How to extract the dominant "Abelian" part of the Wilson loop in higher representations

Ryutaro Matsudo

Graduate School of Science and Engineering, Chiba University

March 27, 2019

Probing the Phase Structure of Strongly Interacting Matter: Theory and Experiment at GSI, Darmstadt

In collaboration with: A. Shibata, S. Kato, and K.-I. Kondo

Magnetic monopoles defined by Abelian projection

Magnetic monopole defined by Abelian projection is one of the candidates of d.o.f which are responsible for non-perturbative phenomena in QCD such as quark confinement and chiral symmetry breaking.

In this talk, I will

1. give a review of Abelian projection, and
2. introduce our study related to the color sources in the higher representations.

Part I

Review of Abelian projection

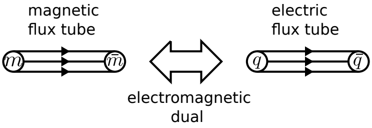
Outline

1. Dual superconductivity picuture
2. Magnetic monopole defined by Abelian projection
3. Numerical evidences of dual superconductivity picutre

Outline

Dual superconductivity picture

- The dual superconductivity picture is a promising scenario for quark confinement proposed by Nambu, 't Hooft and Mandelstam, in which magnetic monopoles play an important role.
- In this scenario, the QCD vacuum is considered as a dual superconductor.
- QCD strings, which are electric flux tubes formed between color sources, are considered as the electromagnetic dual of Abrikosov vortices in a usual superconductor.



Ordinary superconductivity is the result of condensation of Cooper pairs. Therefore we can suppose that, in order to be a dual superconductor, condensation of magnetic monopoles have to occur. Therefore in this scenario, condensation of magnetic monopoles should be a (dis)oerder parameter of the deconfinement transition.

However, the definition of magnetic monopoles in non-Abelian gauge theories is non trivial. ← The Abelian projection gives the definition of monopoles.

Outline

Nontriviality of definition of magnetic monopoles

In non-Abelian gauge theories, the definition of magnetic monopoles is not trivial.

- The naive definition of the conserved magnetic current k^μ and the corresponding conserved charge,

$$k^\mu := \frac{1}{2}\varepsilon^{\nu\mu\rho\sigma}\partial_\nu F_{\rho\sigma},$$
$$Q_M := \int d^3x k^0 = \frac{1}{2}\int_{S^2_\infty} dx^i \wedge dx^j F_{ij},$$

  are not gauge covariant.
- In the case of 't Hooft-Polyakov monopoles, magnetic charge is defined as a topological charge, however the scalar field in the adjoint representation is needed for this definition.

Abelian projection

The Abelian projection, which is proposed by 't Hooft (1981), is a way to define magnetic monopoles in non-Abelian gauge theories. This method consists of three steps.

1. Define a scalar field $\Phi(x)$ in the adjoint representation as a functional of $A_\mu(x)$.
2. Fix the gauge so that $\Phi(x)$ is diagonal everywhere. Note that this is a partial gauge fixing because $U(1)^{N_c-1}$ symmetry remains.
3. By using the diagonal part a_μ of the gauge field, the magnetic current k_μ is defined in the usual way as

$$k_\mu := \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^\nu f^{\rho\sigma},$$

$$f_{\mu\nu} := \partial_\mu a_\nu - \partial_\nu a_\mu.$$

Maximal Abelian projection

Frequently used gauge choice for Abelian projection is the maximal Abelian (MA) gauge, where the functional

$$\int d^4x \sum_a (A^a_\mu(x))^2 \qquad A^a_\mu\text{: an off-diagonal component of gauge field}$$

is minimized. (The space time is Euclidean.)
The corresponding $\Phi(x)$ is defined as

$$\Phi(x) := \underset{\tilde{\Phi}(x)}{\arg\min} \int d^4x (D_\mu \tilde{\Phi}(x) D_\mu \tilde{\Phi}(x)). \qquad D_\mu\text{: the covariant derivative}$$

Outline

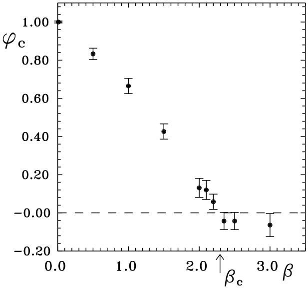
1. Dual superconductivity picuture
2. Magnetic monopole defined by Abelian projection
3. Numerical evidences of dual superconductivity picutre

Condensation of monopoles

The monopole condensate is determined in several ways. For example, in Chernodub et al. (1996), the probability distribution of the monopole creation operator was calculated in $SU(2)$ pure Yang-Mills theory. Then the value of the monopole condensate was determined as the argument of the maximum of the probability distribution.

The transition point is almost the same as that determined by using Polyakov loop average.

Source: Polikarpov (1996)



The calculation was performed on 4×8^3, 4×10^3, 4×12^3, 4×14^3 and 4×16^3 lattices, and data are extrapolated to the infinite volume.

Monopole dominance

We can extract the contribution of monopoles to the averages of several physical quantities such as

- the string tension between heavy quarks (in SU(2) Suzuki, Yotsuyanagi, 1990, in SU(3) Stack et al., 2002),
- the Polyakov line (Suzuki et al., 1995),
- the value of the chiral condensate (Miyamura, 1995).

Then it was confirmed numerically that the contribution of monopoles almost reproduce the correct expectation value of the quantity.

Monopole dominance for the string tension

The confirmation of monopole dominance for the string tension consists of two steps.

1. Firstly the average of the Abelian Wilson loop which is defined as the Wilson loop for a_μ,

$$W_{\text{Abel}} := \operatorname{tr}\exp\left(\oint_C dx^\mu a_\mu\right),$$

   was calculated and it was confirmed that this reproduces the IR behavior of the full Wilson loop, i.e., the full string tension in SU(2) (Suzuki-Yotsuyanagi, 1990) and in SU(3) (Stack-Tucker-Wensley, 2002) in the MA gauge, which is called Abelian dominance.

2. Then the monopole contribution to the Abelian Wilson loop was extracted by applying the Tousaint-DeGrand procedure and it was confirmed that this reproduces the full string tension in SU(2) (Suzuki-Yotsuyanagi, 1990) and in SU(3) (Stack-Tucker-Wensley, 2002) in the MA gauge.

Part II

Abelian dominance for color sources in higher representations

Outline

Outline

Problem of Abelian projection

- For color sources in the fundamental representation, the string tension between color sources is almost fully reproduced by the contribution of magnetic monopoles which is extracted by Abelian projection procedure. This was confirmed in lattice studies.
- For color sources in higher representations, if we adapt the same procedure as fundamental representation naively, the monopole contribution doesn't reproduce the full string tension. For example in the adjoint representation of $SU(2)$, the monopole part of the string tension seems to be zero even in the intermediate region, c.f. Del Debbio et al. (1996).

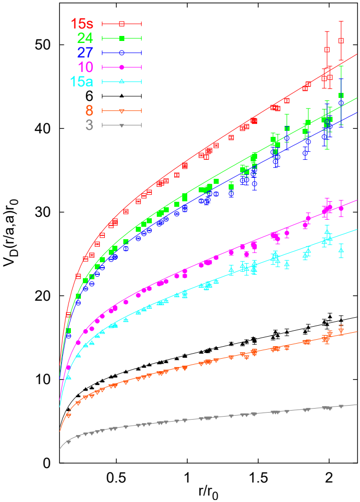
Wilson loops in higher representations

We can use Wilson loops in higher representations to test candidates of confinement mechanism by checking whether they reproduce the following behavior.

The potential between color sources in a higher representation has two characteristic features depending on the distance.

- At intermediate distance, the string tension is proportional to the quadratic Casimir.
- At asymptotic region, due to the screening by gluons, the string tension depends only on the N-ality of the representation.

In $SU(3)$ Source: Bali (2000)



Naively extended Abelian projection in higher reps.

Naively extended Abelian projection does not reproduce the correct behavior of Wilson loops in higher representation.

For example, in the adjoint rep. in $SU(2)$ gauge theory, the average of the naive Abelian projected Wilson loop in the adjoint rep.,

$$W_{\mathrm{adj}}^{\mathrm{Abel}} = \frac{1}{3}\left(\exp\left(ig\oint A^3\right) + \exp\left(-ig\oint A^3\right) + 1\right),$$

approaches $1/3$ other than 0.

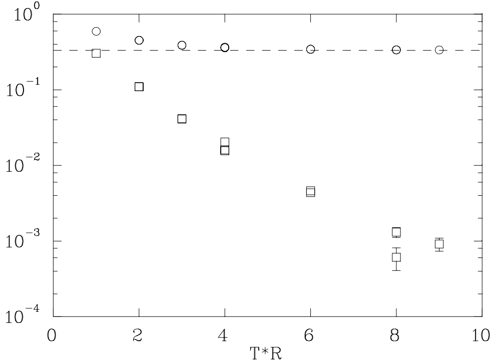


FIG. 7. The adjoint Wilson loop $W_{j=1}$ (2) versus the adjoint diagonal Wilson loop $W^d_{j=1}$ (o) in MA projection. The dashed line corresponds to the asymptotic value for the latter, $W^d_{j=1} = 1/3$.

Source: Poulis (1996)

Outline

The suitable operator to check "the Abelian dominance"

In an arbitrary representation of an arbitrary group, we claim that we should use the highest weight part of the Abelian Wilson loop which is defined by

$$\tilde{W}_R = \exp\left(ig \oint \langle\Lambda| A_\mu |\Lambda\rangle\right), \quad |\Lambda\rangle : \text{ the highest weight state of } R$$

instead of the naively defined Abelian projected Wilson loop in the representation R

$$\begin{aligned} W_R^{\mathrm{Abel}} &= \frac{1}{D_R} \mathrm{tr}_R \exp\left(ig \oint 2\,\mathrm{tr}\left(H_j A_\mu\right) H_j\right) \\ &= \frac{1}{D_R} \sum_\mu \exp\left(ig \oint \langle\mu| A_\mu |\mu\rangle\right), \end{aligned}$$

where the sum is over the whole weights of R and

H_j: the Cartan generatorsD_R: the dimension of R

Example: fund. rep. and adj. rep. in $SU(2)$

In fund. rep. of $SU(2)$,

$$\tilde{W}_F = e^{\frac{1}{2}ig\oint A^3}$$
$$W_F^{\text{Abel}} = \frac{1}{2}\left(e^{\frac{1}{2}ig\oint A^3} + e^{-\frac{1}{2}ig\oint A^3}\right)$$

In adj. rep. of $SU(2)$,

$$\tilde{W}_A = e^{ig\oint A^3},$$
$$W_A^{\text{Abel}} = \frac{1}{3}\left(e^{ig\oint A^3} + e^{-ig\oint A^3} + 1\right)$$

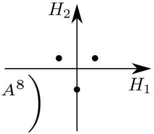
(c.f. Poulis(1996))

Example: fund. rep., adj. rep. and the sextet rep. in $SU(3)$

In fund. rep. of $SU(3)$,

$$\tilde{W}_F = e^{ig\oint\left(\frac{1}{2}A^1+\frac{1}{2\sqrt{3}}A^8\right)}, \qquad (\Lambda = (1/2, 1/2\sqrt{3}))$$

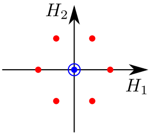
$$W_F^{\rm Abel} = \frac{1}{3}\left(e^{ig\oint\left(\frac{1}{2}A^1+\frac{1}{2\sqrt{3}}A^8\right)} + e^{ig\oint\left(\frac{1}{2}A^1-\frac{1}{2\sqrt{3}}A^8\right)} + e^{ig\oint\frac{1}{\sqrt{3}}A^8}\right)$$



In adj. rep. of $SU(3)$,

$$\tilde{W}_A = e^{ig\oint A^3}, \qquad (\Lambda = (1, 0))$$

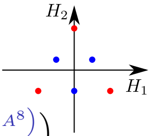
$$\begin{aligned} W_A^{\rm Abel} = & \frac{1}{8}\Big(e^{ig\oint A^3} + e^{-ig\oint A^3} + e^{ig\oint\left(\frac{1}{2}A^3+\frac{\sqrt{3}}{2}A^8\right)} \\ & + e^{-ig\oint\left(\frac{1}{2}A^3+\frac{\sqrt{3}}{2}A^8\right)} + e^{ig\oint\left(\frac{1}{2}A^3-\frac{\sqrt{3}}{2}A^8\right)} \\ & + e^{-ig\oint\left(\frac{1}{2}A^3-\frac{\sqrt{3}}{2}A^8\right)} + 2\Big) \end{aligned}$$



In the sextet rep. of $SU(3)$

$$\tilde{W}_{\mathbf{6}} = e^{i\oint\frac{2}{\sqrt{3}}A^8}, \qquad (\Lambda = (0, 2/\sqrt{3}))$$

$$\begin{aligned} W_{\mathbf{6}}^{\rm Abel} = & \frac{1}{6}\Big(e^{ig\oint\frac{2}{\sqrt{3}}A^8} + e^{ig\oint\left(A^3-\frac{1}{\sqrt{3}}A^8\right)} + e^{ig\oint\left(-A^3-\frac{1}{\sqrt{3}}A^8\right)} \\ & + e^{-ig\oint\frac{1}{\sqrt{3}}A^8} + e^{ig\oint\left(\frac{1}{2}A^3+\frac{1}{2\sqrt{3}}A^8\right)} + e^{ig\oint\left(-\frac{1}{2}A^3+\frac{1}{2\sqrt{3}}A^8\right)}\Big) \end{aligned}$$



Where does this prescription come from?

According to Diakonov-Petrov version of the non-Abelian Stokes theorem, the Wilson loop for the representation R can be written as

Non-Abelian Stokes theorem (Diakonov, Petrov(1989))

$$W_R[A] = \int DU \exp\left(\oint ig \langle\Lambda| A^U |\Lambda\rangle \right)$$
$$= \int DU \exp\left(\int_S ig\, d\left(\langle\Lambda| A^U |\Lambda\rangle \right) \right),$$

where

DU is the product of the Haar measure over the loop or a surface

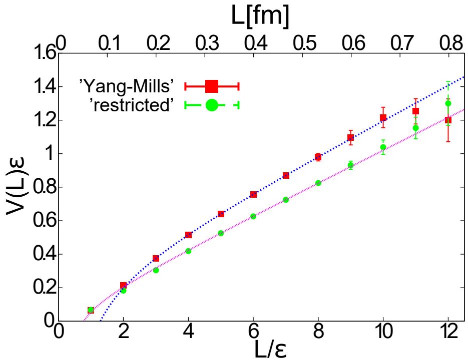
$A^{U^\dagger} := UAU^\dagger + ig^{-1}UdU^\dagger$, and

$|\Lambda\rangle$ is the highest weight state of the representation R.

Outline

$SU(2)$ adj. rep.
24^4lattice
$\beta = 2.5$
HYP smearing

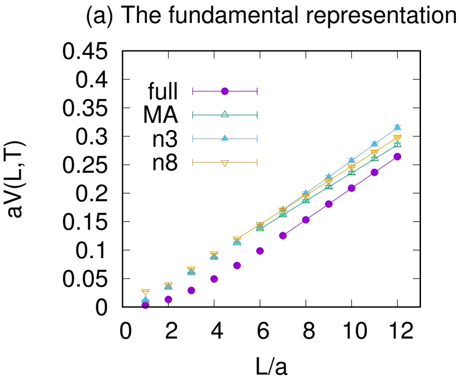
$$V(R,T) = \log \frac{\langle W(R,T)\rangle}{\langle W(R,T+1)\rangle}$$



cf. Poulis (1996), Chernodub-Hashimoto-Suzuki (2004)

$SU(3)$ fund. rep. $(\mathbf{3},\ [1,0])$

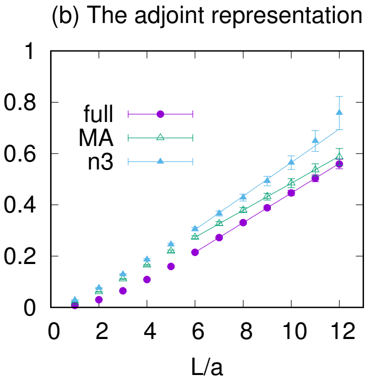
24^4lattice
$\beta = 6.2$
APE smearing
fit range: $3 \le R/a \le 11$



$a^2\sigma^F_{\rm full} \simeq 0.0278$
$a^2\sigma^F_{\rm MA} \simeq 0.0246 \simeq 0.89 a^2\sigma_{\rm full}$
$a^2\sigma^F_{\rm GA1} \simeq 0.0288 \simeq 1.04 a^2\sigma_{\rm full}$
$a^2\sigma^F_{\rm GA2} \simeq 0.0254 \simeq 0.91 a^2\sigma_{\rm full}$
c.f.
Sakumichi-Suganuma (2014)
Perfect Abelian dominance in MA gauge
32^4lattice $\beta = 6.4$

$SU(3)$ adj. rep. $(\mathbf{8}, [1,1])$

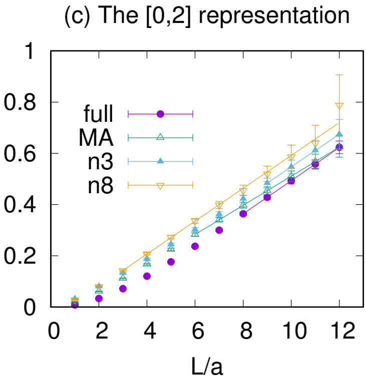
24^4lattice
$\beta = 6.2$
APE smearing
fit range: $3 \leq R/a \leq 8$



$$a^2\sigma^A_{\text{full}} \simeq 0.0576$$
$$a^2\sigma^A_{\text{MA}} \simeq 0.0522 \simeq 0.91 a^2\sigma^A_{\text{full}}$$
$$a^2\sigma^A_{\text{GA2}} \simeq 0.062 \simeq 1.08 a^2\sigma^A_{\text{full}}$$

$SU(3)$ $\mathbf{6}^*$ $([0,2])$

24^4lattice
$\beta = 6.2$
APE smearing
fit range: $3 \leq R/a \leq 8$



$a^2\sigma^{02}_{\text{full}} \simeq 0.0647$
$a^2\sigma^{02}_{\text{MA}} \simeq 0.0569 \simeq 0.91 a^2\sigma_{\text{full}}$
$a^2\sigma^{02}_{\text{GA1}} \simeq 0.0635 \simeq 0.98 a^2\sigma_{\text{full}}$
$a^2\sigma^{02}_{\text{GA2}} \simeq 0.0641 \simeq 0.99 a^2\sigma_{\text{full}}$

Summary

- The naive Abelian projected Wilson loop for a higher representation does not reproduce the correct behavior of the original Wilson loop.
- Through the NAST, we obtain another projected Wilson loop, which is essentially same as the Abelian projected Wilson loop in the fundamental representation, and is different from that in in higher representations.
- According to the lattice simulation, the proposed operator reproduce the correct behavior in the adjoint representation in the $SU(2)$ gauge theory and in the adjoint representation and $\mathbf{6}^*$ in $SU(3)$ gauge theory.

Buckups

The meaning of our prescription

In $SU(2)$, the Wilson loop in the representation j can be written using the untraced Wilson loop W in the fundamental representation. Thus the average is determined if the probability distribution $P(W;C)$ of the untraced Wilson loop is determined as

$$\langle \mathrm{tr}_j W_C \rangle = \int dW P(W;C) \, \mathrm{tr}_j W,$$

$$P(W;C) := \int DU \delta(W, \prod_{l \in C} U_l) e^{-S[U]} / \int DU e^{-S[U]}.$$

In the same way, the (naive) Abelian projected Wilson loop w in the representation j can be expressed as

$$\langle \mathrm{tr}_j V_C \rangle = \int dw \tilde{P}(w;C) \, \mathrm{tr}_j w,$$

$$\tilde{P}(w;C) := \int DU \delta(w, \prod_{l \in C} u_l) e^{-S[U]} / \int DU e^{-S[U]},$$

where u_l is an Abelian projected link variable.
Note that dW is the Haar measure on $SU(2)$ while dw is the Haar measure on $U(1)$.

Because of the gauge invariance of $P(W;C)$, we can express $P(W;C)$ by using the eigenvalue $e^{i\theta}$ of W only as

$$P(W;C) = P(\theta;C),$$

while because we can write $u = \mathrm{diag}(\exp(i\theta), \exp(-i\theta))$, we can express $\tilde{P}(u;C)$ as

$$\tilde{P}(u;C) = \tilde{P}(\theta;C)$$

Then we can write

$$\langle \mathrm{tr}_j\, W_C \rangle = \frac{2}{\pi} \int_0^{\pi} d\theta \sin^2\theta P(\theta;C) F_j(\theta),$$

$$\langle \mathrm{tr}_j\, u_C \rangle = \frac{1}{\pi} \int_0^{\pi} d\theta \tilde{P}(\theta;C) F_j(\theta),$$

$$F_j(\theta) = \sum_{m=-j}^{j} e^{im\theta},$$

where $\sin^2\theta$ comes from the Haar measure of $SU(2)$.
Actually, this factor $\sin^2\theta$ explains the incorrect behaviors of the Abelian projected Wilson loops in higher representations.

Generalized MA gauge

Before showing the numerical results, we introduce generalized MA gauges (c.f. Stack-Tucker-Wensley (2002)).
The gauge fixing functional of the MA gauge is the form of a mass term for the gauge fields.

$$\int \mathrm{tr}\left(A_\mu^a A_\mu^a\right) \quad (a \text{ denotes off-diagonal components})$$

In $SU(3)$ case, we can generalize it as

$$\int \left({m_1}^2\left((A_\mu^1)^2+(A_\mu^2)^2\right)+{m_2}^2\left((A_\mu^4)^2+(A_\mu^5)^2\right)+{m_3}^2\left((A_\mu^6)^2+(A_\mu^7)^2\right)\right).$$

In the following we use

$$m_1=0,\ m_2=m_3=m,\ \text{(GA1)}$$
$$m_1=2m,\ m_2=m_3=m.\ \text{(GA2)}$$

GA1 is special because the symmetry breaking pattern is different from the MA gauge as $SU(3) \rightarrow U(2)$. Therefore we cannot use GA1 in every case, for example we can use it in fund. rep. and $\mathbf{6}^*$ and cannot use it in adj. rep.