Quarkonium Equilibrium — Duke Approach

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EMMI Rapid Reaction Task Force (RRTF)
Suppression and (re)generation of quarkonium in heavy-ion collisions at the LHC

GSI, Dec. 18, 2019

Overview

- QCD is time reversible (no theta term), why approach equilibrium?
- Open quantum system and transport

Open Quantum System

$$H = H_S + H_E + H_I$$

$$\rho(t=0) = \rho_S \otimes \rho_E$$

System & environment

(Heavy quark pairs & QGP)

Unitary evolution

Time reversible

$$U(t,0)(\rho_S \otimes \rho_E)U^\dagger(t,0)$$

System & environment

Trace out (integrate out) environment

$$\rho_S(t=0)$$

System(heavy quark pairs)

Non-unitary

Time irreversible

$$\mathrm{Tr}_E\left[U(t,0)(\rho_S \otimes \rho_E)U^\dagger(t,0)\right]$$

Time-irreversibility comes from monotonicity of relative entropy under partial trace

From Open Quantum System to Transport Equation

Lindblad equation: weak coupling expansion to second order

$$\rho_S(t) = \rho_S(0) - i\Big[tH_S + \sum_{a,b}\sigma_{ab}(t)L_{ab}, \rho_S(0)\Big] + \sum_{a,b,c,d}\gamma_{ab,cd}\Big(L_{ab}\rho_S(0)L^\dagger_{cd} - \frac{1}{2}\{L^\dagger_{cd}L_{ab}, \rho_S\}\Big)$$

Markovian approximation
(system relaxation time >>
environment correlation time)

Wigner transform

$$f_{nl}(\boldsymbol{x}, \boldsymbol{k}, t) \equiv \int \frac{d^3k'}{(2\pi)^3} e^{i\boldsymbol{k}'\cdot\boldsymbol{x}} \langle \boldsymbol{k} + \frac{\boldsymbol{k}'}{2}, nl, 1|\rho_S(t)|\boldsymbol{k} - \frac{\boldsymbol{k}'}{2}, nl, 1\rangle$$

Boltzmann transport equation

$$\frac{\partial}{\partial t} f_{nls}(\boldsymbol{x}, \boldsymbol{k}, t) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f_{nls}(\boldsymbol{x}, \boldsymbol{k}, t) = \mathcal{C}^{(+)}_{nls}(\boldsymbol{x}, \boldsymbol{k}, t) - \mathcal{C}^{(-)}_{nls}(\boldsymbol{x}, \boldsymbol{k}, t)$$

Rate equation

XY T.Mehen, arXiv:1811.07027

From Open Quantum System to Transport Equation

Lindblad equation: **Correction to Hamiltonian**

$$\rho_S(t) = \rho_S(0) - i\left[tH_S + \boxed{\sum_{a,b}\sigma_{ab}(t)L_{ab},}\,\rho_S(0)\right] + \sum_{a,b,c,d}\gamma_{ab,cd}\left(\boxed{L_{ab}\rho_S(0)L^\dagger_{cd}} - \boxed{\frac{1}{2}\{L^\dagger_{cd}L_{ab},\rho_S\}}\right)$$

Markovian approximation
(system relaxation time >> environment correlation time)

Wigner transform

$$f_{nl}(\boldsymbol{x},\boldsymbol{k},t) \equiv \int \frac{d^3k'}{(2\pi)^3} e^{i\boldsymbol{k}'\cdot\boldsymbol{x}} \langle \boldsymbol{k} + \frac{\boldsymbol{k}'}{2}, nl, 1|\rho_S(t)|\boldsymbol{k} - \frac{\boldsymbol{k}'}{2}, nl, 1\rangle$$

Recombination **Dissociation**

Boltzmann transport equation

$$\frac{\partial}{\partial t} f_{nls}(\boldsymbol{x},\boldsymbol{k},t) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f_{nls}(\boldsymbol{x},\boldsymbol{k},t) = \boxed{\mathcal{C}^{(+)}_{nls}(\boldsymbol{x},\boldsymbol{k},t)} - \boxed{\mathcal{C}^{(-)}_{nls}(\boldsymbol{x},\boldsymbol{k},t)}$$

Rate equation

XY T.Mehen, arXiv:1811.07027

Dissociation $-\gamma_{ab,cd}\frac{1}{2}\{L^{\dagger}_{cd}L_{ab},\rho_S(0)\}$

$$\gamma_{ab,cd} = \int d^3R_1 \int d^3R_2 \sum_{i_1,i_2,b_1,b_2} \int_0^t dt_1 \int_0^t dt_2\, C_{\boldsymbol{R}_1 i_1 b_1, \boldsymbol{R}_2 i_2 b_2}(t_1,t_2)$$

$$\langle \boldsymbol{k}_1, n_1 l_1, 1 | \langle S(\boldsymbol{R}_1, t_1) | r_{i_1} | O^{b_1}(\boldsymbol{R}_1, t_1)\rangle | \boldsymbol{p}_{\mathrm{cm}}, \boldsymbol{p}_{\mathrm{rel}}, a_1\rangle$$

$$\langle \boldsymbol{p}_{\mathrm{cm}}, \boldsymbol{p}_{\mathrm{rel}}, a_1 | \langle O^{b_2}(\boldsymbol{R}_2, t_2) | r_{i_2} | S(\boldsymbol{R}_2, t_2)\rangle | \boldsymbol{k}_3, n_3 l_3, 1\rangle$$

$$\langle \Psi_{\boldsymbol{p}_{\mathrm{rel}}} | r_{i_2} | \psi_{n_3 l_3}\rangle \delta^{a_1 b_2} e^{-i(E_{\boldsymbol{k}_3} t_2 - \boldsymbol{k}_3\cdot \boldsymbol{R}_2)} e^{i(E_{\boldsymbol{p}} t_2 - \boldsymbol{p}_{\mathrm{cm}}\cdot \boldsymbol{R}_2)}$$

Weakly-coupled plasma

$$\frac{T_F}{N_C} g^2 \langle E^{b_1}_{i_1}(\boldsymbol{R}_1, t_1) E^{b_2}_{i_2}(\boldsymbol{R}_2, t_2)\rangle_T$$

$$= \frac{T_F}{N_C} g^2 \delta^{b_1 b_2} \int \frac{d^4 q}{(2\pi)^4} e^{iq_0(t_1-t_2) - i\boldsymbol{q}\cdot(\boldsymbol{R}_1 - \boldsymbol{R}_2)} (q_0^2 \delta_{i_1 i_2} - q_{i_1} q_{i_2}) n_B(q_0)(2\pi)\mathrm{sign}(q_0)\delta(q_0^2 - \boldsymbol{q}^2)$$

Markovian approximation:

$t \longrightarrow \infty$ when doing time integral —> time length * energy conservation

Maybe we can compare this between transport and quantum evolutions?

Recombination $\gamma_{ab,cd} L_{ab} \rho_S(0) L^\dagger_{cd}$

$$t\int \frac{d^3p_{\rm cm}}{(2\pi)^3}\frac{d^3p_{\rm rel}}{(2\pi)^3}\frac{d^3q}{(2\pi)^3 2q}(1+n_B(q))\sum_{a,i}(2\pi)^4\delta^3(\boldsymbol{k}-\boldsymbol{p}_{\rm cm}+\boldsymbol{q})\delta(-|E_{nl}|+q-\frac{\boldsymbol{p}^2_{\rm rel}}{M})$$

$$\frac{2T_F}{3N_C}q^2g^2\underbrace{\langle\psi_{nl}|r_i|\Psi_{\boldsymbol{p}_{\rm rel}}\rangle}_{p_{\rm rel}\sim a_B^{-1}\sim Mv}\int d^3r\,\underbrace{\psi_{nl}(\boldsymbol{r})r_i\Psi^*_{\boldsymbol{p}_{\rm rel}}}_{\text{no recombination if r >> size of quarkonium}}(\boldsymbol{r})f_{Q\bar{Q}}(\boldsymbol{x},\boldsymbol{p}_{\rm cm},\boldsymbol{r},\boldsymbol{p}_{\rm rel},a,t=0)$$

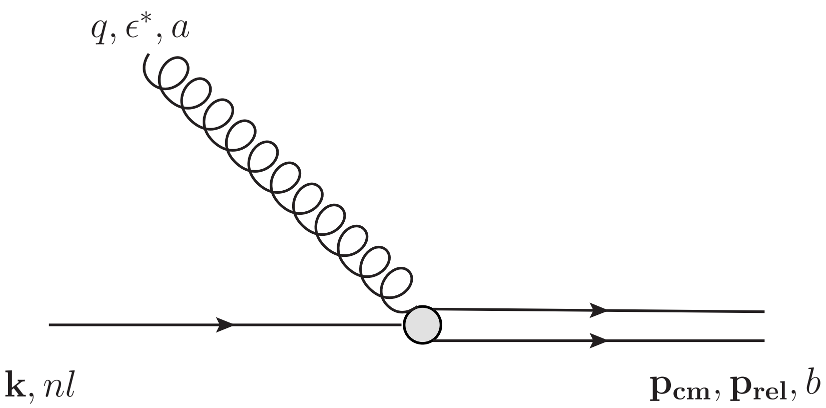
When can we take distribution function out?

Uniformly distributed when r < Bohr radius a_B **Diffusion length >> Bohr radius**

$$p_{\rm rel}\ll\frac{Mv}{\alpha_s^2v^2}$$

BBGKY hierarchy $f_{Q\bar{Q}}(\boldsymbol{x},\boldsymbol{p}_{\rm cm},\boldsymbol{r},\boldsymbol{p}_{\rm rel},a,t)=\frac{1}{9}f_Q(\boldsymbol{x}_1,\boldsymbol{p}_1,t)f_{\bar{Q}}(\boldsymbol{x}_2,\boldsymbol{p}_2,t)$

Equilibrium in Transport Equations



$$C_{nl}^{(-)} \equiv \int \frac{d^3p_{\rm cm}}{(2\pi)^3}\frac{d^3p_{\rm rel}}{(2\pi)^3}\frac{d^3q}{(2\pi)^3 2q} n_B(q)(2\pi)^4\delta^3(\boldsymbol{k}-\boldsymbol{p}_{\rm cm}+\boldsymbol{q})\delta(E_k-E_p+q)$$
$$\frac{2}{3}C_F q^2 g^2 |\langle\psi_{nl}|\boldsymbol{r}|\Psi_{\boldsymbol{p}_{\rm rel}}\rangle|^2 f_{nl}(\boldsymbol{x},\boldsymbol{k},t\quad)$$

$$C_{nl}^{(+)} \equiv \frac{1}{9}\int \frac{d^3p_{\rm cm}}{(2\pi)^3}\frac{d^3p_{\rm rel}}{(2\pi)^3}\frac{d^3q}{(2\pi)^3 2q}(1+n_B(q))f_Q(\boldsymbol{x}_1,\boldsymbol{p}_1,t)f_{\bar{Q}}(\boldsymbol{x}_2,\boldsymbol{p}_2,t)$$
$$(2\pi)^4\delta^3(\boldsymbol{k}-\boldsymbol{p}_{\rm cm}+\boldsymbol{q})\delta(-|E_{nl}|+q-\frac{\boldsymbol{p}_{\rm rel}^2}{M})\frac{2}{3}C_F q^2 g^2|\langle\psi_{nl}|\boldsymbol{r}|\Psi_{\boldsymbol{p}_{\rm rel}}\rangle|^2$$

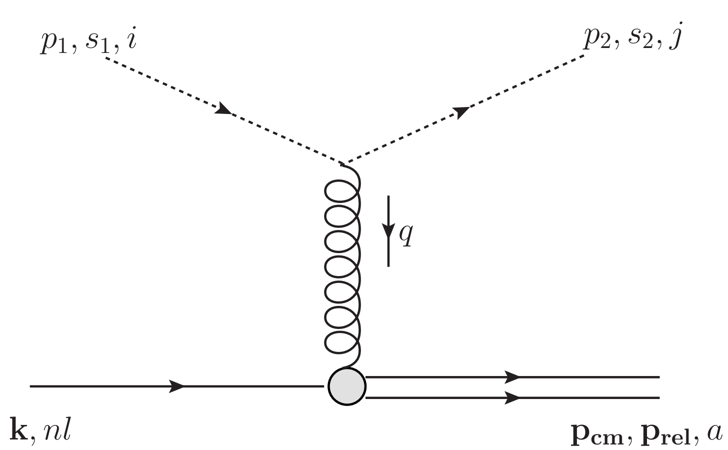
Equilibrium built in the rates:

$$E_k = -|E_{nl}|$$
$$E_p = \frac{\boldsymbol{p}_{\rm rel}^2}{M}$$

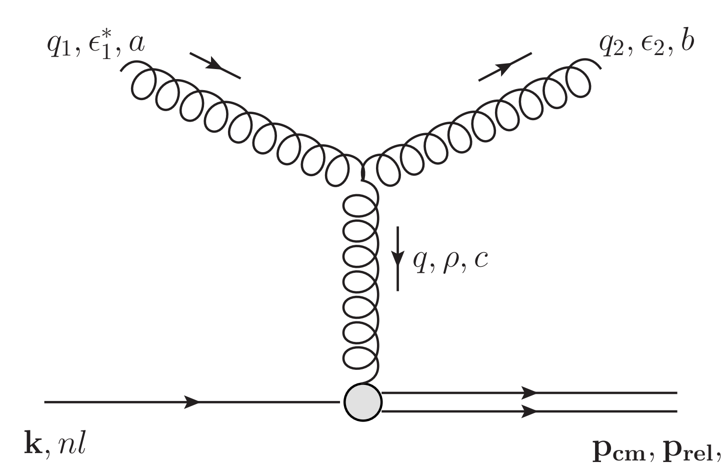
same amplitude squared

$$n_B f_{nl} \sim (1+n_B)f_Q f_{\bar{Q}}$$

Similarly for NLO processes:



$$n_B(1+n_B)f_{nl} \sim (1+n_B)n_B f_Q f_{\bar{Q}}$$
$$n_F(1-n_F)f_{nl} \sim (1-n_F)n_F f_Q f_{\bar{Q}}$$

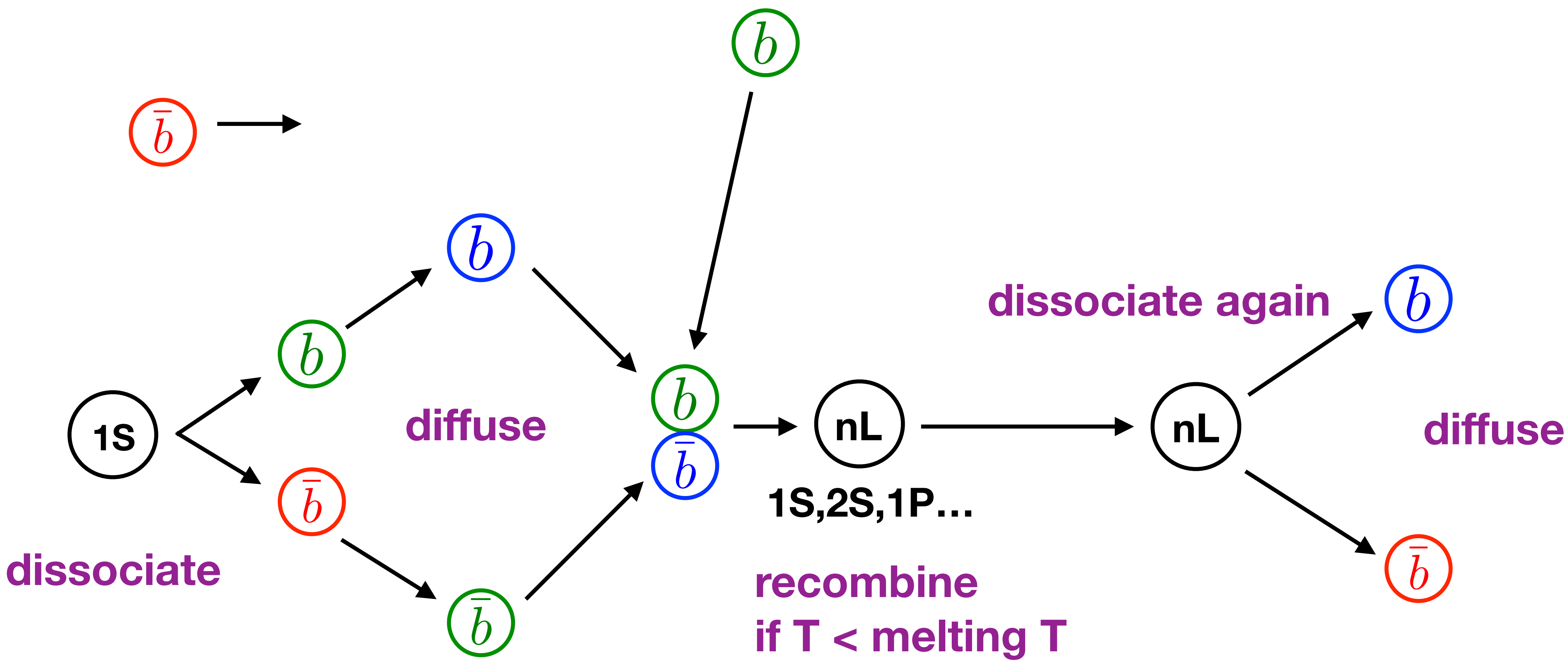


Coupled with Transport of Open Heavy Flavor

heavy quark
$$(\frac{\partial}{\partial t} + \dot{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{x}}) f_Q(\boldsymbol{x}, \boldsymbol{p}, t) = \mathcal{C}_Q - \mathcal{C}_Q^+ + \mathcal{C}_Q^-$$

anti-heavy quark
$$(\frac{\partial}{\partial t} + \dot{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{x}}) f_{\bar{Q}}(\boldsymbol{x}, \boldsymbol{p}, t) = \mathcal{C}_{\bar{Q}} - \mathcal{C}_{\bar{Q}}^+ + \mathcal{C}_{\bar{Q}}^-$$

each well-defined quarkonium state nl = 1S, 2S,1P etc.
$$(\frac{\partial}{\partial t} + \dot{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{x}}) f_{nls}(\boldsymbol{x}, \boldsymbol{p}, t) = \mathcal{C}_{nls}^+ - \mathcal{C}_{nls}^-$$



Approach Equilibrium

Setup:

- QGP box w/ const T = 250 MeV, Υ(1S) state & b quarks, total b flavor = 50 (fixed), box side = 10 fm
- Initial momenta sampled from thermal or uniform distributions
- Turn on/off heavy quark transport
- Compare hidden b flavor fraction with thermal equilibrium

$$N_i^{\mathrm{eq}} = g_i \mathrm{Vol} \int \frac{d^3p}{(2\pi)^3} \lambda_i e^{-E_i(p)/T}$$

$$N_b^{\mathrm{eq}} + N_{\Upsilon(1S)}^{\mathrm{eq}} = N_{b,\mathrm{tot}}$$

$$g_b = g_{\bar{b}} = 3 \times 2 = 6 \qquad \lambda_b = \lambda_{\bar{b}}$$

$$g_{\Upsilon(1S)} = 3(4) \qquad \lambda_b^2 = \lambda_{\Upsilon(1S)}$$

$$E_i(p) = \sqrt{M_i^2 + p^2}, \qquad M_i + \frac{p^2}{2M_i}$$

$$M_{nl} = 2M_b - |E_{nl}|$$

XY, B.Müller arXiv:1709.03529

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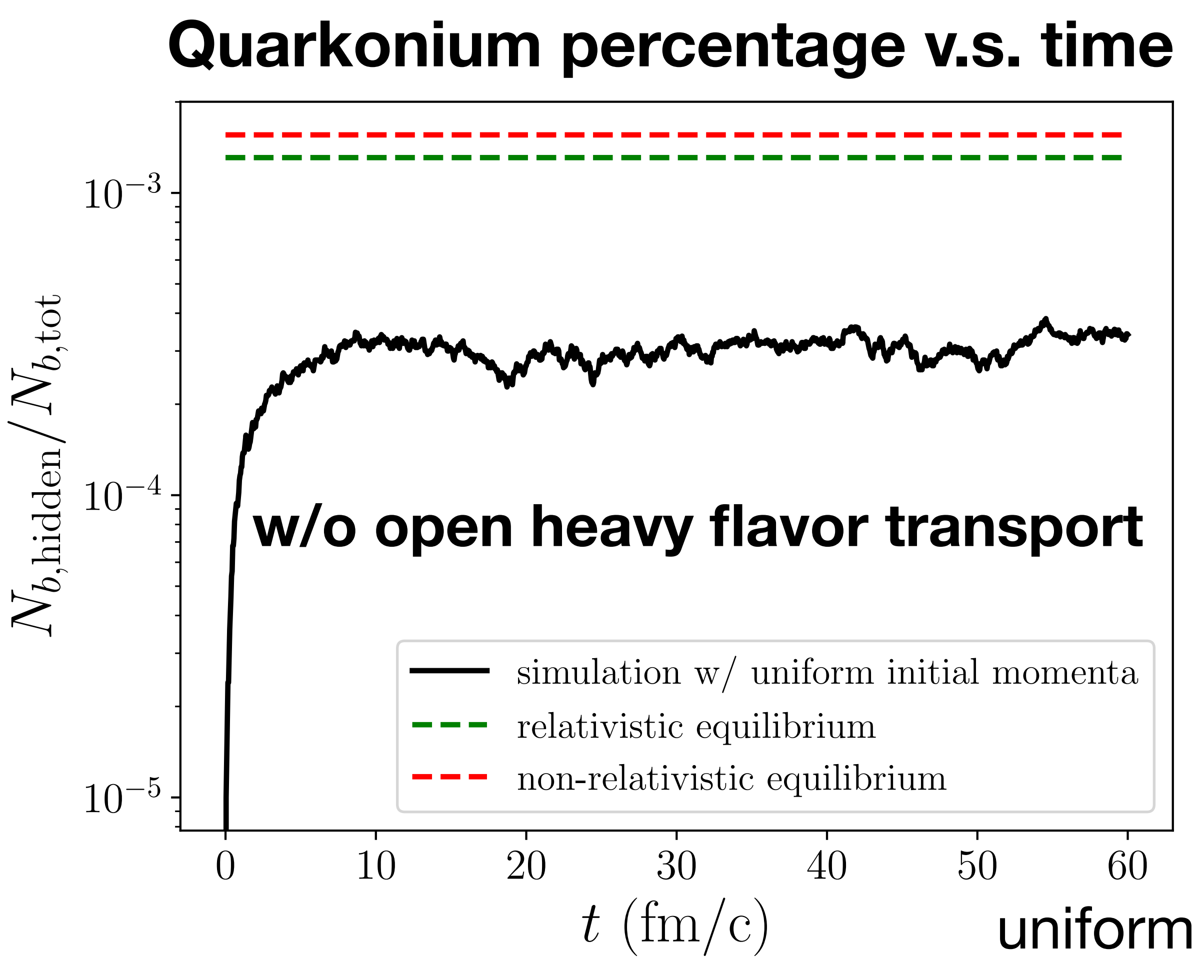
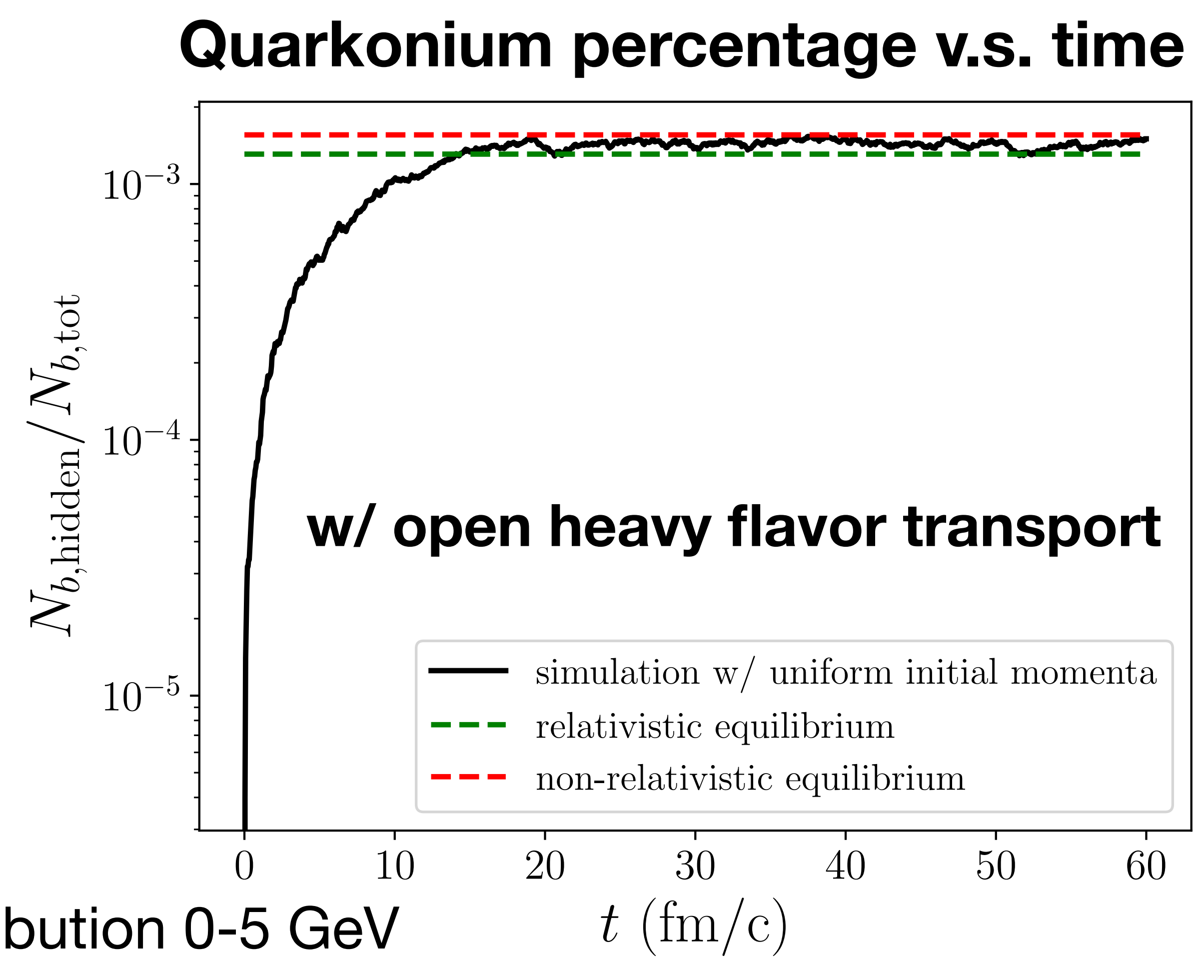




uniform distribution 0-5 GeV
T=250 MeV

Dissociation-recombination interplay drives to detailed balance

Heavy quark energy gain/loss necessary to drive kinetic equilibrium of quarkonium

Perspective from random matrix theory /eigenstate thermalization hypothesis

$$O^\dagger \boldsymbol{r}\cdot g\underset{\substack{\downarrow \\ \text{“random”}}}{\boldsymbol{E}} S$$

off-diagonal element

Conclusions

- Connection between transport and quantum evolutions
- Demonstrate numerically how system approaches equilibrium in coupled transport equations
- The equilibrium fraction of hidden heavy flavor depends on: total number of open quarks (open + hidden), binding energy, QGP volume, spin, how many states are included

Backup

Open Quantum System: Towards Equilibrium

Relative entropy $$S(\rho||\sigma) \equiv \mathrm{Tr}(\rho \ln \rho) - \mathrm{Tr}(\rho \ln \sigma)$$

Monotonicity of relative entropy under partial trace

$$\rho = \rho_S \otimes \rho_E \qquad\qquad S(\rho_S||\sigma_S) \leq S(\rho||\sigma)$$
$$\sigma = \sigma_S \otimes \sigma_E$$

Equilibrium state defined by

$$\rho_S^{\mathrm{eq}} = \mathrm{Tr}_E\{U(t)(\rho_S^{\mathrm{eq}} \otimes \rho_E^{\mathrm{eq}})U^\dagger(t)\}$$

Open quantum system evolution (assume QGP in thermal equilibrium)

$$\rho_S(t) = \mathrm{Tr}_E\{U(t)(\rho_S(t=0) \otimes \rho_E^{\mathrm{eq}})U^\dagger(t)\}$$

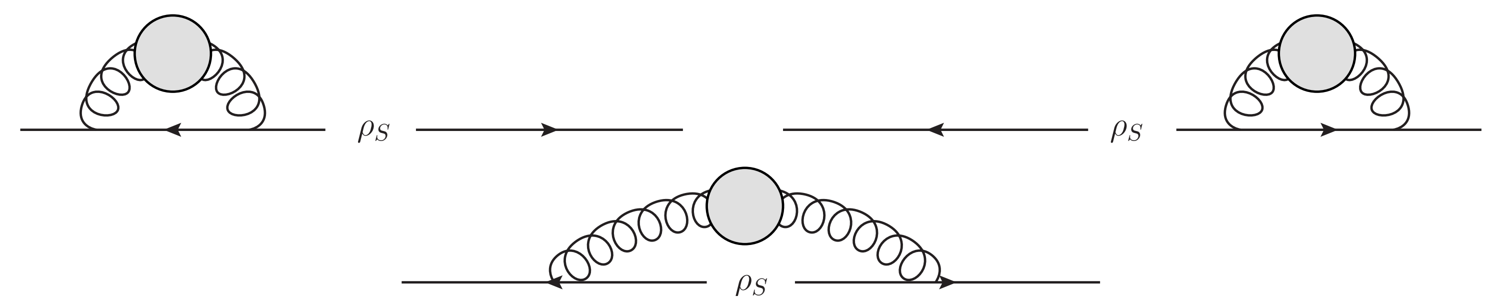
We can show

$$S(\rho_S(t)||\rho_S^{\mathrm{eq}}) \leq S(\rho_S(t=0)||\rho_S^{\mathrm{eq}})$$

From Open Quantum System to Transport Equation

Weak coupling to 2nd order: Lindblad equation

$$\rho_S(t) = \rho_S(0) - i\Big[tH_S + \sum_{a,b}\sigma_{ab}(t)L_{ab}, \rho_S(0)\Big] + \sum_{a,b,c,d}\gamma_{ab,cd}\Big(L_{ab}\rho_S(0)L^{\dagger}_{cd} - \frac{1}{2}\{L^{\dagger}_{cd}L_{ab}, \rho_S\}\Big)$$



$$H_I = \sum_{\alpha} O^{(S)}_{\alpha} \otimes O^{(E)}_{\alpha}$$

$$\gamma_{ab,cd}(t) \equiv \sum_{\alpha,\beta}\int_0^t dt_1 \int_0^t dt_2 C_{\alpha\beta}(t_1,t_2)\langle a|O^{(S)}_{\beta}(t_2)|b\rangle\langle c|O^{(S)}_{\alpha}(t_1)|d\rangle^*$$

$$\sigma_{ab}(t) \equiv \frac{-i}{2}\sum_{\alpha,\beta}\int_0^t dt_1 \int_0^t dt_2 C_{\alpha\beta}(t_1,t_2)\mathrm{sign}(t_1 - t_2)\langle a|O^{(S)}_{\alpha}(t_1)O^{(S)}_{\beta}(t_2)|b\rangle$$

Boltzmann transport equation

$$C_{\alpha\beta}(t_1,t_2) \equiv \mathrm{Tr}_E(O^{(E)}_{\alpha}(t_1)O^{(E)}_{\beta}(t_2)\rho_E)$$

$|a\rangle$ Eigenstates of H_S

$$L_{ab} = |a\rangle\langle b|$$

Backup: Numerical Implementation

- **Test particle Monte Carlo** $f(\boldsymbol{x}, \boldsymbol{p}, t) = \sum_i \delta^3(\boldsymbol{x} - \boldsymbol{y}_i(t))\delta^3(\boldsymbol{p} - \boldsymbol{k}_i(t))$

- Each time step: read in hydro-cell velocity, temperature; consider diffusion, dissociation, recombination in particle's rest frame and boost back

- If specific process occurs, sample incoming medium particles and outgoing particles from differential rates, conserving energy momentum

- Recombination term contains $f_Q(\boldsymbol{x}_1, \boldsymbol{p}_1, t) f_{\bar{Q}}(\boldsymbol{x}_2, \boldsymbol{p}_2, t)$

For each HQ, search anti-HQ within a radius, weighted sum

$$f_Q(\boldsymbol{x}_1, \boldsymbol{p}_1, t) f_{\bar{Q}}(\boldsymbol{x}_2, \boldsymbol{p}_2, t) = \sum_{i,j} \frac{e^{-(\boldsymbol{y}_i - \bar{\boldsymbol{y}}_j)^2/2a_B^2}}{(2\pi a_B)^{3/2}} \delta^3\left(\frac{\boldsymbol{x}_1 + \boldsymbol{x}_2}{2} - \frac{\boldsymbol{y}_i + \bar{\boldsymbol{y}}_j}{2}\right) \delta^3(\boldsymbol{p}_1 - \boldsymbol{k}_i) \delta^3(\boldsymbol{p}_2 - \bar{\boldsymbol{k}}_j)$$