Remarks on transverse cooling theory

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Electromagnetism: General Hamiltonian (curved coordinates)

$$H(x, p_x, y, p_y, s, p_s; t) = c\left[m^2c^2 + \frac{(p_s - QeA_s)^2}{(1+x/\rho)^2} + (p_x - QeA_x)^2 + (p_y - QeA_y)^2\right]^{1/2}$$

- Use s as independent variable
- Linear optics assuming $p_x \ll p_s, p_y \ll p_s$
- Define kinetic momentum $p(E) = \frac{E^2}{c^2} - m^2c^2$
- New Hamiltonian (the mother of them all)

$$H(x, p_x, y, p_y, t, -E; s) = -QeA_s + \left(1 + \frac{x}{\rho}\right)c\left[p(E) + \frac{(p_x - QeA_x)^2}{2p(E)} + \frac{(p_y - QeA_y)^2}{2p(E)}\right]$$

New Variables

1^{st} canonical transformation: Introduce dispersion function
2^{nd} canocical transformation: Introduce beta functions

$$H(\psi_x, J_x, \psi_y, J_y, \theta, -J_s; s) = -\frac{J_s}{R} - \frac{J_x}{\beta_x} - \frac{J_y}{\beta_y} - \frac{Qe}{p_s}\Delta A_s(\psi_x, J_x, \psi_y, J_y, \theta, -J_s)$$

$$\frac{\delta p}{p_s} = \frac{2\pi J_s}{C_s} - 1$$

$$\theta = \frac{s - s_0}{R} + \theta_0$$

$$x = D(s)\frac{\delta p}{p_s}(J_s) + \sqrt{2 J_x \beta_x}\sin\psi_x$$

$$y = \sqrt{2 J_y \beta_y}\sin\psi_y$$

$$t = \frac{1}{v}(\theta R + \delta C) - t_\beta$$

Particle Arrival Time

$$t = \frac{1}{v}\left(\theta R + \delta C\right) - t_\beta$$

$$\delta C = \frac{\delta p}{p_s}\int_{s_0}^{s} D(s)\rho(s)$$

$$t_\beta = \frac{\sqrt{2J_x}}{v}\left[D\frac{(s)}{\sqrt{\beta_x}}\left(\cos\psi_x - \frac{\beta'_x}{2}\sin\psi_x\right) - D'\sqrt{\beta_x}\sin\psi_x\right]$$

Example:

CR Palmer pu (D=6m, β_x=10.5 m, v=0.83 c):

$$\frac{1}{v}\frac{\sqrt{2J_x}}{\sqrt{\beta_x}}D(s) = 100 \text{ ps}$$

second amplitude proportional to sin ψ_x (α_x = -1.22, D' = 0.26):

-129 ps – 48 ps = -177 ps

absolute all-over amplitude: 207 ps

Arrival oscillations at Palmer pu (freq. modulation)

take t_β = 100 ps

$2 J_x = 200 * 10^{-6}$

Reduction of signal at revolution harmonics: $Z^{1/2} \rightarrow Z^{1/2} J_0(\Omega t_\beta)$

Contamination of signal at sidebands: $Z^{1/2} J_1(\Omega t_\beta)$

f [GHz]	1	1.5	2
$J_0(\Omega\, t_\beta)$	0.62	0.25	-0.09
$J_1(\Omega\, t_\beta)$	0.52	0.58	0.47

Hamiltonian Dynamics: Longitudinal

$$H(\psi_x, J_x, \psi_y, J_y, \theta, -J_s; s) = -\frac{J_s}{R} - \frac{J_x}{\beta_x} - \frac{J_y}{\beta_y} - \frac{Qe}{p_s} \Delta A_s(\psi_x, J_x, \psi_y, J_y, \theta, -J_s)$$

$$\frac{dJ_s}{ds} = Qe \frac{\partial(\Delta A_s)}{\partial \theta} = Qe \frac{\partial(\Delta A_s)}{\partial t} \frac{\partial t}{\partial \theta}$$

$$-\frac{\partial(\Delta A_s)}{\partial t} = E_s$$

$$\frac{\partial t}{\partial \theta} = \frac{R}{v}$$

Hamiltonian Dynamics: Vertical

$$H(\psi_x, J_x, \psi_y, J_y, \theta, -J_s; s) = -\frac{J_s}{R} - \frac{J_x}{\beta_x} - \frac{J_y}{\beta_y} - \frac{Qe}{p_s}\Delta A_s(\psi_x, J_x, \psi_y, J_y, \theta, -J_s)$$

$$\frac{d J_y}{d s} = Qe \frac{\partial(\Delta A_s)}{\partial \psi_y} = \frac{Qe}{p_s} \frac{\partial(\Delta A_s)}{\partial y} \frac{\partial y}{\partial \psi_y}$$

$$\frac{\partial y}{\partial \psi_y} = \sqrt{2 J_y \beta_y} \cos \psi_y$$

Panofsky-Wenzel (Fourier transform):

$$\frac{\partial(\Delta A_s)}{\partial y} \propto \frac{i}{\Omega} \frac{\partial E_s}{\partial y}$$

Hamiltonian Dynamics: Horizontal

$$H(\psi_x, J_x, \psi_y, J_y, \theta, -J_s; s) = -\frac{J_s}{R} - \frac{J_x}{\beta_x} - \frac{J_y}{\beta_y} - \frac{Qe}{p_s}\Delta A_s(\psi_x, J_x, \psi_y, J_y, \theta, -J_s)$$

$$\frac{dJ_x}{ds} = Qe\frac{\partial(\Delta A_s)}{\partial\psi_x} = \frac{Qe}{p_s}\left[\frac{\partial(\Delta A_s)}{\partial x}\frac{\partial x}{\partial\psi_x} + \frac{\partial(\Delta A_s)}{\partial t}\frac{\partial t}{\partial\psi_x}\right]$$

$$\frac{\partial t}{\partial\psi_x} = \frac{\partial t_\beta}{\partial\psi_x} = \frac{\sqrt{2J_x}}{v}\left[\frac{D(s)}{\sqrt{\beta_x}}\left(-\sin\psi_x - \frac{\beta'_x}{2}\cos\psi_x\right) - D'\sqrt{\beta_x}\cos\psi_x\right]$$

In addition to the Panofsky-Wenzel effect, there is a change of emittance due to longitudinal kicks at non-zero dispersion. In the framework of Hamiltonian dynamics, this effect is connected to the oscillation of arrival times due to betatron oscillations.

Fokker Planck Equation

a continuity equation in action space

$$\frac{\partial\Psi\left(J_x, J_y, \delta p/p; t\right)}{\partial t} + \nabla \bullet \Phi\left(J_x, J_y, \delta p/p; t\right) = 0$$

flux in action space $\Phi_m = -F_m \Psi + \frac{1}{2} D_{mn} \frac{\partial \Psi}{J_n}$

drift (coherent cooling effect) $F_m = \frac{1}{\tau}\left\langle \Delta J_m \right\rangle$

diffusion (incoherent cooling effect) $D_{mn} = \frac{1}{\tau}\left\langle\left(\Delta J_m \Delta J_n\right)\right\rangle$

Assumptions and Simplifications

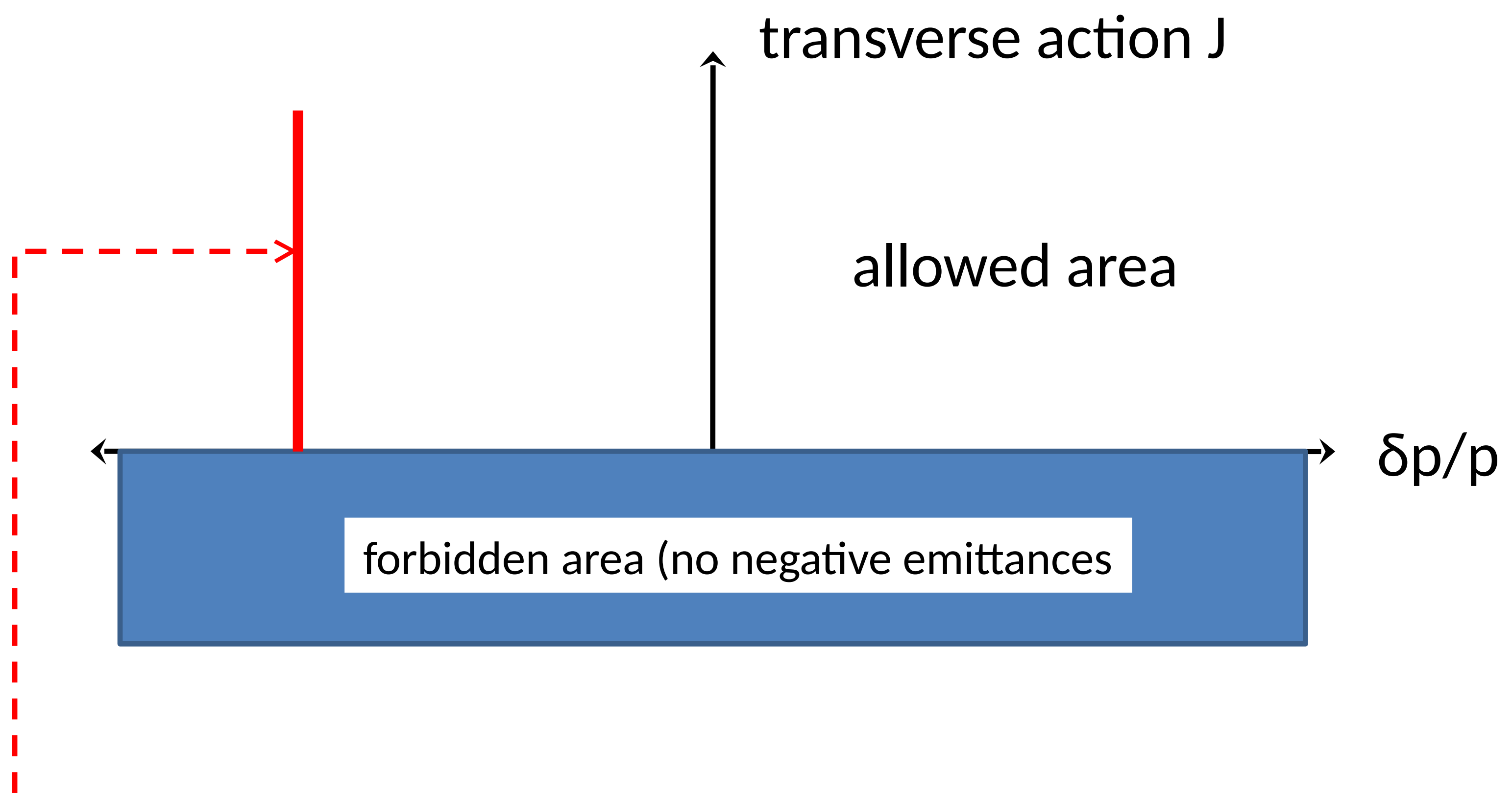
- Zero dispersion at pick-ups and kickers
- Linear response for pick-ups and kickers
- No Schottky overlap
- No Chromaticity

pick-up response: $V_p(\Omega) = \frac{Qe\omega}{2} S_p(x, y, \Omega) j_p(x, y, \Omega)$

accelerating voltage: $U_k(\Omega) = S_k(x, y, \Omega) V_k(\Omega)$

transverse sensitivity: $S(x, y, \Omega) = x S'(0,0,\Omega)$

2D Phase space



Assume that all particles in the 1D distribution function $\Psi(J)$ of transverse actions are at a given δp/p

Pick-up Signal for Transverse Cooling

$$V_p(\Omega) \propto \frac{QeZ_l\sqrt{2J_x\beta_x}\;\partial S_p / \partial x}{4i}$$

at the frequencies $\omega_{m,\pm} = (m \pm Q_x)\,\omega_{rev}$

in the following: $\dfrac{\partial S_{pk}}{\partial x} := S'_{pk}$

Voltage Power Densities (1)

The voltage spectral power density *C(Ω)* is by definition the Fourier transform of the voltage autocorrelation $R(t) = \langle V(\tau) V(t+\tau) \rangle$.

In a coasting beam it does not depend on *τ*,

i.e. *V(t)* is a *stationary process*.

The dimension of *C(Ω)* is V^2s.

Voltage Power Densities (2)

Voltage power density due to Schottky noise:

$$C_p(\Omega) = \frac{(QeZ_l)^2 \omega \beta_x \left| S_p^{'} \right|^2}{16\pi |m\eta|} \langle J \rangle \psi(\delta p / p)$$

average emittance $$\langle J \rangle = \frac{\int_0^\infty J_x \Psi_x \, dJ}{\int_0^\infty \Psi_x \, dJ}$$

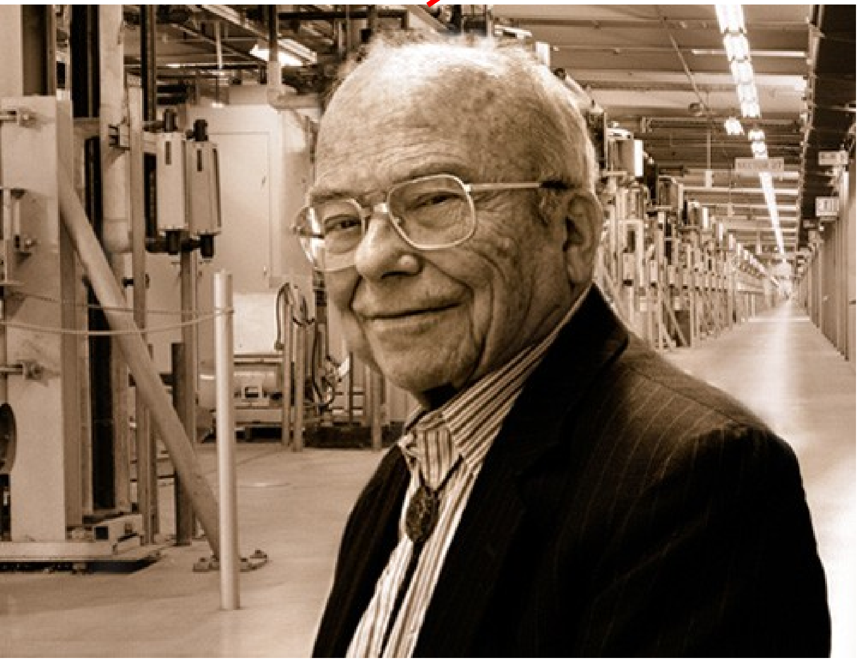
Voltage Power Densities (3)

Voltage power density due to thermal noise:

$$C_n(\Omega) = \frac{1}{2} Z_l k_B T_{eff}$$

Transverse System Gain

$$g_{\perp} = \frac{(Qe)^2 \omega Z_l}{8\pi\, p\,\Omega} \sqrt{\beta_p \beta_k} S_k^{'} g_k G g_p S_p^{'}$$



frequency dependent!

Cooling Coefficient

$$C=\frac{\omega}{2\pi}\sum_{m,\pm} \pm g_{\perp} \exp\left[i\left(\left(\mu_k-\mu_p\right)-\left(m\pm Q\right)\eta_{pk}\,\omega\, T_{pk}\frac{\delta p}{p}\right)\right]$$

betatron phase advance

undesired mixing

Schottky Noise Coefficient

Schottky power density depends on longitudinal distribution

$$S = \frac{\omega \psi(\delta p / p)}{2\pi} \sum_{m,\pm} \left| \frac{g_{\perp}^{2}}{m\eta} \right|$$

Schottky power density decreases with harmonics

Thermal Heating Coefficient

$$H = \frac{4k_B T_{eff}}{(Qe)^2 Z_l \beta_p \left| g_p \right|^2} \sum_{m,\pm} \left| \frac{g_\perp^2}{S_p^{'}} \right|$$

Fokker Planck flux

$$\Phi = -F\Psi + \frac{1}{2} D \frac{\partial \Psi}{J}$$

$$F = -C J$$

$$D = \left(S \langle J \rangle + H\right) J$$

The terms C (cooling), S (Schottky noise), and H (thermal noise) do not depend on J.

Therefore the flux is proportional to J.

No flux towards J<0!

Equilibrium Distribution

condition: $\frac{\partial \Psi}{\partial t} = 0$

The equilibrium distribution turns out to be an exponential:

$$\Psi(J) = \Psi_0 \exp\left(-\frac{2C - S}{H} J\right)$$

with average emittance $\langle J \rangle_\infty = \frac{H}{2C - S}$

Time Dependent Solution

$$\Psi(J,t)=\alpha(t)N\exp\left[-\alpha(t)J\right]$$

It turns out that this ansatz is a time-dependent solution to the transverse Fokker-Planck equation if

$$\frac{\dot{\alpha}}{\alpha}+\left[-C+\frac{S+\alpha H}{2}\right]=0$$

If C, S, and H are constant in time, then

$$\langle J\rangle(t)=\left[\langle J\rangle_0-\langle J\rangle_\infty\right]\exp\left(-\frac{t}{\tau}\right)+\langle J\rangle_\infty$$

with $\tau=C-\frac{S}{2}$ not the instantaneous cooling rate!