Functional Methods in Strongly Correlated Systems
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Sine-Gordon models and 1D quantum fluids

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based on arXiv:1812.01908 (PRL 2019) with Romain Daviet
arXiv:1903.12374

One-dimensional quantum fluids

↕ (bosonization)

sine-Gordon model + non-integrable perturbations

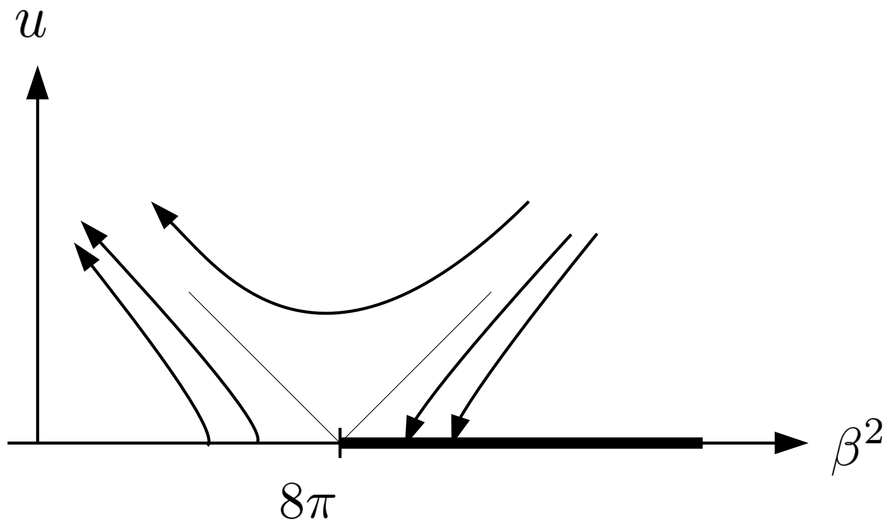
Outline

- sine-Gordon model
    - perturbative RG & exact results
    - non-perturbative functional RG
    - spectrum in the massive phase
    - the Lukyanov-Zamolodchikov conjecture
- Disordered one-dimensional bosons
    - bosonization and replica formalism
    - strong-disorder fixed point: Bose-glass phase
    - metastable states: pinning, “shocks” and “avalanches”
    - conductivity
- Conclusion

Sine-Gordon model

- Euclidean action $$S[\varphi] = \int d^2r \;\; \frac{1}{2}(\nabla\varphi)^2 - u\cos(\beta\varphi)$$

- perturbative RG:



Perturbative RG
BKT flow

- exact results
  - $\beta^2 > 8\pi$: massless phase
  - $\beta^2 < 8\pi$: massive phase with (anti)soliton excitation $Q=\pm 1$

$$M_{\rm sol} = \Lambda \frac{2\Gamma\left(\frac{K}{2-2K}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2-2K}\right)} \left[\frac{\Gamma(1-K)}{\Gamma(K)} \frac{\pi u}{2\Lambda^2}\right]^{\frac{1}{2-2K}} \qquad K = \beta^2/8\pi \ : \text{Luttinger parameter}$$

$\beta^2 < 4\pi$: soliton-antisoliton bound state (breather) $Q=0$

$$M_1 = 2M_{\rm sol} \sin\left(\frac{\pi}{2}\frac{K}{1-K}\right)$$

FRG approach to sine-Gordon model

[R. Daviet & ND, arXiv:1812.01908, PRL'19]

- infrared regulator

$$S[\varphi] \to S[\varphi] + \int_q J(-q)\varphi(q) + \frac{1}{2}\int_q \varphi(-q) R_k(q)\varphi(q)$$

- effective action

$$\phi(q) = \langle \varphi(q) \rangle$$

$$\Gamma_k[\phi] = -\ln \mathcal{Z}_k[J] + \int_q J(-q)\phi(q) - \frac{1}{2}\int_q \phi(-q) R_k(q)\phi(q)$$

- Wetterich's equation

$$\partial_k \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k \left(\Gamma_k^{(2)}[\phi] + R_k\right)^{-1}\right\} \quad \text{with} \quad \Gamma_\Lambda[\phi] = S[\phi]$$

- derivative expansion

$$\Gamma_k[\phi] = \int_r \frac{1}{2} Z_k(\phi)(\nabla\phi)^2 + U_k(\phi)$$

Previous works on SG model: Nagi *et al.* 2009, Pangon 2012, Bacsó *et al.* 2015

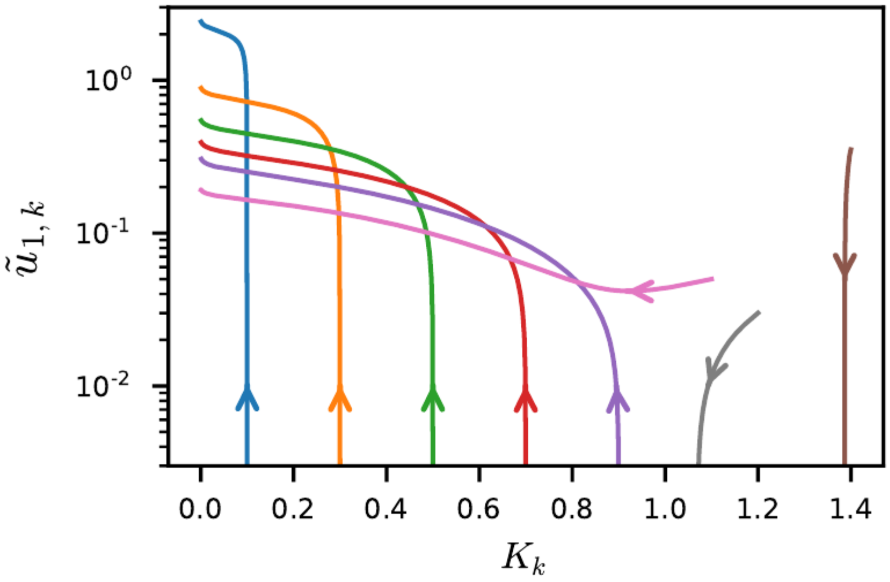
- phase diagram

dimensionless variables

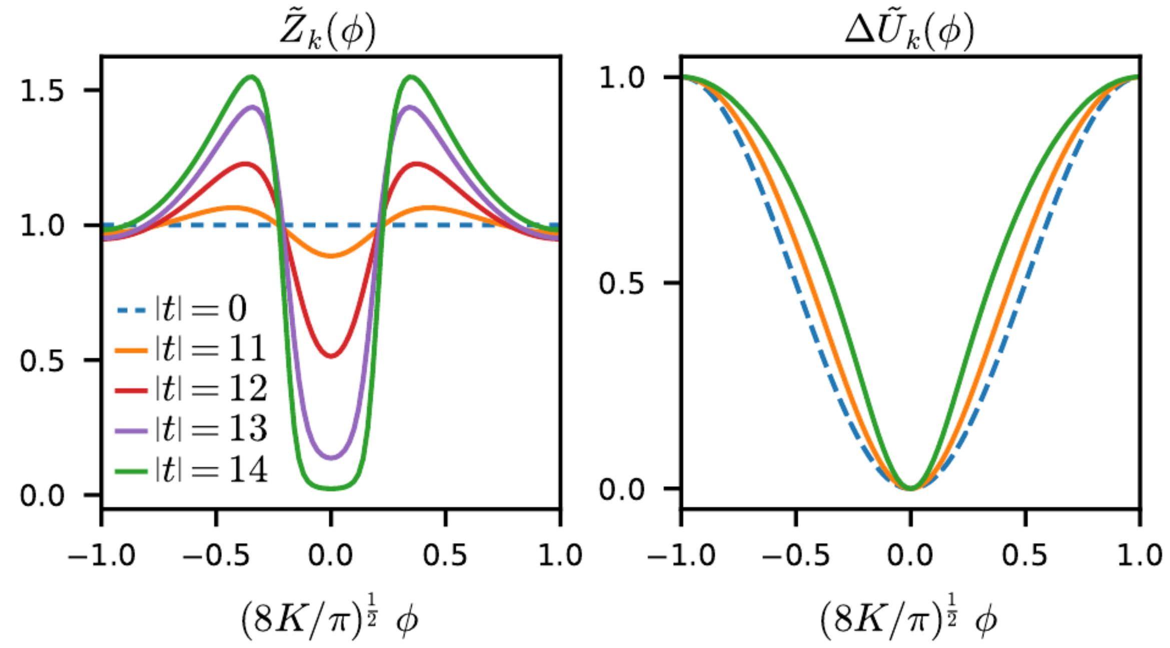
$$\tilde{Z}_k(\phi) = \frac{Z_k(\phi)}{Z_k} \quad \text{with} \quad Z_k = \langle Z_k(\phi) \rangle_\phi$$

$$\tilde{U}_k(\phi) = \frac{U_k(\phi)}{Z_k k^2} = -\sum_{n=0}^{\infty} \tilde{u}_{n,k} \cos(n\beta\phi)$$

$$K_k = \frac{K}{Z_k} \quad \text{running Luttinger parameter}$$



- massive phase



$$\lim_{k\to 0} \tilde{Z}_k(\phi) = \tilde{Z}^*(\phi) \qquad \lim_{k\to 0} \tilde{U}_k(\phi) = \tilde{U}^*(\phi)$$

spectrum in topological sector Q=0: soliton + antisoliton (free pair or bound state)

$$\Gamma_k^{(2)}(p,\phi) = Z_k(\phi)p^2 + U_k''(\phi)$$

$$M^2 = \lim_{k\to 0} \frac{U_k''(0)}{Z_k(0)} = \lim_{k\to 0} k^2 \frac{\tilde{U}_k''(0)}{\tilde{Z}_k(0)} \simeq \lim_{k\to 0} \frac{k^2 \tilde{U}^{*\prime\prime}(0)}{\tilde{Z}_k(0)} = \begin{cases} (2M_{\rm sol})^2 \\ M_1^2 \end{cases}$$

NB: Ising model: $\lim_{k\to 0} U_k''(0)$ and $\lim_{k\to 0} Z_k(0)$ are finite

- soliton and breather masses

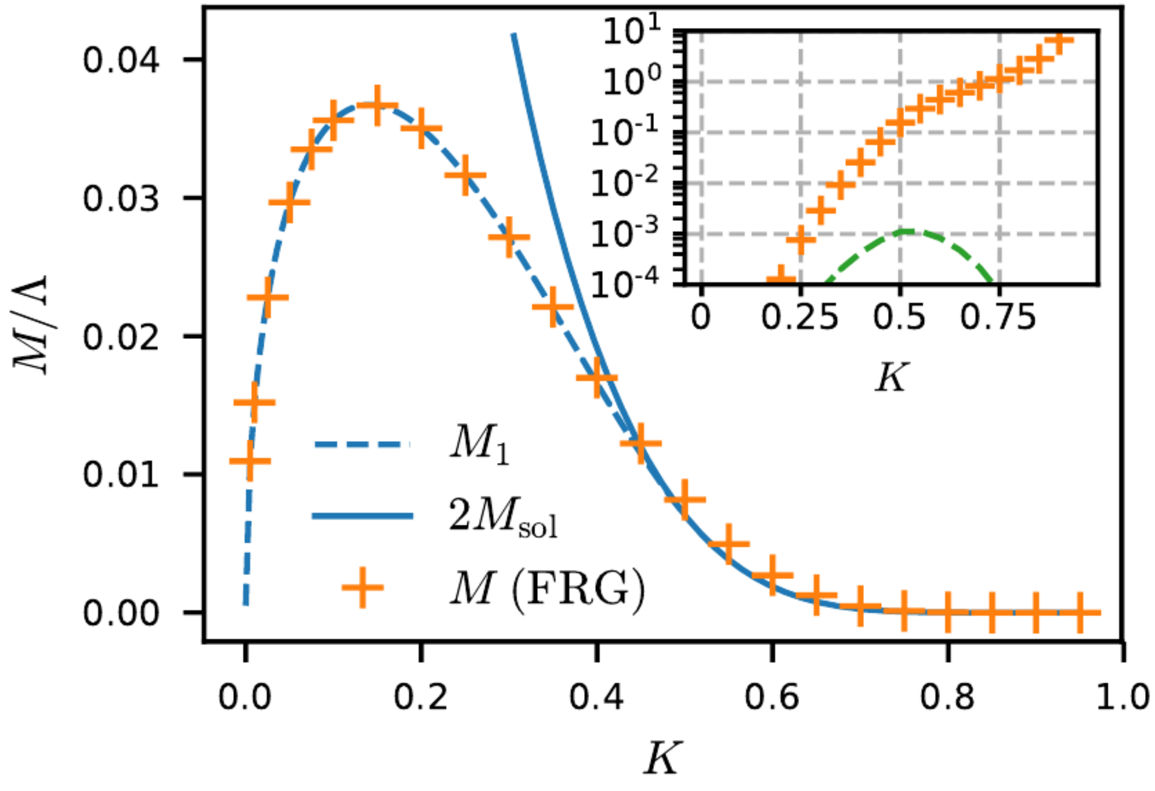


Figure 3. Mass M of the lowest excitation as obtained from the FRG approach. The solid and dashed lines show the exact values of $2M_{\rm sol}$ and M_1 (the latter being defined only for $K \leq 1/2$) [Eqs. (8,9)]. The inset shows the relative (crosses) and absolute (dashed line) errors of the FRG result.

Lukyanov-Zamolodchikov conjecture

Nucl. Phys. B 493, 571 (1997)

$$\langle e^{i\sqrt{2\pi K}n\varphi}\rangle = \left[\frac{\Gamma(1-K)}{\Gamma(K)}\frac{\pi u}{2\Lambda^2}\right]^{\frac{n^2K}{4-4K}} \exp\left\{\int_0^\infty \frac{dt}{t}\left[\frac{\sinh^2(nKt)}{2\sinh(Kt)\sinh(t)\cosh[(1-K)t]} - \frac{n^2K}{2}e^{-2t}\right]\right\}$$

where $K=\beta^2/8\pi$ (massive phase: $K<1$)

exact for $K\to 0$, $K=1/2$ (noninteracting Thirring model) and $n=2$ (free energy)

- FRG approach

$$S[\varphi] \to S[\varphi] + \int_r \left(h^* e^{i\sqrt{2\pi K}n\varphi} + \text{c.c.}\right)$$

$$\Gamma_k[\phi; h^*, h] = \int_r \frac{1}{2}Z_k(\phi, h^*, h)(\nabla\phi)^2 + U_k(\phi, h^*, h)$$

$$\langle e^{i\sqrt{2\pi K}n\varphi}\rangle = -\frac{\partial U_k(\phi=0, h^*, h)}{\partial h^*}\bigg|_{h^*=h=0}$$

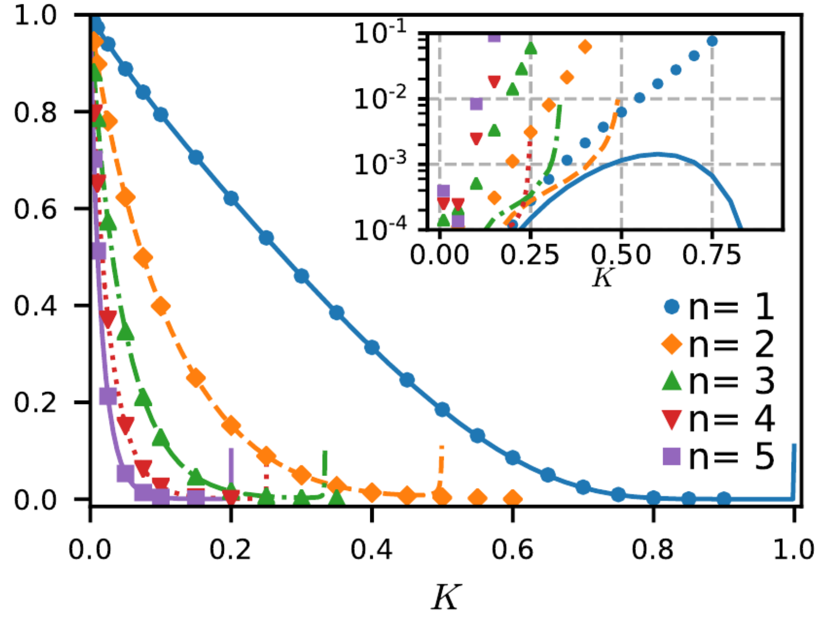


Figure 4. Expectation value $\langle e^{in\sqrt{2\pi K}\varphi}\rangle$ as obtained from FRG (symbols) vs the Lukyanov-Zamolodchikov conjecture (2) valid for $K < 1/n$ (lines). The inset shows the relative (symbols) and absolute (lines) disagreements between the FRG results and the conjecture, respectively $\epsilon_{\rm rel}$ and $\epsilon_{\rm abs}$. ($\Lambda = 1$ and $u/\Lambda^2 = 10^{-4}$.)

Lukyanov-Zamolodchikov conjecture confirmed with 1% accuracy

One-dimensional Bose fluid

- Hamiltonian

$$
\begin{aligned}
H = \quad & \int dx \;\; \psi^\dagger(x)\left(-\frac{\partial_x^2}{2m} - \mu\right) + g\left(\psi^\dagger(x)\psi(x)\right)^2 && \text{Luttinger liquid (superfluid)} \\
& +V_{\text{lattice}}(x)\;\psi^\dagger(x)\psi(x) && \text{Mott insulator} \\
& +V_{\text{disorder}}(x)\;\psi^\dagger(x)\psi(x) && \text{Bose glass} \\
& +\cdots
\end{aligned}
$$

- bosonization [Haldane 1981]

$$
\psi(x) = e^{i\theta(x)}\sqrt{\rho(x)}
$$

$$
\rho(x) = \rho_0 - \frac{1}{\pi}\partial_x\varphi(x) + 2\rho_2\cos(2\pi\rho_0 x + 2\varphi(x)) + \cdots
$$

$$
[\theta(x), \partial_y\varphi(y)] = i\pi\delta(x-y)
$$

- Luttinger liquids

$$
H_{\text{LL}} = \int dx \frac{v}{2\pi}\left\{K(\partial_x\theta)^2 + \frac{1}{K}(\partial_x\varphi)^2\right\}
$$

- beyond luttinger liquids: sine-Gordon models + non-integrable perturbations

Interacting bosons in a random potential

[ND, arXiv:1903.12374]

- Hamiltonian

$$H = H_{\rm LL} + \int dx\, V(x)\rho(x)$$
$$= H_{\rm LL} + \int dx\, V(x) \left(\rho_0 - \frac{1}{\pi}\partial_x\varphi + 2\rho_2 \cos(2\pi\rho_0 x + 2\varphi) + \cdots \right)$$
$$\text{with} \quad \begin{cases} \overline{V(x)} = 0 \\ \overline{V(x)V(x')} = D\delta(x-x') \end{cases}$$

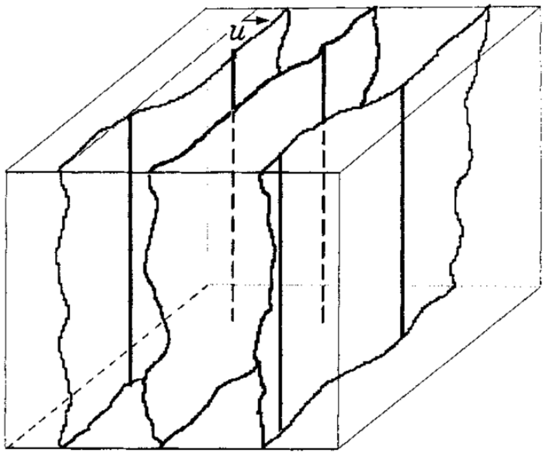
- replica formalism (n copies of the system)

$$\overline{Z^n} = 1 + n\overline{\ln Z} + \cdots = \int \mathcal{D}[\varphi]\, e^{-S[\varphi]}$$
$$S[\varphi] = \sum_{a=1}^{n} \int dx \int_0^\beta d\tau \frac{v}{2\pi K} \left\{ (\partial_x\varphi_a)^2 + \frac{(\partial_\tau\varphi_a)^2}{v^2} \right\}$$
$$-\mathcal{D} \sum_{a,b=1}^{n} \int dx \int_0^\beta d\tau\, d\tau' \cos[2\varphi_a(x,\tau) - 2\varphi_b(x,\tau')]$$

- analogy with classical disordered systems: *r=(x,y=vτ)*

$$S[\varphi] = \sum_{a=1}^{n} \int d^2r \frac{1}{2\pi K} (\nabla \varphi_a)^2 - \mathcal{D} \sum_{a,b=1}^{n} \int dx \int dy\, dy' \cos[2\varphi_a(x,y) - 2\varphi_b(x,y')]$$

describes 2D elastic manifolds in a 3D disordered medium with correlated disorder at temperature *T=πK*

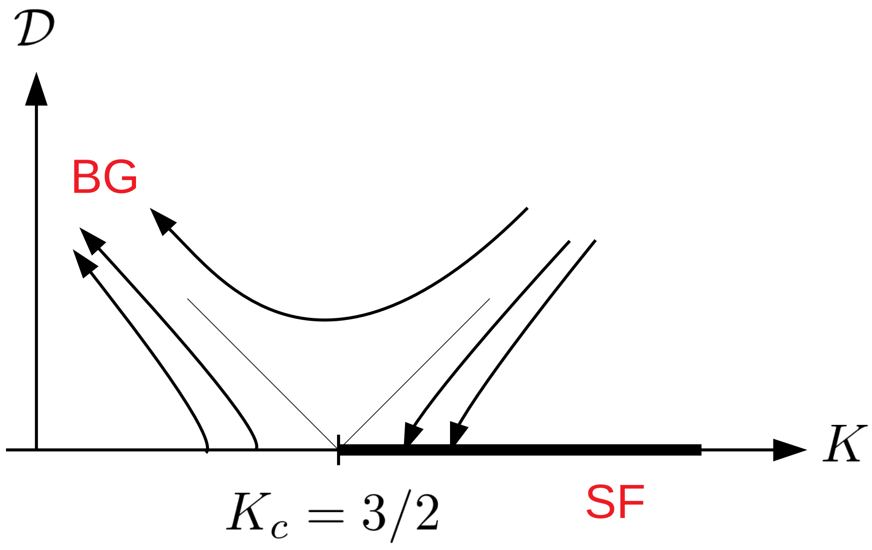


- *5-ε* expansion: *Balents 1993, etc.*
- Gaussian Variational Method with spontaneous replica symmetry breaking: Giamarchi and Le Doussal 1996

Perturbative RG [Giamarchi, Schulz 1988, Ristivojevic *et al.* 2012]

- phase diagram

$$\frac{dK}{dl} = -K^2 \frac{\mathcal{D}}{\pi v^2}$$
$$\frac{d\mathcal{D}}{dl} = (3 - 2K)\mathcal{D}$$
(BKT flow)



- Bose-glass phase [Fisher *et al.* 1989]

compressibility: $d\kappa/dl = 0, \quad \kappa > 0$

localized phase: $\xi_{\text{loc}} \sim \mathcal{D}^{-\frac{1}{3-2K}}$

high-frequency conductivity: $\sigma(\omega \gg v/\xi_{\text{loc}}) \sim \omega^{2K-4}$

FRG approach to disordered (classical) systems

- long history… Fisher 1985, Narayan, Balents, Nattermann, Chauve, Le Doussal, Wiese, etc.

  Metastable states: pinning, “shocks” and “avalanches”, chaotic behavior, aging, etc.

- non-perturbative (Wetterich’s) formulation: Tissier & Tarjus 2004- (RFIM)

- truncation of the replicated effective action

$$\Gamma_k[\phi] = \sum_a \Gamma_{1,k}[\phi_a] - \frac{1}{2}\sum_{a,b}\Gamma_{2,k}[\phi_a,\phi_b] + \cdots \quad \text{(free replica sum expansion)}$$

$$\Gamma_{1,k}[\phi_a] = \int dx \int_0^\beta d\tau \left\{ \frac{Z_x}{2}(\partial_x\phi_a)^2 + \phi_a \Delta_k(-\partial_\tau)\phi_a \right\},$$

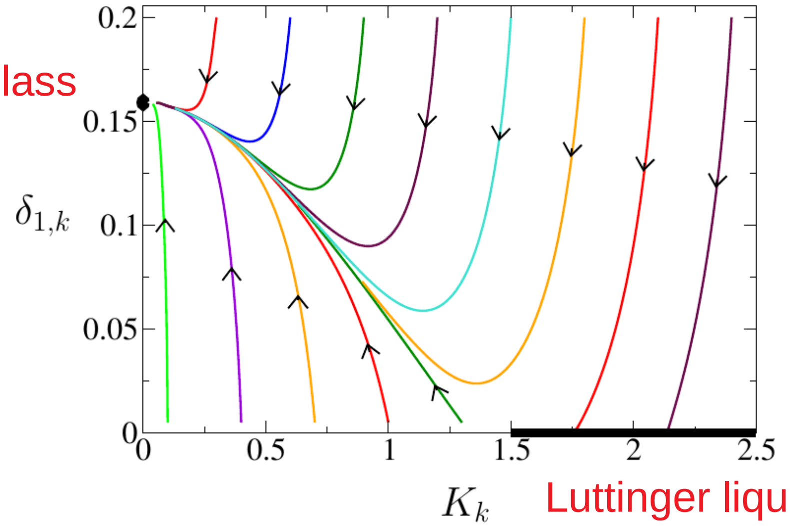
$$\Gamma_{2,k}[\phi_a,\phi_b] = \int dx \int_0^\beta d\tau\, d\tau'\, V_k(\phi_a(x,\tau) - \phi_b(x,\tau'))$$

$$\text{with initial conditions: } Z_x = \frac{v}{\pi K}, \quad \Delta_\Lambda(i\omega) = Z_x\omega^2/v^2, \quad V_\Lambda(u) = 2\mathcal{D}\cos(2u)$$

$$\text{velocity: } \Delta_k(i\omega) = Z_x\omega^2/{v_k}^2 + \mathcal{O}(\omega^4)$$

$$\text{Luttinger parameter: } K_k = v_k/\pi Z_x$$

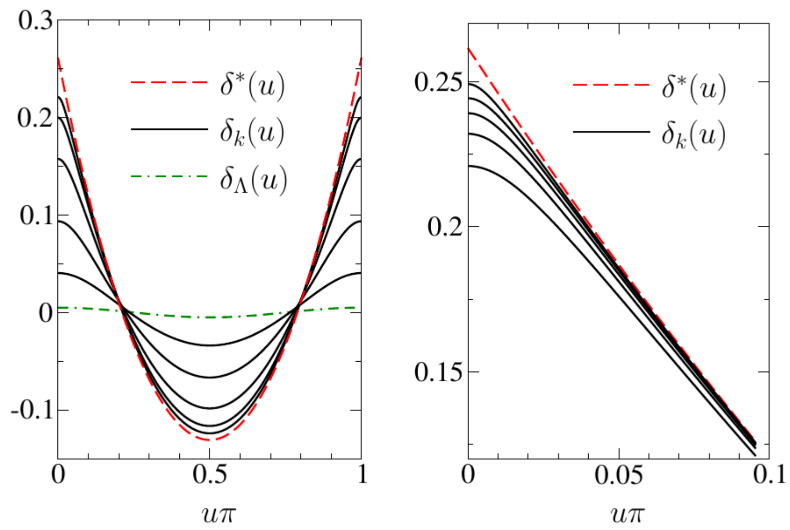
- phase diagram



- Bose-glass fixed point:

$$K^* = 0, \quad K_k \sim k^\theta \quad \text{with} \quad \theta = 1/2 \quad \text{(no quantum fluctuations, hence pinning)}$$

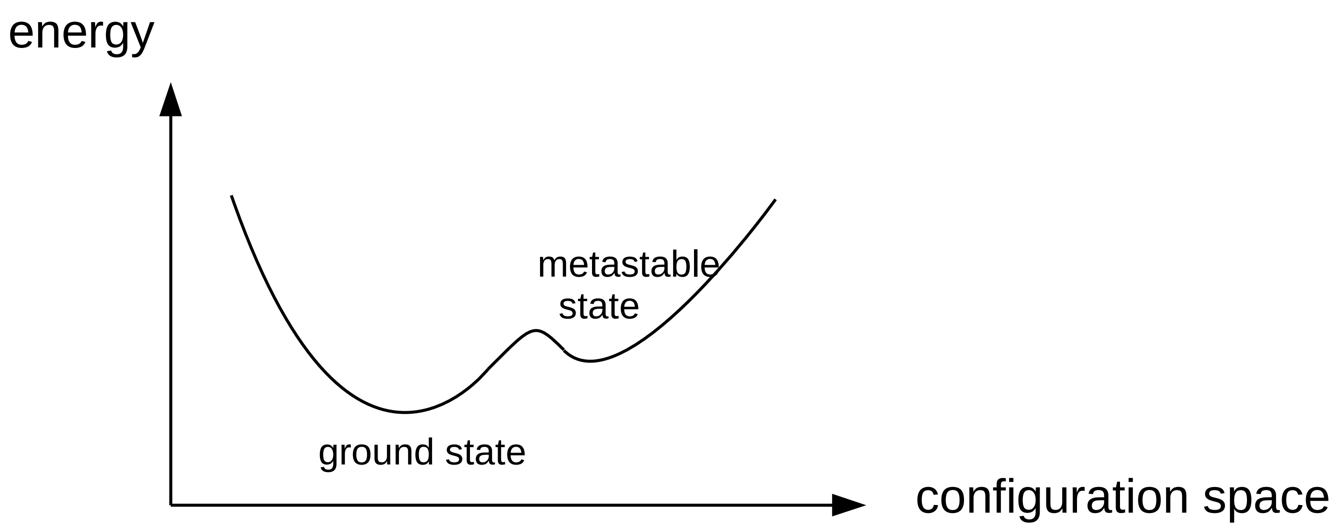
$$\delta^*(u) = -\frac{K^2}{v^2} \lim_{k\to 0} \frac{V_k''(u)}{k^3} = \frac{1}{2a_2}\left[\left(u - \frac{\pi}{2}\right)^2 - \frac{\pi^2}{12}\right] \quad \text{for} \quad u \in [0,\pi]$$



cusp
and quantum boundary layer
(controlled by $K_k \sim k^\theta$)

Physics of the cusp and the boundary layer: metastable states

[Balents *et al.* 1996, Le Doussal, etc.]

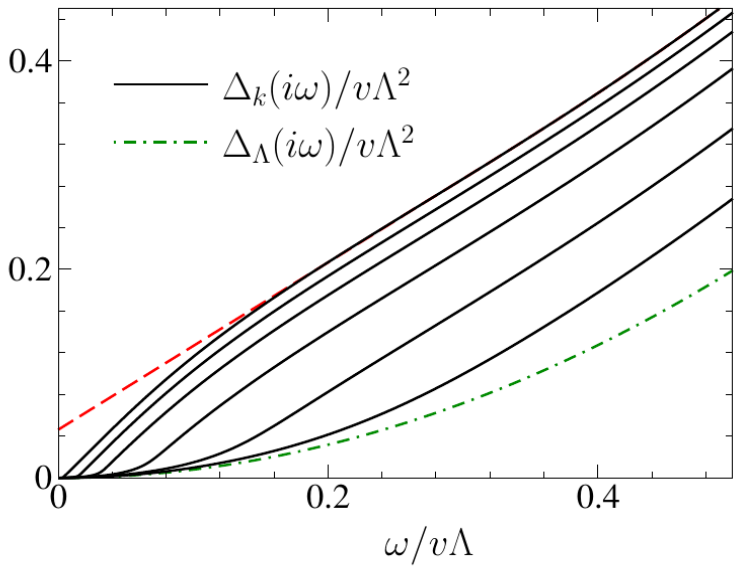


- cusp: the ground state varies discontinuously, as a function of an external "force", whenever it becomes degenerate with a metastable state: "shocks" or "avalanches".
- quantum boundary layer: quantum fluctuations (K>0) lead to quantum tunneling between nearly degenerate states and a rounding of the cusp in a boundary layer.

→(rare) superfluid regions with significant density fluctuations and reduced phase fluctuations (Griffiths phase) [Fisher et al. 1989, Pollet et al. 2009, etc.]

Conductivity

- self-energy



$$\partial_k \Delta_k(i\omega_n) = \delta_k''(0)(\cdots) \sim k^{-\theta}(\cdots)$$

$$\lim_{k\to 0} \Delta_k(i\omega_n) = \begin{cases} 0 & \text{if} \quad \omega_n = 0 \text{ (statistical tilt symmetry)} \\ A + B|\omega_n| & \text{if} \quad \omega_n \neq 0 \end{cases}$$

- localization/pinning length: $\xi_{\mathrm{loc}} \sim A^{-1/2}$

- conductivity $\sigma(\omega) = \left. \frac{\omega_n}{\pi^2 \Delta(i\omega_n)} \right|_{i\omega_n \to \omega + i0^+} \simeq \frac{1}{\pi^2 A^2}(B\omega^2 - Ai\omega)$

Conclusion

- FRG is a very powerful method to study the sine-Gordon model and its (non-integrable) generalizations.

- In the sine-Gordon model, FRG allows us to compute the masses of the (anti)soliton and breather in the massive phase, in very good agreement with exact results, and to confirm the Lukyanov-Zamolodchikov conjecture.

- For 1D disordered bosons, FRG gives a fairly complete picture of the Bose-glass phase and reveals (some of) its glassy properties: pinning, “shocks” and “avalanches”.