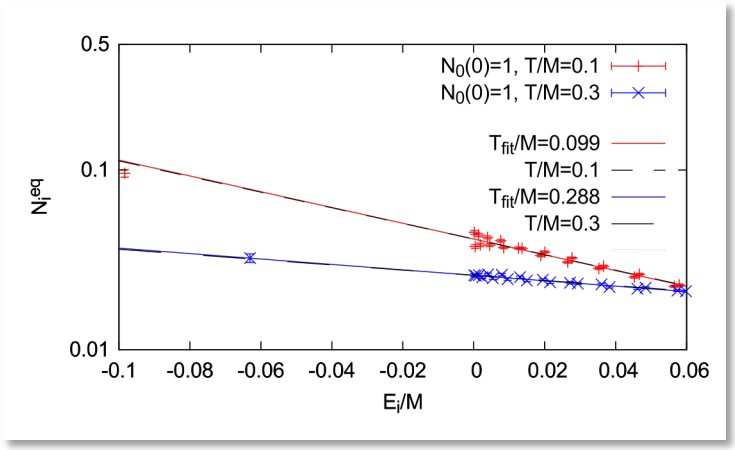

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Wigner transform

$$f_p(R) = \int d^3r \exp\left[-i\frac{\vec{p}\cdot\vec{r}}{\hbar}\right]\rho(R,r)$$

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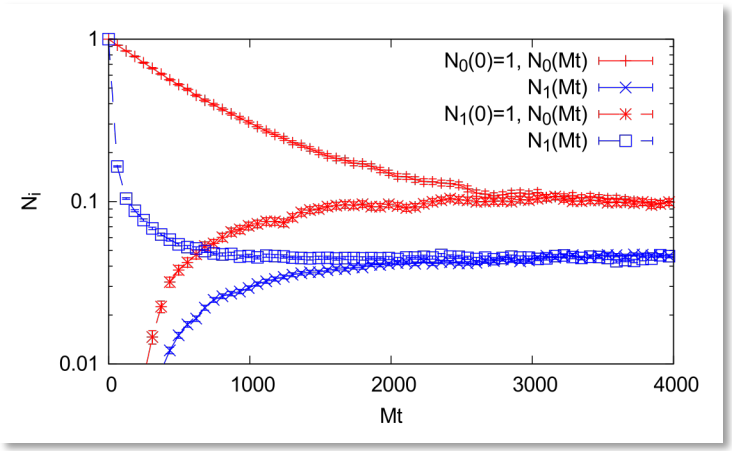

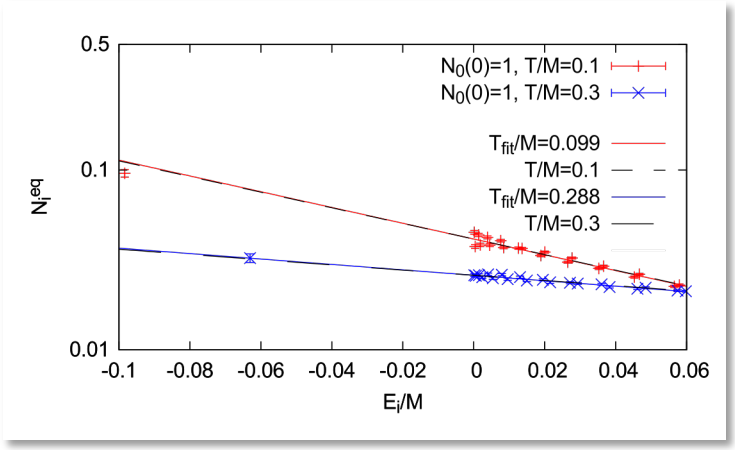

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Classical limit h->0

$$\nabla_R\nabla_r\hbar \quad V/\hbar \quad F^1/\hbar^2 \quad F^2/\hbar^0$$

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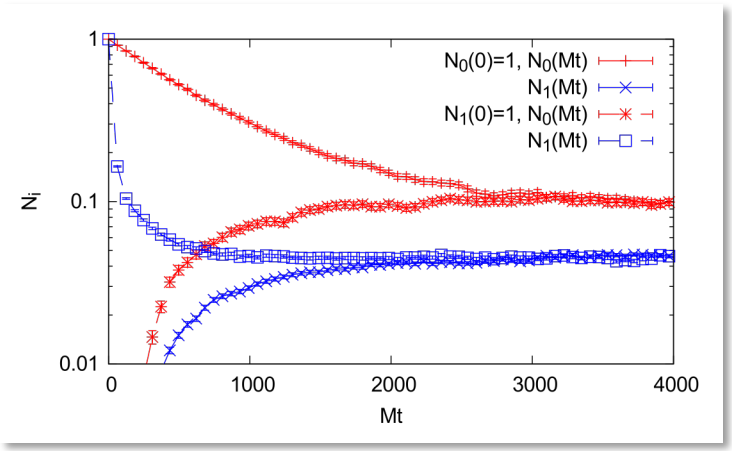

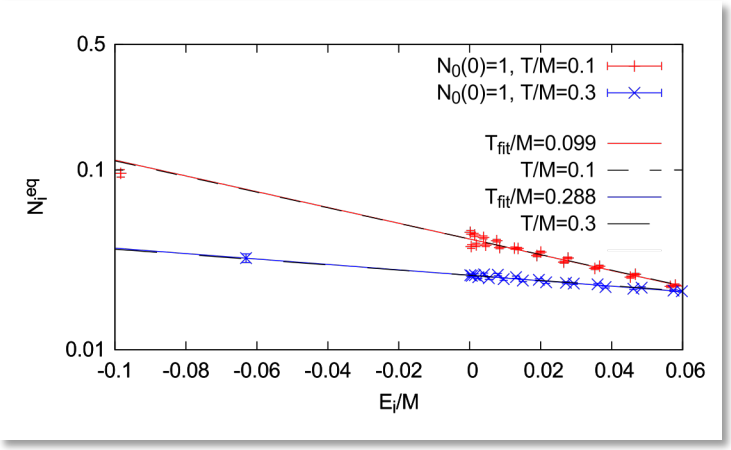

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$$f_p^{\rm eq}(R) \propto \exp\left[-\frac{1}{T}\left\{\frac{p^2}{M} + V(R)\right\}\right]$$

Wigner transform

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$\nabla_R \nabla_r \hbar$ $V/\hbar$ $F^1/\hbar^2$ $F^2/\hbar^0$

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We recover Boltzmann in the classical limit

The QCD real-time interquark potential

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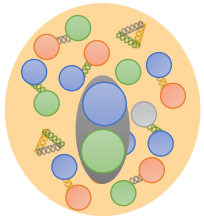
- Exploit $\frac{T}{m_Q} \ll 1, \ \frac{\Lambda_{\rm QCD}}{m_Q} \ll 1$ to treat heavy quarks non-relativistically

Relativistic T>0 field theory



Brambilla et.al. Rev.Mod.Phys. 77 (2005) 1423

QCD	NRQCD
Dirac fields	**Pauli fields**
$Q(x), \bar{Q}(x)$	$\psi, \psi^\dagger, \chi, \chi^\dagger$

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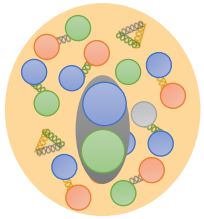

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$$L_{\text{pNRQCD}} = \int d^3\mathbf{r}\,\text{Tr}\Big[\psi_S^\dagger\big[i\partial_0 - \big(\frac{\mathbf{P}^2}{2M} + V_S^{(0)} + \mathcal{O}(\frac{1}{m_Q})\big)\big]\psi_S + \psi_O^\dagger\big[iD_0 - \big(\frac{\mathbf{P}^2}{2M} + V_O^{(0)} + \mathcal{O}(\frac{1}{m_Q})\big)\big]\psi_O\Big]$$
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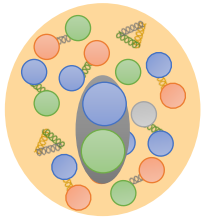

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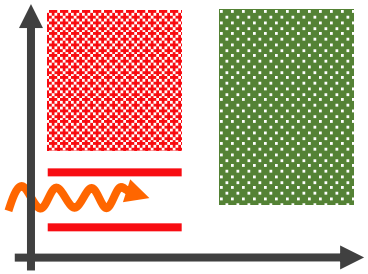

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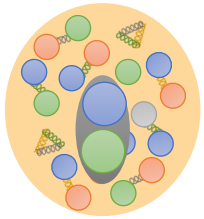

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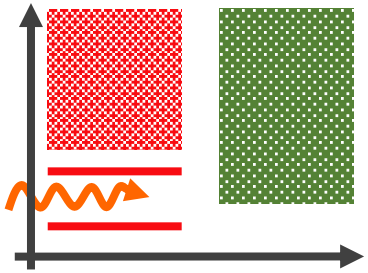

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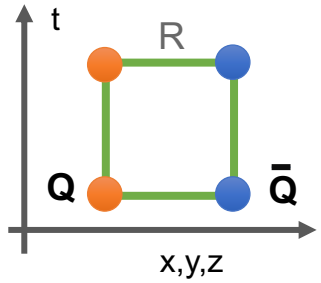
- Matching $V^{(0)}_S$ to underlying QCD in the infinite mass limit: Wilson loop

$$\langle \psi_S(R,t)\psi_S^*(R,0)\rangle_{\text{pNRQCD}} \equiv W_\square(R,t) = \left\langle \text{Tr}\left[\exp\left(-ig\int_\square dx^\mu A_\mu(x)\right)\right]\right\rangle_{\text{QCD}}$$



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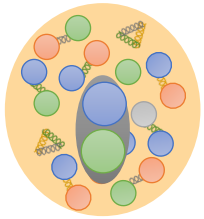

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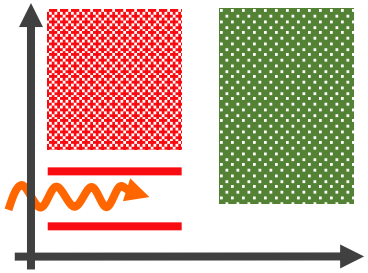

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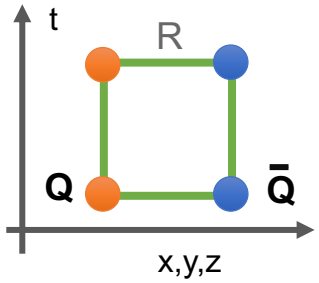


- Wilson loop: potential emerges at late times - multi gluon interaction

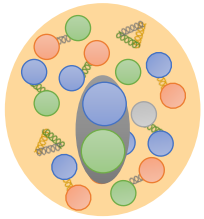
$$i\partial_t\langle\psi_S(t)\psi_S(0)\rangle = \big[V^{(0)}(R) + \Theta(R,t)\big]\langle\psi_S(t)\psi_S(0)\rangle$$

The QCD real-time interquark potential

University of Stavanger

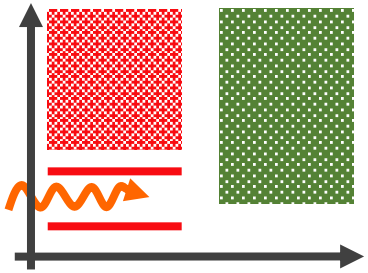
- Exploit $\frac{T}{m_Q} \ll 1, \ \frac{\Lambda_{QCD}}{m_Q} \ll 1$ to treat heavy quarks non-relativistically

Relativistic T>0 field theory



Brambilla et.al. Rev.Mod.Phys. 77 (2005) 1423

QCD	NRQCD	pNRQCD
Dirac fields	Pauli fields	Singlet/Octet
$Q(x), \bar{Q}(x)$	$\psi, \psi^\dagger, \chi, \chi^\dagger$	$\psi_S(R,t), \psi_O(R,t)$

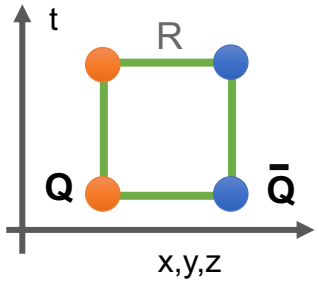


$$L_{\text{pNRQCD}} = \int d^3\mathbf{r}\,\text{Tr}\Big[\psi_S^\dagger\big[i\partial_0 - \big(\frac{\mathbf{P}^2}{2M} + V_S^{(0)} + \mathcal{O}(\frac{1}{m_Q})\big)\big]\psi_S + \psi_O^\dagger\big[iD_0 - \big(\frac{\mathbf{P}^2}{2M} + V_O^{(0)} + \mathcal{O}(\frac{1}{m_Q})\big)\big]\psi_O\Big]$$

$$+V_A(r)\,\text{Tr}\Big[\psi_O^\dagger \mathbf{r}g\mathbf{E}\psi_S + \psi_S^\dagger \mathbf{r}g\mathbf{E}\psi_O\Big] + \mathcal{O}(r^2, \frac{1}{m_Q})$$

- Matching $V^{(0)}_S$ to underlying QCD in the infinite mass limit: Wilson loop

$$\langle\psi_S(R,t)\psi_S^*(R,0)\rangle_{\text{pNRQCD}} \equiv W_\square(R,t) = \left\langle \text{Tr}\left[\exp\left(-ig\int_\square dx^\mu A_\mu(x)\right)\right]\right\rangle_{\text{QCD}}$$



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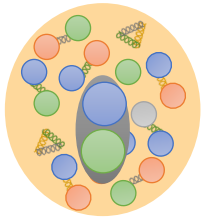

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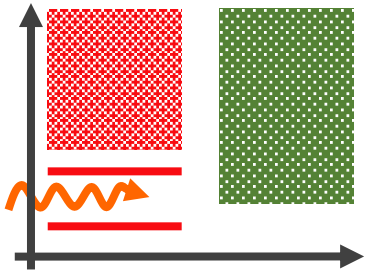

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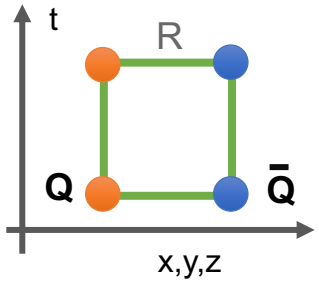

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- To access this real-time quantity from the lattice: spectral function reconstructions

Non-perturbative evaluation of V(R)

University of Stavanger

- How to connect to the Euclidean domain: **spectral functions**

A.R., T.Hatsuda & S.Sasaki
PRL 108 (2012) 162001

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$$V^{QCD}(R) = \lim_{t\to\infty} \frac{\int_{-\infty}^{\infty} d\omega\, \omega\, e^{-i\omega t}\, \rho_{\square}(R,\omega)}{\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \rho_{\square}(R,\omega)}$$

A.R., T.Hatsuda & S.Sasaki PoS LAT2009 (2009) 162

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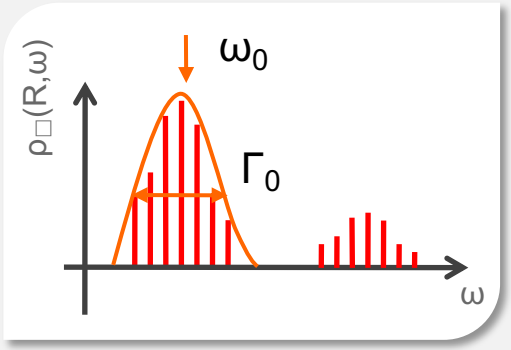

well defined V(R) if low lying Breit-Wigner present in Wilson loop spectral function

For technical details see
Y.B., A.R. PRD86 (2012) 051503

$$V(R) = \omega_0(R) - i\Gamma_0(R)$$

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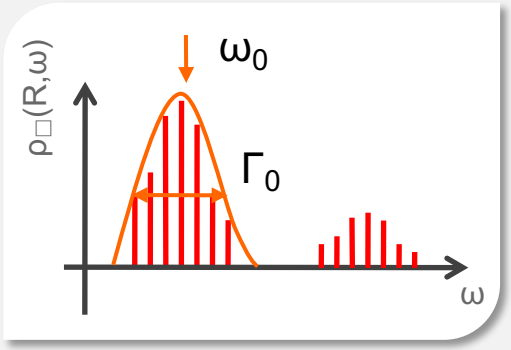

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Spectral Reconstruction

- In case of usual $\Delta W/W=10^{-2}$ statistical un-certatinty in $W_{\square}$: **Bayesian inference**

incorporate prior information to regularize the inversion task (BR method)

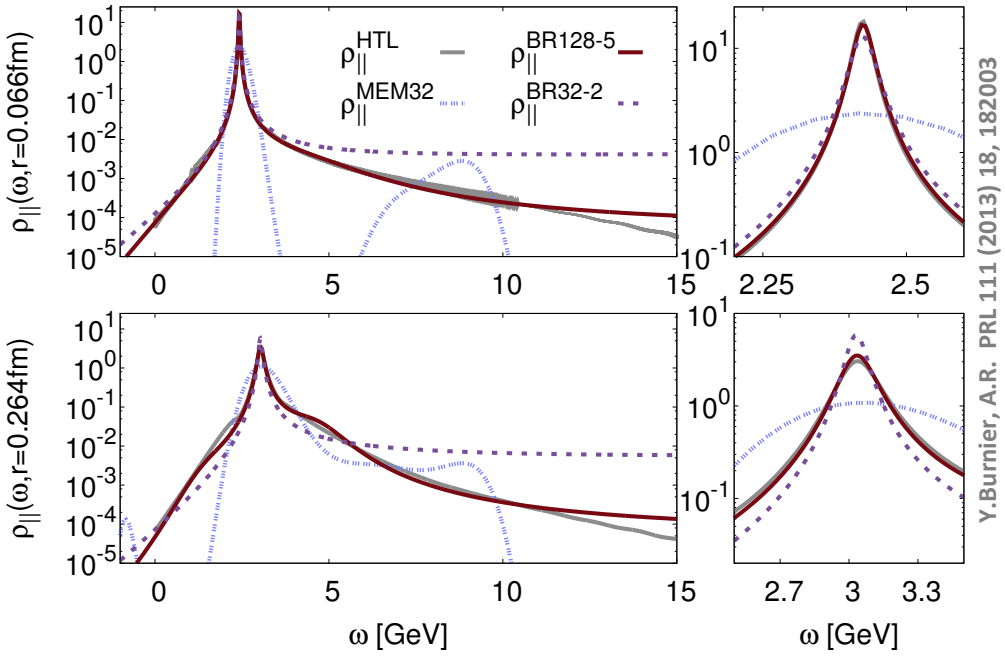
- In case of **small $\Delta W/W<10^{-3}$** statistical un-certatinty in $W_{\square}$ also **Pade approximation**

exploit the analyticity of the Wilson correlator to extract spectra

Bayesian spectral reconstruction

University of Stavanger

- HTL predicts a well defined Breit-Wigner peak in the Wilson correlator spectrum
- Found that MEM regulator produces Gaussian like peaks even for underlying BW
- Developed a novel regulator that actually reproduces BW peaks



MEM

Regulator from axioms justified in 2d image recognition lead to entropy

$$S_{SJ} = \int d\omega \left(\rho - m - \rho\frac{\rho}{m}\right)$$

M. Jarell, J.E. Gubernatis, Phys.Rept. 269 (1996) 133

additional ad-hoc smoothing via SVD

R.K. Bryan E. Biophysics J. 1990, see A.R. J.Comp.Phys. 238 (2013)

BR method

Regulator guarantees positivity, smoothness, independence of units

$$S_{BR} = \int d\omega \left[1 - \frac{\rho}{m} + \log\left(\frac{\rho}{m}\right)\right]$$

Y.Burnier, A.R. PRL 111 (2013) 18, 182003

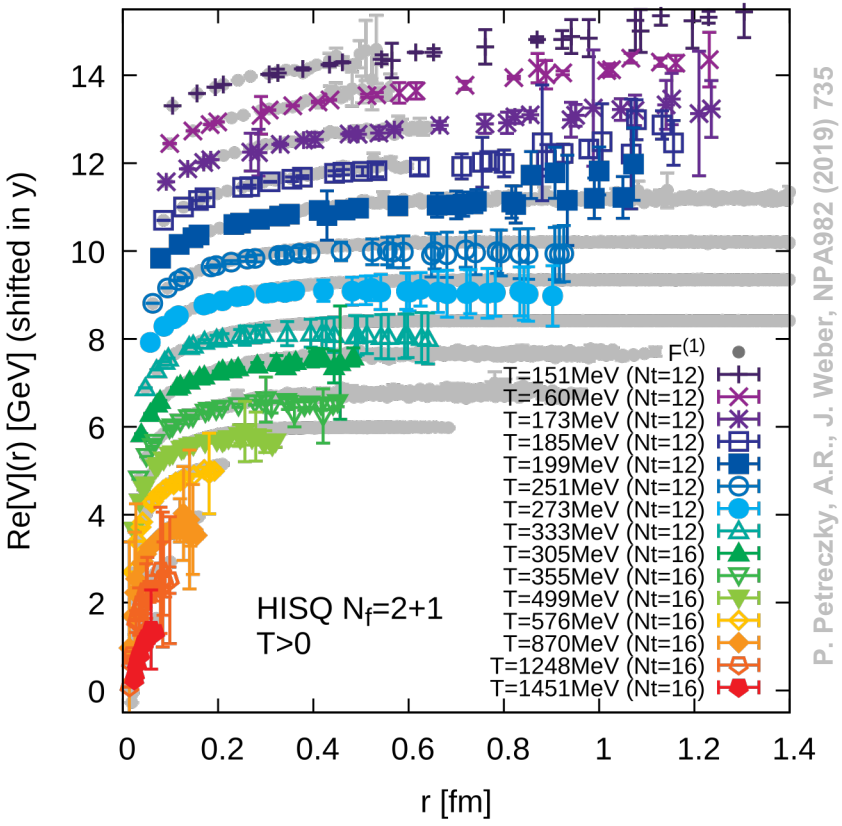
Latest results on the lattice potential

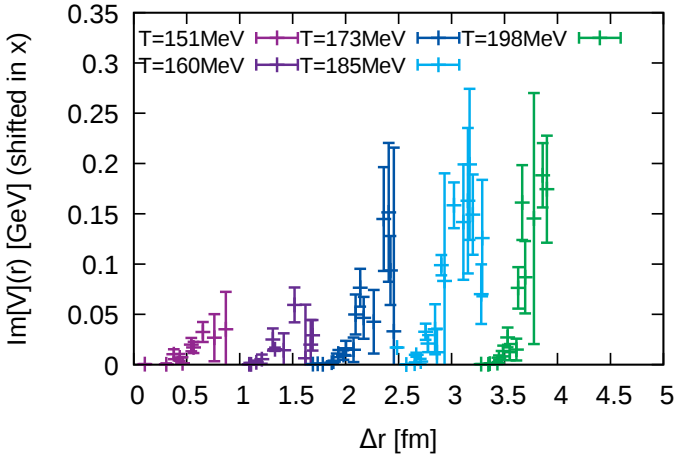
University of Stavanger

- Lattices with dynamical u,d,s quarks (HISQ action, HotQCD & TUMQCD)

  A. Bazavov et.al. PRD97 (2018) 014510, HotQCD PRD90 (2014) 094503

  - realistic m_{π}~161MeV (T=151-1451MeV)
  - fixed box (N_s=48 - **N_τ=12, N_τ=16**) & very **high statistics** 4000-9000 realizations





- Smooth transition from Cornell @ T=0 to Debye screened @ $T>T_C$
- Finite Im[V] above T_c present

A word on potential models

University of Stavanger

- In the age of a 1st principles lattice V(R): analytic parametrization & interpretation

see e.g. D.Lafferty and A.R. arXiv:1906.00035

T=0 Cornell + generalized Gauss law

$$V_{Q\bar{Q}}^{T=0}(R) = V_C(R) + V_S(R) = -\frac{\alpha_S}{r} + \sigma r + c$$

$$\mathcal{G}_a[V(R)] = \vec{\nabla}\left(\frac{\vec{\nabla}V(R)}{R^{a+1}}\right) = -4\pi q \delta^{(3)}(\vec{R})$$

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$$V^{med}(\mathbf{p}) = V^{vac}(\mathbf{p})/\epsilon(\mathbf{p})$$

$$\epsilon^{-1}(\vec{p}, m_D) = \frac{p^2}{p^2 + m_D^2} - i\pi T \frac{p m_D^2}{(p^2 + m_D^2)^2}$$

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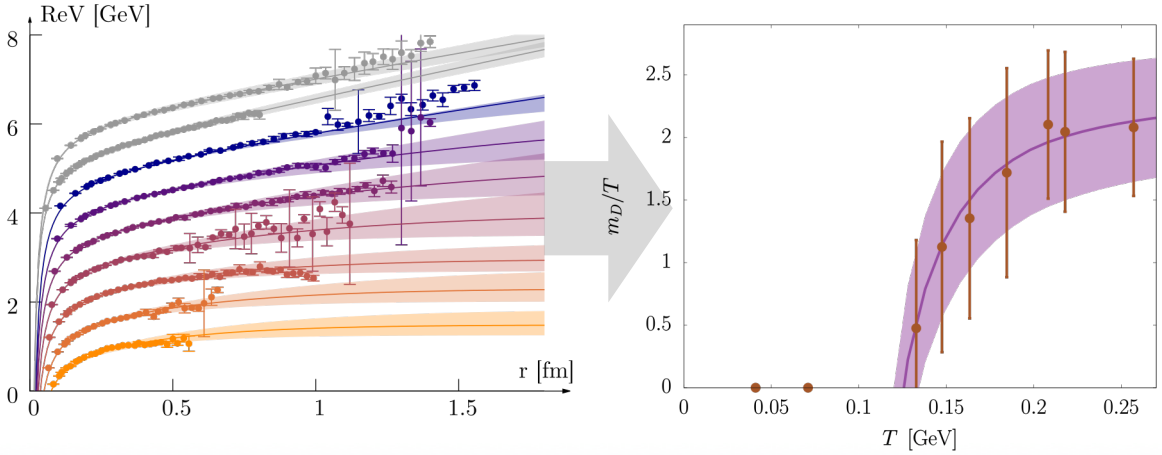

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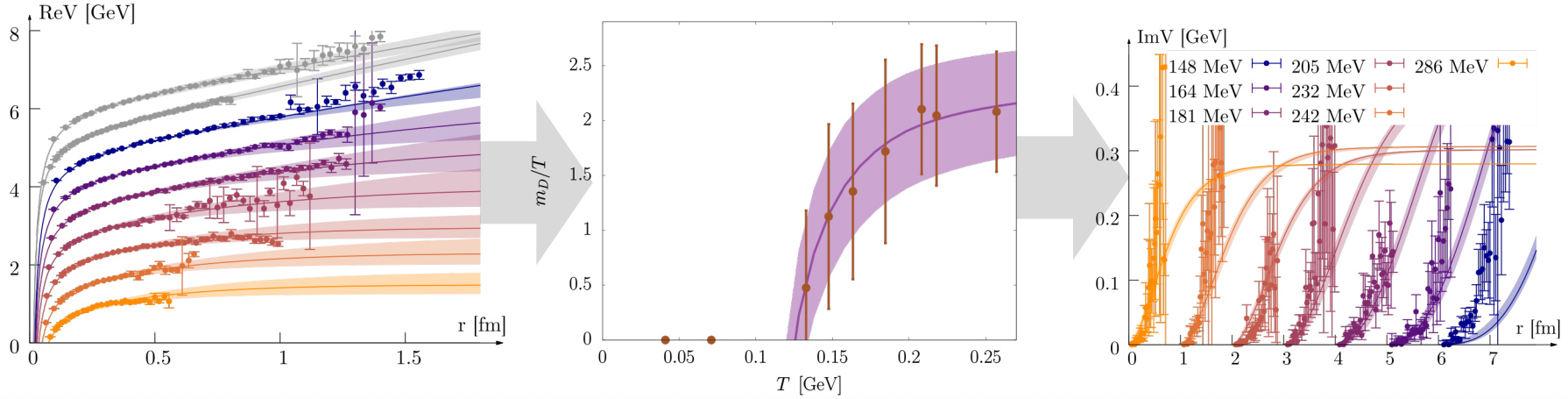

Continuum corrected spectral functions

University of Stavanger

- Need to manually correct for lattice artefacts: T_C and string tension not yet physical

Y.Burnier, O.Kaczmarek, A.R. JHEP 1512 (2015) 101

Pheno fit to T=0 quarkonium

$$V_{Q\bar{Q}}^{T=0}(R) = -\frac{\alpha_S}{r} + \sigma r + c$$

Continuum corrected Debye mass

$$m_D^{\text{phys}}(t = T/T_c^{\text{lat}}) = \frac{m_D^{\text{lat}}(t)}{\sqrt{\sigma^{\text{lat}}(t)}}\sqrt{\sigma^{\text{cont}}}$$

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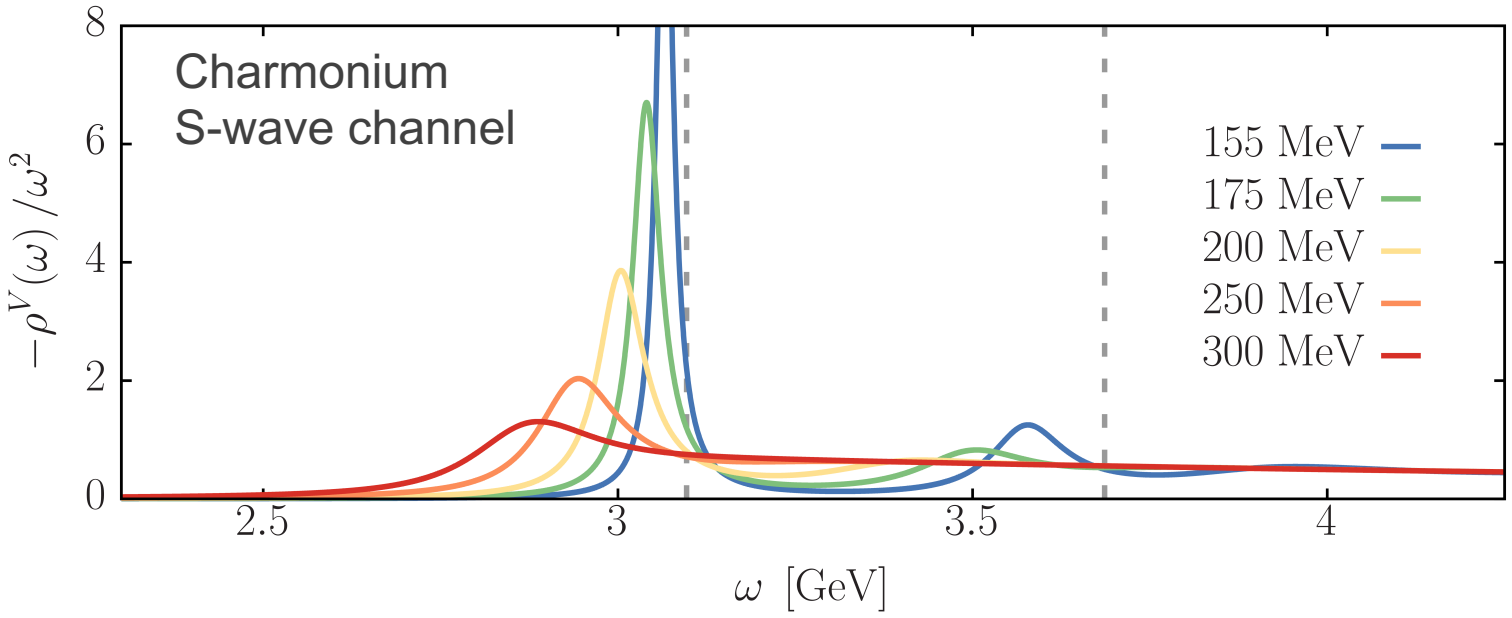

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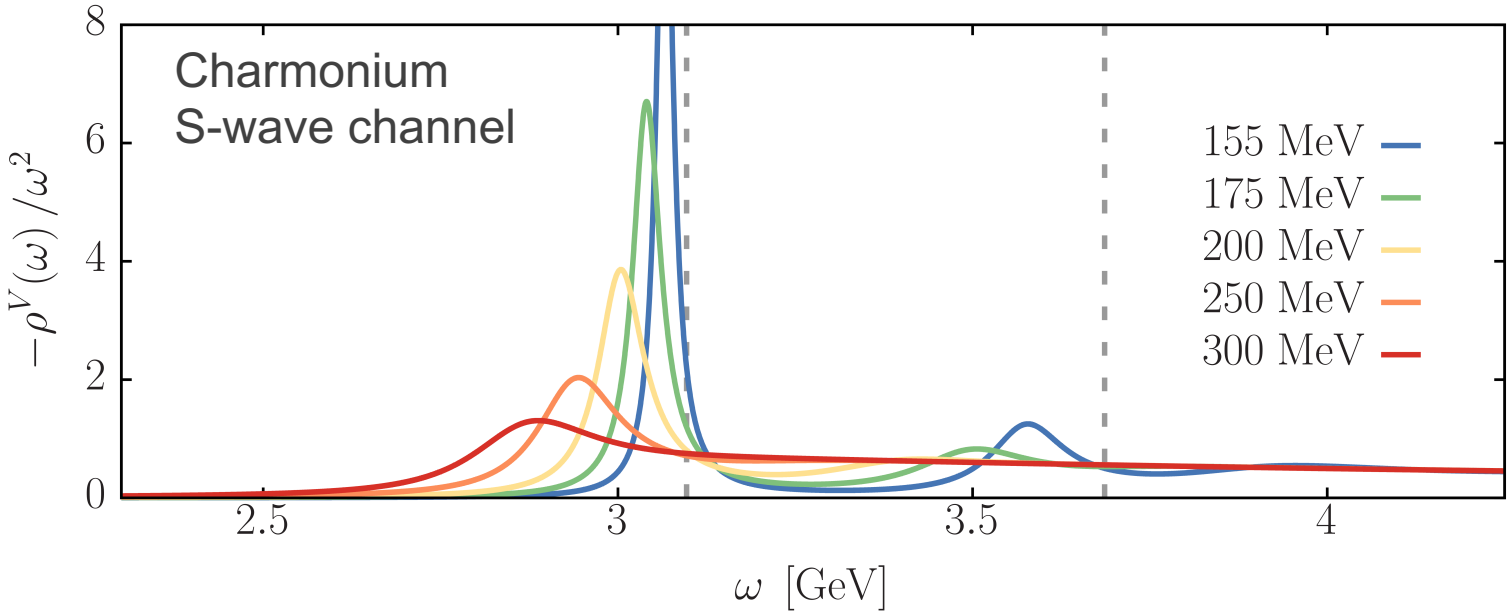
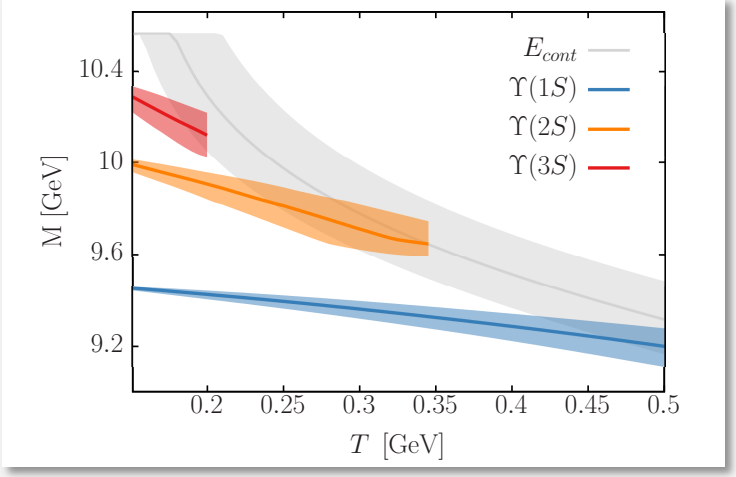


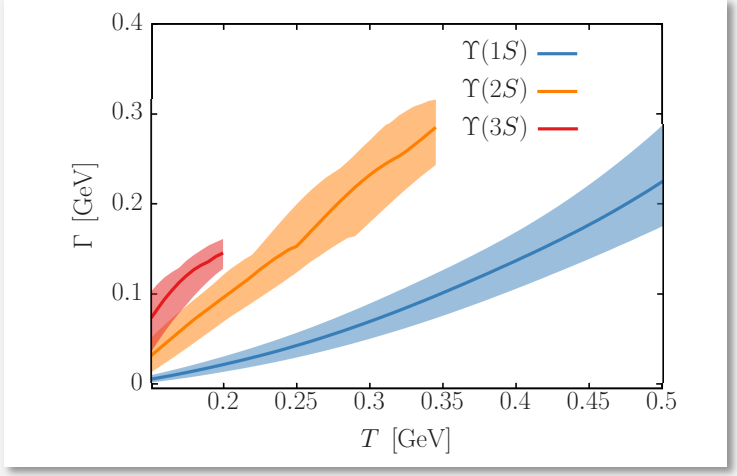
- Shifting to lower mass and broadening hierarchichally ordered with vacuum E_{bind}

Thermal masses and widths

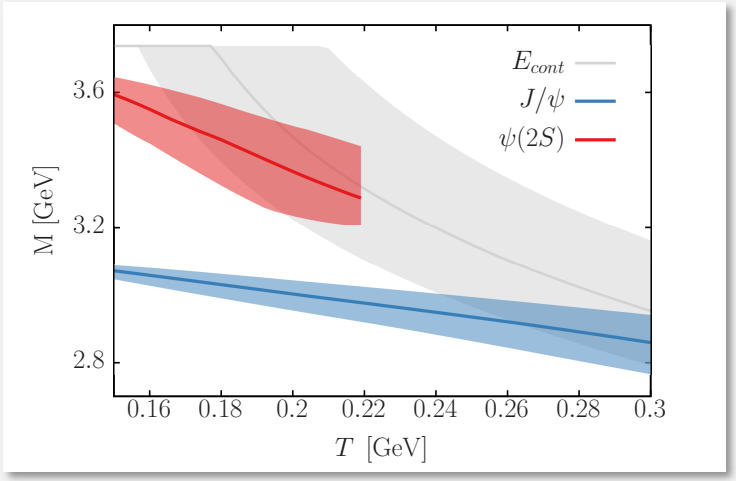
University of Stavanger

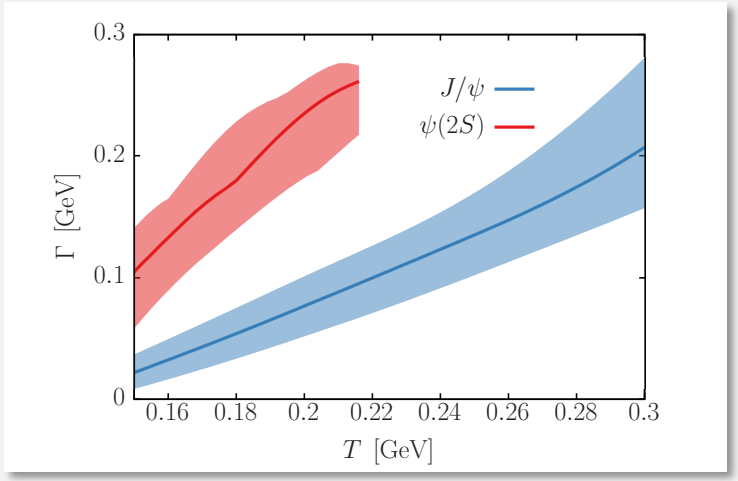
Bottomonium





Charmonium





Conclusion

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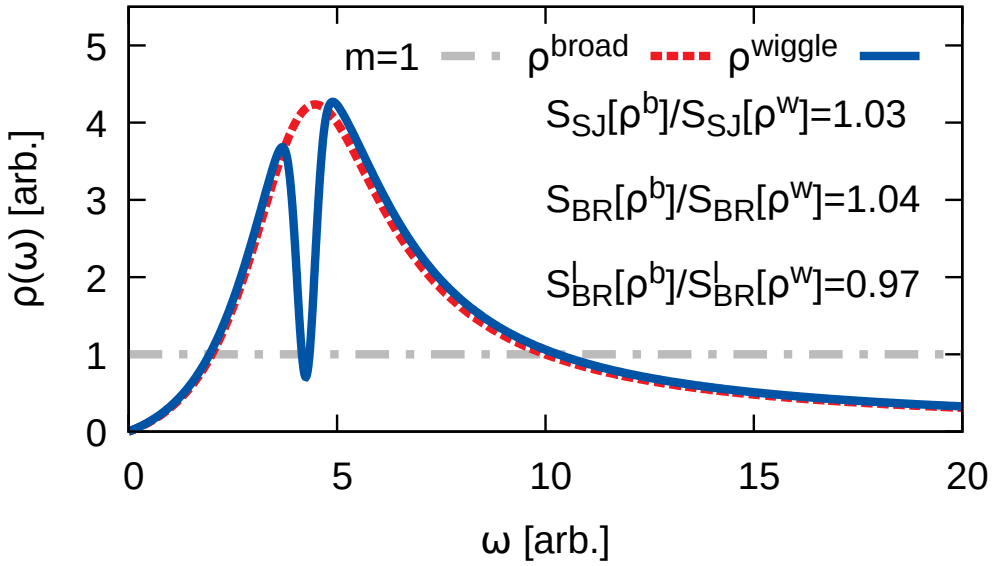
- **Dissipative** Lindblad dynamics recover **universal thermal fixed point**
- Static **inter-quark potential** defined from the **real-time Wilson loop**
- Novel Bayesian **BR method** designed to **accurately extract BW peaks**
- Current **state-of-the-art extraction**: Debye screened Re[V] & Im[V] > 0
- Potential model as **efficient parametrization** of Re[V] and Im[V]
- **Precision spectral functions** from the continuum corrected potential

Local regulators do not avoid ringing

University of Stavanger



- Development of regulators based on different prior information: MEM & BR

Y.Burnier, A.R.
PRL 111 (2013) 18, 182003

$$S_{BR} = \alpha \int d\omega \left(1 - \frac{\rho}{m} + log\left[\frac{\rho}{m}\right]\right)$$

BR prior enforces:
ρ positive definite,
smoothness of ρ,
result independent of units

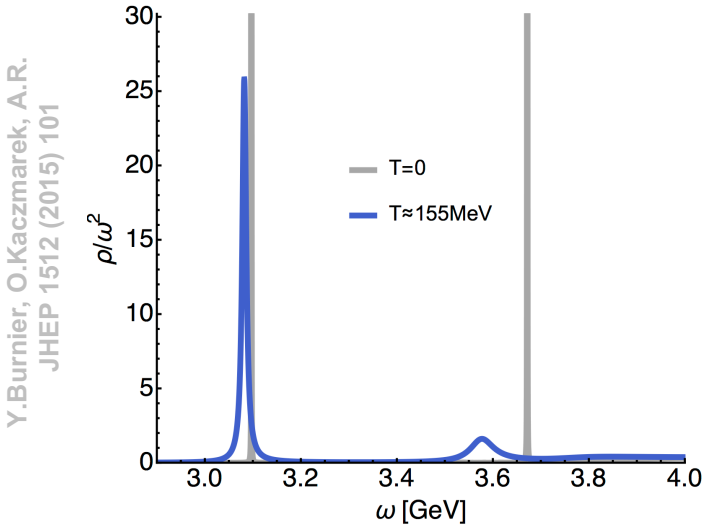
- BR prior: better accuracy in sharp peak structures than MEM but prone to ringing
- MEM avoids ringing through additional ad-hoc smoothing via SVD

C.Fischer, J. Pawlowski,
A.R., C. Welzbacher
PRD98 (2018) 014009

$$S_{BR}^{smooth} = \alpha \int d\omega \left(\kappa \left(\frac{\partial \rho}{\partial \omega}\right)^2 + 1 - \frac{\rho}{m} + log\left[\frac{\rho}{m}\right]\right)$$

Towards the ψ' to J/ψ ratio

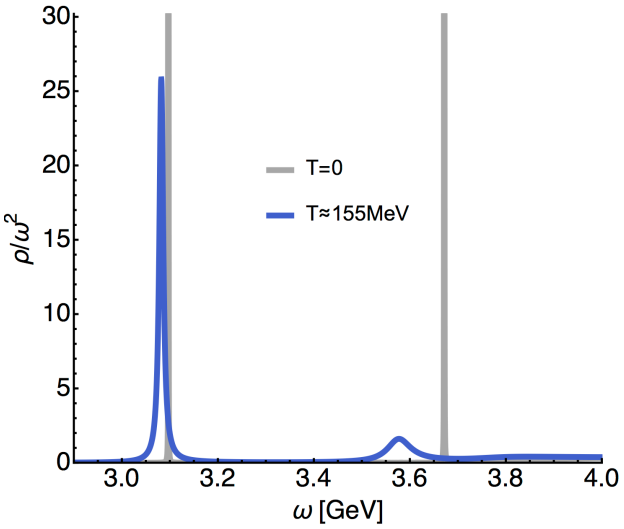
University of Stavanger



- Assume instantaneous freezeout: T>0 states convert to real vacuum particles at around T_C

Towards the ψ' to J/ψ ratio

University of Stavanger



Y.Burnier, O.Kaczmarek, A.R. JHEP 1512 (2015) 101

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(to leading order $\rho(P) = \rho(p_0^2 - \mathbf{p}^2)$)

Towards the ψ' to J/ψ ratio

University of Stavanger

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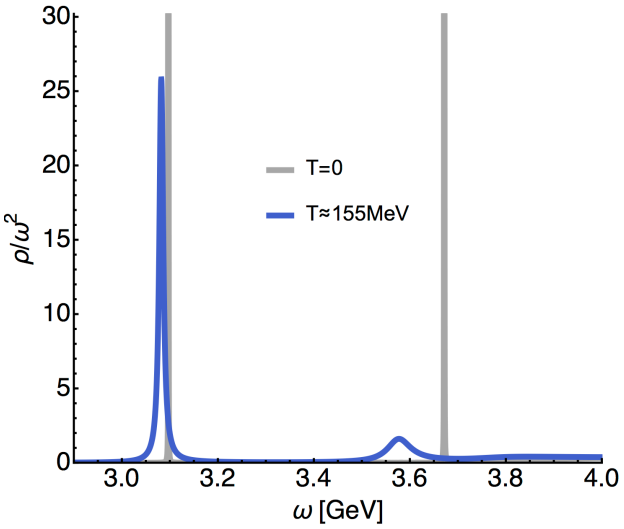

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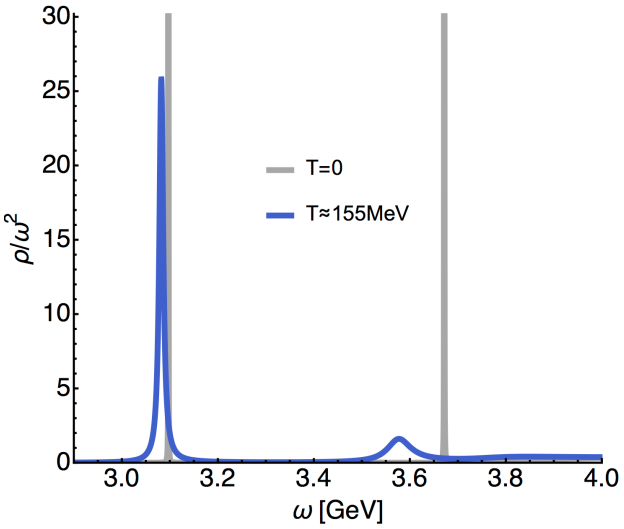
Towards the ψ' to J/ψ ratio

University of Stavanger



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Y.Burnier, O. Kaczmarek, A.R.
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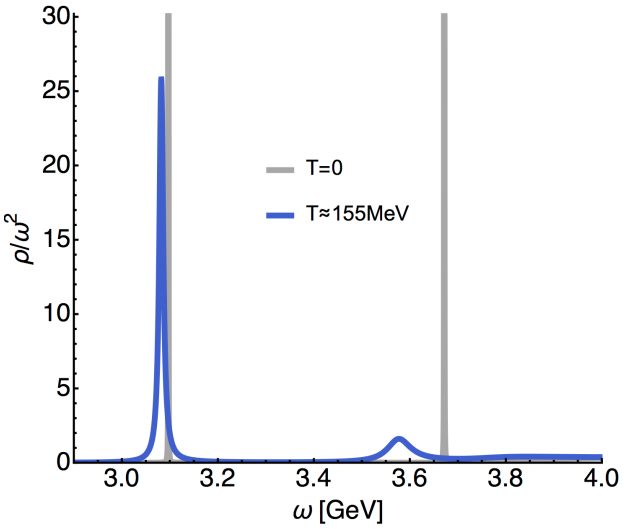
Towards the ψ' to J/ψ ratio

University of Stavanger



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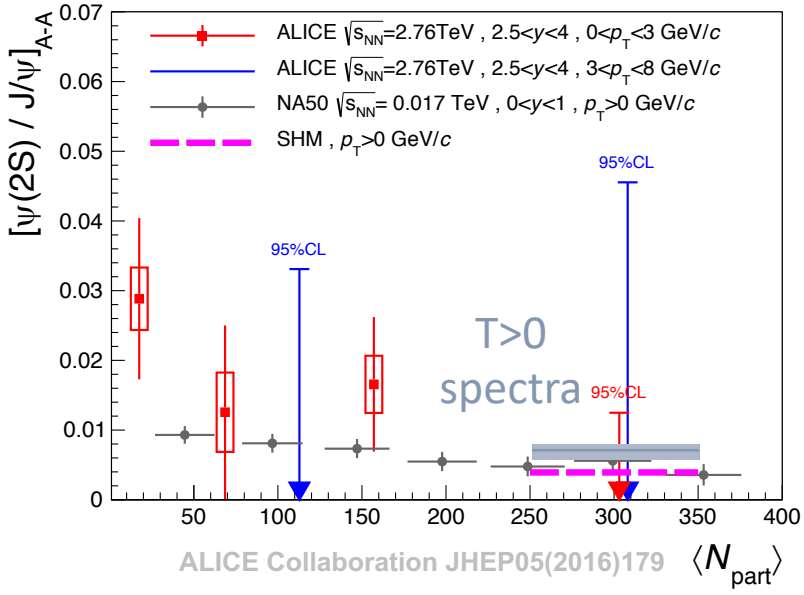
Y.Burnier, O. Kaczmarek, A.R. JHEP 1512 (2015) 101



ALICE Collaboration JHEP05(2016)179

Extracting the potential

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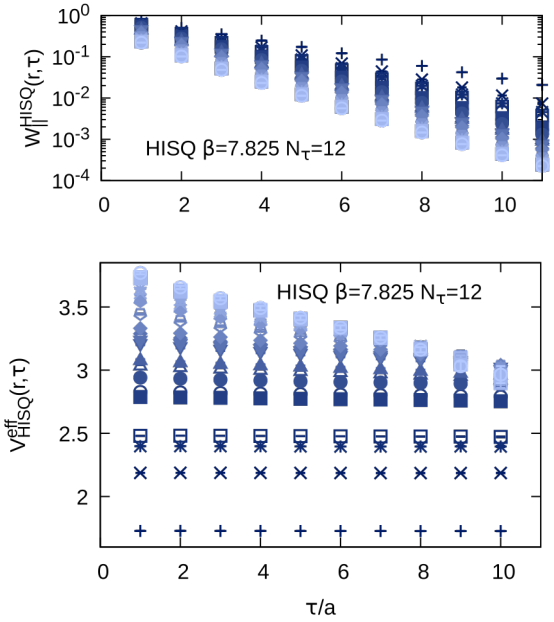
- Example: Pade based reconstructions at β=7.825 T=407MeV N_τ=12



Extracting the potential

University of Stavanger

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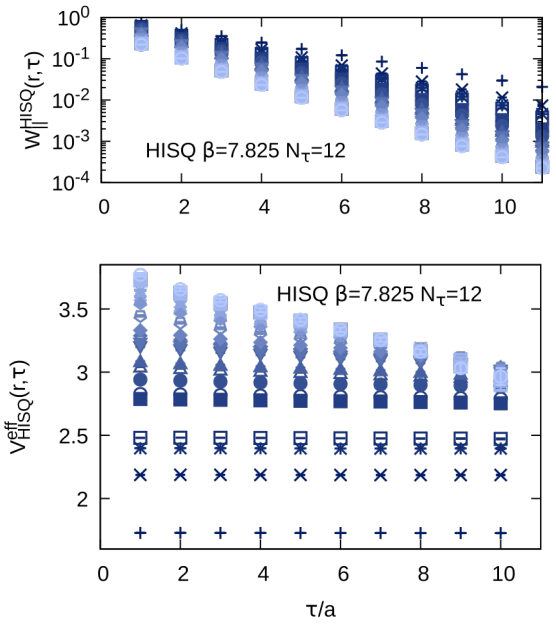
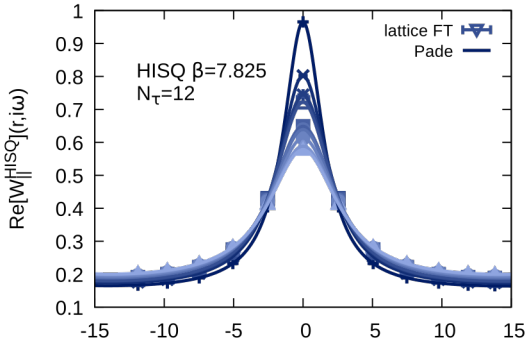
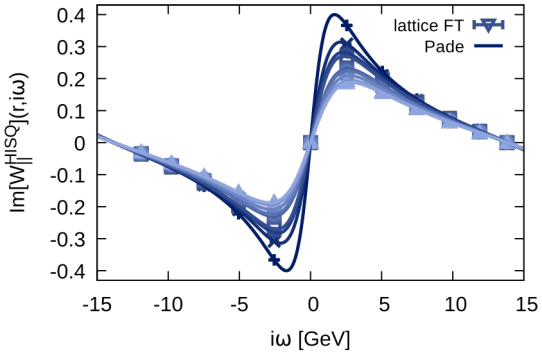


DFT





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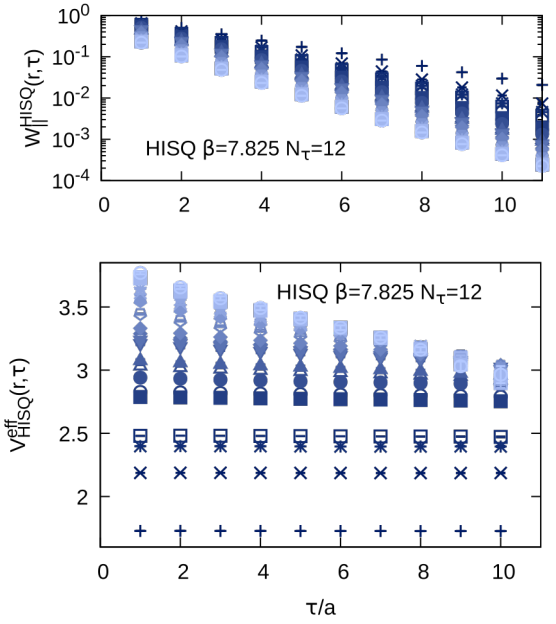

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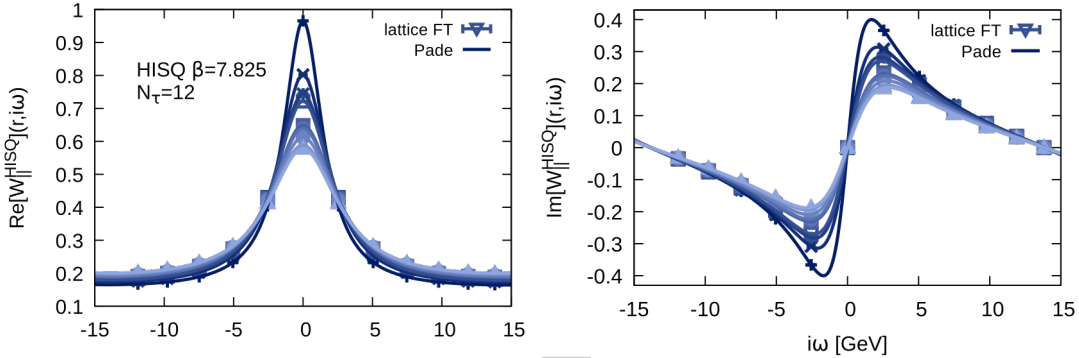
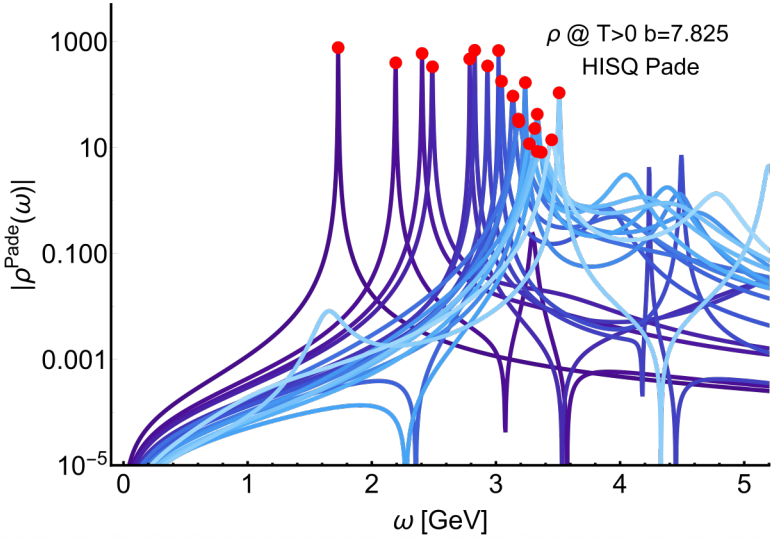


$$\rho(\omega) \approx -\frac{1}{\pi}\mathrm{Im}[R_{N_\tau}(\omega)]$$



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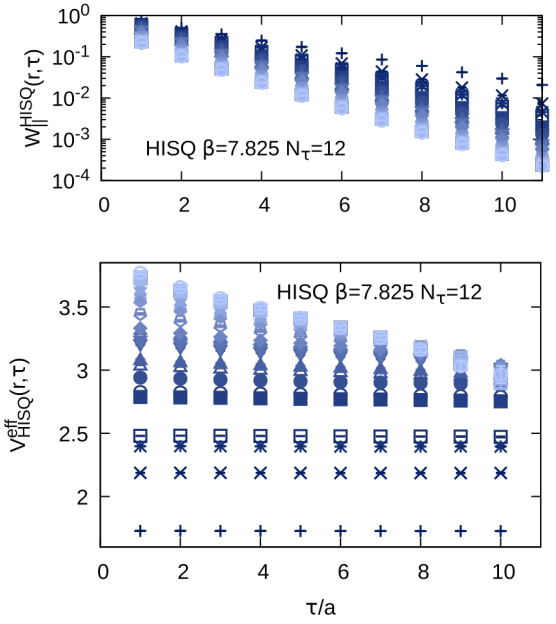

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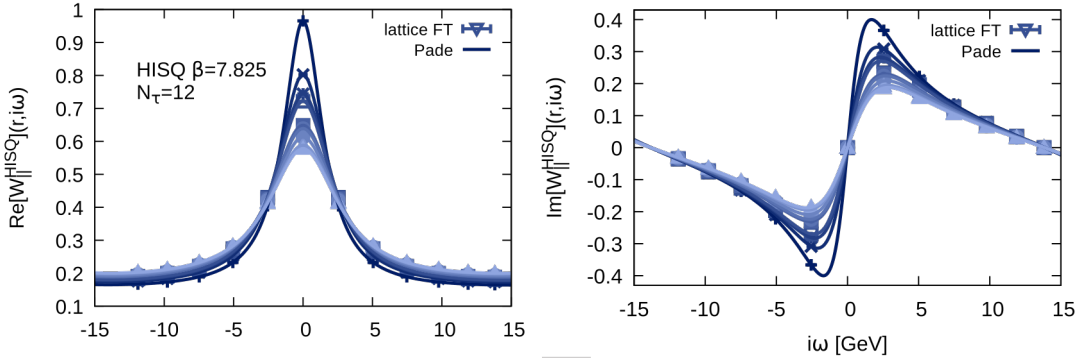

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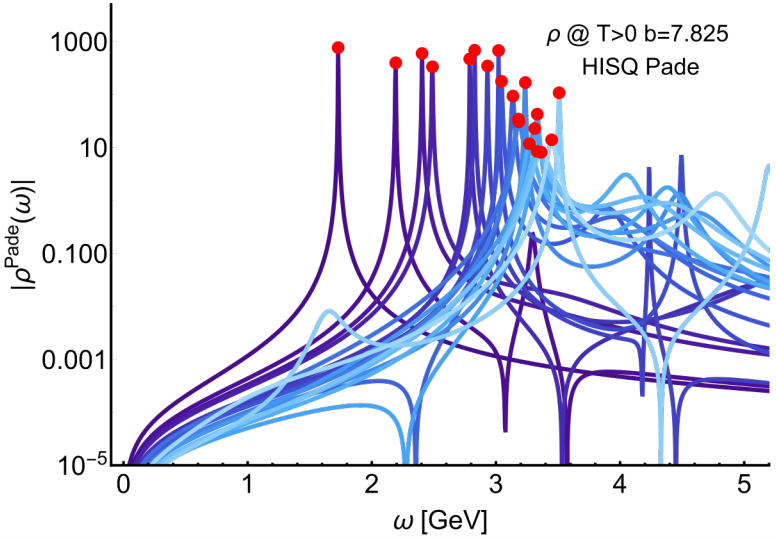

- Always find well defined lowest peak: potential picture appears viable
- Beware of Pade artifacts besides peak: e.g. positivity violation, spikes