Transport Properties of the QGP from Lattice QCD

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Outline

- why η/s?
- Kubo formula, the spectral functions
- the bulk sum rule and ζ near T_c
- HQET approach to heavy quark diffusion
- conclusions.

All lattice data shown are obtained in $N_{\rm f}=0$ QCD: gluon plasma.

Viscosity: dissipative fluid dynamics

A small perturbation $T_{0z} = T_{0z}(t, x)$ of a fluid around equilibrium satisfies the diffusion equation

$$\partial_t T_{0z} - D\partial_x^2 T_{0z} = 0 \,, \qquad D = \frac{\eta}{e+p} \qquad \text{(shear mode)}$$

A sound wave with wavelength $\lambda = 2\pi/k_z$ is damped as

$$T_{0z}(t,k) \propto e^{-\frac{1}{2}(\frac{4}{3}\eta+\zeta)k^2 t/(e+p)} \qquad \text{(sound mode)}\,.$$

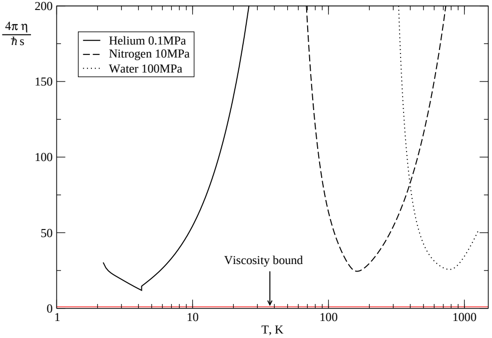
T_{0k}= momentum density	$\eta =$ shear viscosity	$\zeta =$ bulk viscosity

Why η/s ?

in a heavy-ion collision, the relaxation time τ_R should be small compared to the expansion rate $\Gamma_{\rm exp}$ for hydrodynamics to be applicable

- $\tau_R \geq \frac{4}{3(1-v_s^2)} \cdot \frac{\eta}{Ts} \approx 2\frac{\eta}{Ts}$ (causality bound Rischke et al [0907.3906])
- validity of hydrodynamics $\Rightarrow$ $\boxed{\frac{\eta}{s}\frac{\Gamma_{\rm exp}}{T} \ll 1.}$

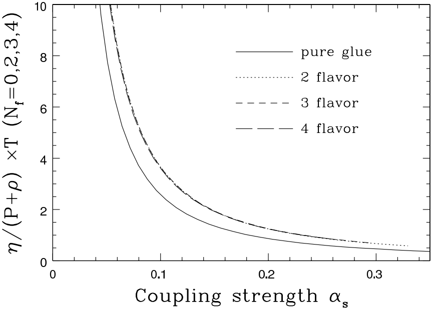
η/s for ordinary substances



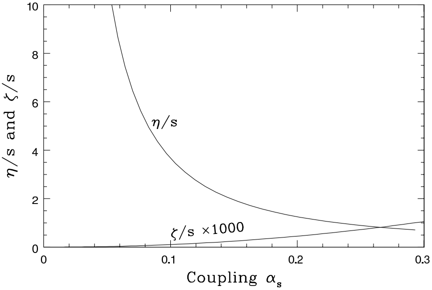
- "all theories with a classical gravity dual description satisfy $\eta/s = 1/(4\pi)$"
- most substances have much larger values

Kovtun, Son, Starinets PRL 94:111601, 2005

Perturbative QCD results



Arnold, Moore, Yaffe [SEWM 04]



Arnold, Dogan, Moore 06

for $\alpha_s = 0.25$: $\eta/s \approx 1.0$
$\zeta \approx 0.001\eta$

$gg \to ggg$ processes may significantly lower η/s C. Greiner & Xu 07

Calculating transport coefficients in lattice QCD: the case of sound damping

To calculate the viscosity, we will study the damping of sound waves in the QGP. Sound waves are longitudinal fluctuations in the pressure of the fluid.

- $\langle T_{33} \rangle_{\rm eq}$ = pressure in the z-direction
- let $-\frac{1}{\pi}\rho(\omega, \mathbf{q}, T)$ be the imaginary part of the retarded correlator of T_{33}

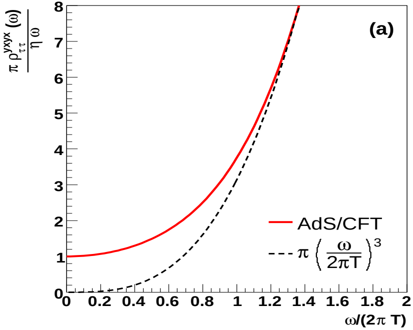
$$\Rightarrow \quad \boxed{(\tfrac{4}{3}\eta + \zeta)(T) = \lim_{\omega\to 0} \tfrac{\pi}{\omega}\rho(\omega, \mathbf{q} = \mathbf{0}, T)} \quad \textbf{Kubo formula}$$

The correlator $C(x_0)$ of T_{33}, computable on the lattice, is related to ρ by

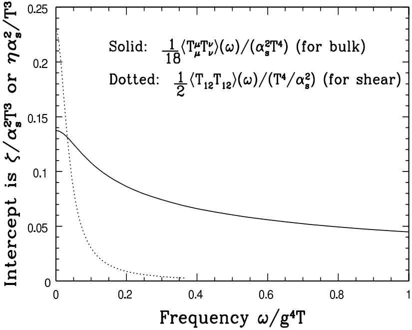
$$C(x_0, \mathbf{q}, T) = \int_0^\infty d\omega\, \rho(\omega, \mathbf{q}, T)\, \frac{\cosh \omega(1/2T - x_0)}{\sinh \omega/2T} \qquad \text{[Karsch, Wyld, '86]}$$

ρ is called the "spectral function". The same applies in the shear channel with the substitutions $T_{33} \to T_{13}$ and $(\frac{4}{3}\eta + \zeta) \to \eta$.

Spectral function at weak coupling and in AdS/CFT (shear channel, $\mathbf{q} = 0$)



[Teaney 06]



Weak coupling [Moore, Saremi 2008]

free field theory: $\rho_{12,12}(\omega, T) = \frac{d_A}{10(4\pi)^2}\frac{\omega^4}{\tanh\frac{\omega}{4T}} + \left(\frac{2\pi}{15}\right)^2 d_A T^4\,\omega\delta(\omega)$

Q: is the QCD spectral function qualitatively like the red or the dotted curve?

Ward identities of translation invariance

Ward Identities deriving from $\left\{ \begin{array}{l} \partial_0 T_{00} + \partial_k T_{0k} = 0 \\ \partial_k T_{0k} + \partial_\ell T_{k\ell} = 0 \end{array} \right. \Rightarrow \qquad (\mathbf{q} = q\hat{e}_3)$

$$\int d^3\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}}\, \langle T_{00}(x_0,\mathbf{x})\, T_{00}(0)\rangle = \int_0^\infty d\omega \quad \rho_{\rm snd}(\omega,\mathbf{q})\, \frac{\cosh\omega(\frac{1}{2}L_0 - x_0)}{\sinh\frac{1}{2}\omega L_0}$$

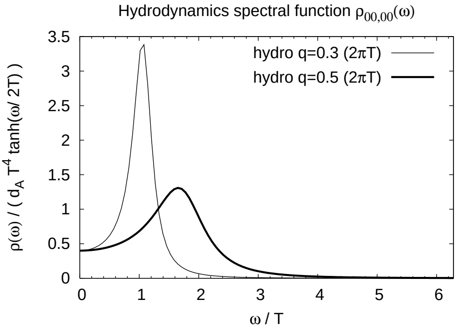
$$\int d^3\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}}\, \langle T_{00}(x_0,\mathbf{x})\, T_{03}(0)\rangle = \frac{-i}{q}\int_0^\infty d\omega \quad \omega\, \rho_{\rm snd}(\omega,\mathbf{q})\, \frac{\sinh\omega(\frac{1}{2}L_0 - x_0)}{\sinh\frac{1}{2}\omega L_0}$$

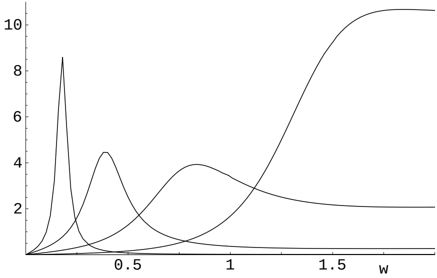
$$\left.\begin{array}{l} -\int d^3\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}}\, \langle T_{03}(x_0,\mathbf{x})\, T_{03}(0)\rangle \\ -\int d^3\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}}\, \langle T_{00}(x_0,\mathbf{x})\, T_{33}(0)\rangle \end{array}\right\} = \frac{1}{q^2}\int_0^\infty d\omega\, \omega^2\, \rho_{\rm snd}(\omega,\mathbf{q})\, \frac{\cosh\omega(\frac{1}{2}L_0 - x_0)}{\sinh\frac{1}{2}\omega L_0}$$

$$\int d^3\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}}\, \langle T_{03}(x_0,\mathbf{x})\, T_{33}(0)\rangle = \frac{i}{q^3}\int_0^\infty d\omega\, \omega^3\, \rho_{\rm snd}(\omega,\mathbf{q})\, \frac{\sinh\omega(\frac{1}{2}L_0 - x_0)}{\sinh\frac{1}{2}\omega L_0}$$

$$\int d^3\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}}\, \langle T_{33}(x_0,\mathbf{x})\, T_{33}(0)\rangle = \frac{1}{q^4}\int_0^\infty d\omega\, \omega^4\, \rho_{\rm snd}(\omega,\mathbf{q})\, \frac{\cosh\omega(\frac{1}{2}L_0 - x_0)}{\sinh\frac{1}{2}\omega L_0}$$

Sound channel spectral function $\rho_{00,00}(\omega, \mathbf{q})$





Hydrodynamics spectral function

$v_s^2 = \frac{1}{3}$, $s = \frac{3}{4} s_{SB}$, $T\Gamma_s = \frac{1}{3\pi}$

valid at small ω, q

strongly coupled $\mathcal{N} = 4$ SYM

$\frac{2\rho}{\pi d_A T^4}$ for $q/(2\pi T) = 0.3, 0.6, 1.0, 1.5$

(Kovtun, Starinets 2006)

Hydrodynamics predictions for the spectral functions at small $(\omega, \mathbf{q} = q\hat{e}_3)$

▷ **1st order prediction for hydrodynamic pole structure:** (Teaney 06)

sound channel: $$\frac{\rho_{00,00}(\omega,\mathbf{q})}{\omega} \overset{\omega,q\to 0}{\sim} \frac{\Gamma_s}{\pi} \frac{(e+P)\, q^4}{(\omega^2 - v_s^2 q^2)^2 + (\Gamma_s \omega q^2)^2},$$

$$\Gamma_s = \frac{\frac{4}{3}\eta+\zeta}{e+P}$$ = sound attenuation length

▷ **2nd order corrections to sound dispersion relation:** (Baier et al 07)

$$\omega = v_s(q) q, \qquad v_s(q) = v_s \left\{ 1 + \frac{\Gamma_s}{2} q^2 \left(\tau_\Pi - \frac{\Gamma_s}{4 v_s^2} \right) + \mathrm{O}(q^4) \right\}$$

τ_Π is the relaxation time

Sum rule in the bulk channel

Which spectral functions are analytic at $\omega = 0$?

Recall the small $(\omega, \mathbf{k})$ hydrodynamic prediction [Teaney 06]

$$\frac{\rho_{33,33}(\omega,\mathbf{k},T)}{\omega} = \frac{e+p}{\pi}\frac{\Gamma_s\omega^4}{(\omega^2 - c_s^2k^2)^2 + (\Gamma_s\omega k^2)^2}.$$

NB. $\forall f$ s.t. $\int_{-\infty}^{\infty} f(x)dx = 1$, $\frac{1}{\epsilon}f(x/\epsilon)$ provides a rep$^{\text{n}}$ of the δ-function. Here

$$f(x) = \frac{1}{\pi(b^2-1)}\left[\frac{x^4}{(x^2-b^2)^2+x^2} - 1\right], \quad b \equiv c_s/(\Gamma_s k), \quad \epsilon \equiv \Gamma_s k^2.$$

$$\Rightarrow \qquad \boxed{\lim_{\mathbf{k}\to 0} \frac{\rho_{33,33}(\omega,\mathbf{k},T)}{\omega} = \frac{e+p}{\pi}\Gamma_s + (e+p)c_s^2\delta(\omega) + \dots}$$

$\Rightarrow$ separate the smooth part of the spectral function from the singular term,

$$\rho_{\mu\mu,\nu\nu}(\omega,\mathbf{0},T) = \rho^{\text{smooth}}_{\mu\mu,\nu\nu}(\omega,\mathbf{0},T) + \frac{e+p}{c_s^2}(3c_s^2-1)^2\omega\delta(\omega), \quad \frac{\rho^{\text{smooth}}_{\mu\mu,\nu\nu}(0,\mathbf{0},T)}{\omega} = \frac{9\zeta}{\pi}.$$

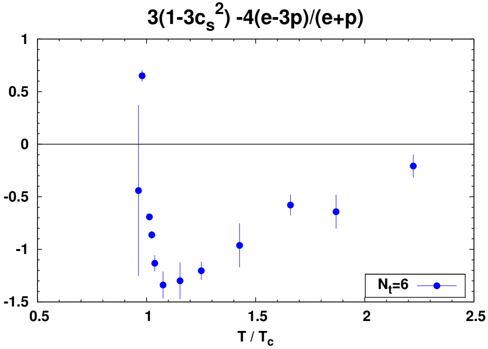
Equivalently, work with $T_{\mu\mu} + (3c_s^2 - 1)T_{00}$, whose spectral function = $\rho^{\text{smooth}}_{\mu\mu,\nu\nu}$.

Sum rule in the bulk channel

$$\int d^4x \langle T_{\mu\mu}(x) T_{\nu\nu}(0)\rangle_{T-0} = \int_0^\infty \frac{d\omega}{\omega} \Delta\rho_{\mu\mu,\nu\nu}(\omega, \mathbf{0}, T) = T^5 \frac{\partial}{\partial T} \frac{e-3p}{T^4}$$

Taking into account the δ-function: [Son, Romatschke 09]

$$\int_0^\infty \frac{d\omega}{\omega} \Delta\rho^{\text{smooth}}_{\mu\mu,\nu\nu}(\omega, \mathbf{0}, T) = 3(1-3c_s^2)(e+p) - 4(e-3p).$$



With the δ-function in place, the sum rule no longer suggests a large bulk viscosity just above T_c.

However, if relaxation time is long, the transport peak makes only a small contribution to the LHS of the sum rule.

[HM 09]

An alternative to calculate the velocity of sound

no derivative: $$p(T) = -f(T) = \frac{T}{V} \log\{Z(T)/Z(0)\}$$ (for V large)

one derivative: $$e = T^2 \frac{\partial}{\partial T} \frac{p}{T}, \qquad s = \frac{\partial p}{\partial T}$$

two derivatives: $$c_v = \frac{\partial e}{\partial T}, \qquad v_s^2 = \frac{s}{c_v}$$

Alternative method based on exact **lattice sum rules** [HM 07]:

$$\begin{aligned} \frac{c_v}{T^3} &= \frac{a^4}{T^4} \sum_x \langle T_{\mu\mu}(x) T_{\nu\nu}(0) \rangle^{\mathrm{conn}}_{T-0} \\ &+ \frac{3s}{T^3} + 4\frac{e-3p}{T^4} - \frac{d^2\beta/d(\log a)^2}{d\beta/d\log a} \frac{\langle T^{\mathrm{g}}_{\mu\mu} \rangle}{T^4} - \frac{d^2\kappa/d(\log a)^2}{d\kappa/d\log a} \frac{\langle T^{\mathrm{f}}_{\mu\mu} \rangle}{T^4} \,. \end{aligned}$$

No T-derivatives required!

$$\begin{aligned} a^4 T^{\mathrm{g}}_{\mu\mu}(x) &= \frac{d\beta}{d\log a} \sum_{\mu<\nu} \frac{\mathrm{Re\,Tr}}{N_c} \{1 - U_{\mu\nu}(x)\} \\ a^4 T^{\mathrm{f}}_{\mu\mu}(x) &= -\frac{d\kappa}{d\log a} \sum_{\mu} \bar\psi(x) U_\mu(x)(1-\gamma_\mu)\psi(x+a\hat\mu) + \bar\psi(x+a\hat\mu) U_\mu(x)^{-1}(1+\gamma_\mu)\psi(x) \end{aligned}$$

A note on the "reconstructed" correlator

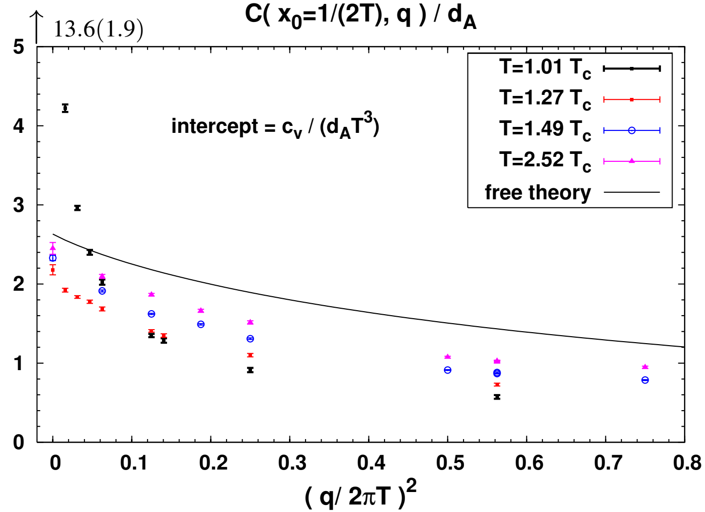
$$
\begin{aligned}
C_{\rm rec}(t,T) &\overset{L_0=1/T}{\equiv} \int_0^\infty d\omega \rho(\omega,T=0)\frac{\cosh\omega(\frac{L_0}{2}-t)}{\sinh \omega L_0/2} \\
&= 2T\int_0^\infty d\omega\ \omega\rho(\omega,T=0)\sum_{n\in\mathbf{Z}}\frac{e^{i\omega_n t}}{\omega^2+\omega_n^2} \\
&= 2\int_0^\infty d\omega\ \omega\rho(\omega,T=0)\sum_{m\in\mathbf{Z}}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\frac{e^{i\omega'(t+mL_0)}}{\omega^2+\omega'^2} \\
&= \int_0^\infty d\omega \rho(\omega,T=0)\sum_{m\in\mathbf{Z}} e^{-\omega|t+mL_0|} \\
&= \sum_{m\in\mathbf{Z}} C(T=0,|t+mL_0|)
\end{aligned}
$$

It is not necessary to reconstruct $\rho(\omega,T=0)$ in order to evaluate $C_{\rm rec}(t,T)$.

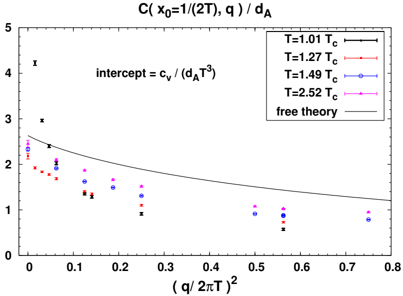
$$
\Rightarrow \text{solve}\ \ [C-C_{\rm rec}](t,T) = \int_0^\infty d\omega \Delta\rho(\omega,T)\frac{\cosh\omega(\frac{L_0}{2}-t)}{\sinh\omega L_0/2} \quad \text{for} \quad \Delta\rho.
$$

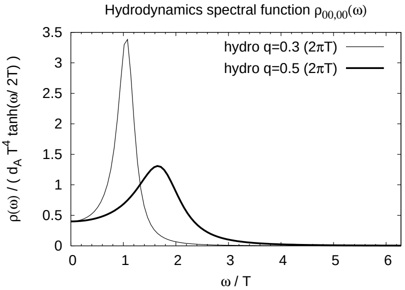
NB. $\Delta\rho \overset{\omega\to\infty}{\longrightarrow} 0$ and satisfies the bulk sum rule.

Bulk viscosity near T_c: $\langle \int d^3x\, e^{i\mathbf{q}\cdot\mathbf{x}}\, T_{00}(x_0,\mathbf{x})\, T_{00}(0)\rangle$



Bulk viscosity near T_c: $\langle \int d^3x\, e^{i\mathbf{q}\cdot\mathbf{x}}\, T_{00}(x_0,\mathbf{x})\, T_{00}(0)\rangle$





$$C(x_0 = \frac{L_0}{2}) \qquad = \quad T^{-4}\int_0^\infty d\omega \frac{\rho(\omega)}{\tanh(\omega/2T)} \cdot \frac{1}{\cosh(\omega/2T)}\,.$$

- much larger slope at $q=0$ for $T=1.01T_c$ than at other temperatures $\Rightarrow$ sound waves with wavelength $\lambda = 8/T = 5.8$fm are strongly damped

see also [Kharzeev, Tuchin 07; Karsch, Hübner, Pica 08; Kunihiro et al.]

Heavy-Quark Momentum Diffusion Constant in the QGP [Caron-Huot, Laine, Moore 09]

The correlators to be computed on the lattice are ($L_0 = 1/T$)

$$C_E(x_0) = -\frac{1}{3}\sum_{i=1}^{3} \frac{\langle \mathrm{Re\,Tr}\,\{W_0(L_0,x_0)\widehat{F}_{0i}(x_0)W_0(x_0,0)\widehat{F}_{0i}(0)\}\rangle}{\mathrm{Re\,Tr}\,\{W_0(L_0,0)\}},$$

$$C_B(x_0) = \frac{1}{6}\sum_{k,l=1}^{3} \frac{\langle \mathrm{Re\,Tr}\,\{W_0(L_0,x_0)\widehat{F}_{kl}(x_0)W_0(x_0,0)\widehat{F}_{kl}(0)\}\rangle}{\mathrm{Re\,Tr}\,\{W_0(L_0,0)\}}.$$

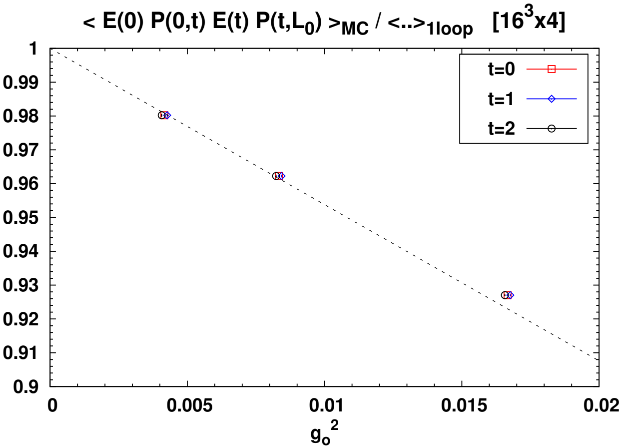
- $W_0(x_0', x_0)$ is the parallel transporter from time x_0 to time x_0'.
- the denominator is the Polyakov loop in the fundamental representation.
- the correlator obeys a spectral representation

$$C_E(x_0) = \int_0^\infty \rho(\omega)\frac{\cosh\omega(\frac{1}{2}L_0 - x_0)}{\sinh\frac{\omega L_0}{2}}d\omega.$$

- The momentum diffusion constant is then given by

$$\kappa(T) = \lim_{\omega\to 0}\frac{2T}{\omega}\,\rho(\omega).$$

Electric field correlators in finite-temperature HQET



- weak-coupling prediction for the spectral function: smooth on the scale T, not on scale T^2/M.

Concluding remarks

1. bulk sum rule can be used to constrain the spectral function, as long as one keeps track of the exact δ-functions in $\rho(\omega)$.

2. $[C - C_{\rm rec}]$ is more sensitive to the low-frequency part of $\rho(\omega)$ (but positivity is lost)

3. does the system become very dissipative near the (weakly 1st order) phase transition?

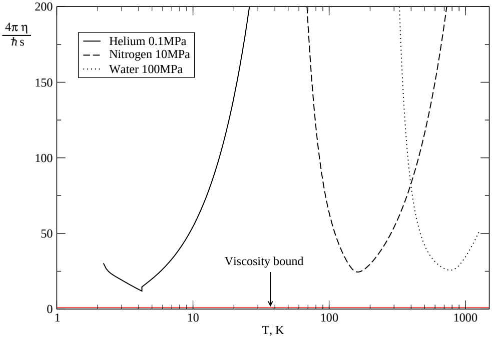
Backup slides

Why η/s ?

in a heavy-ion collision, the relaxation time τ_R should be small compared to the expansion rate Γ_{exp} for hydrodynamics to be applicable [see e.g. talk by U. Heinz]

- $\tau_R \approx \frac{\eta}{Ts}$ in an ultrarelativistic plasma
- validity of hydrodynamics $\Rightarrow$ $\boxed{\frac{\eta}{s}\frac{\Gamma_{\text{exp}}}{T} \ll 1.}$

η/s for ordinary substances

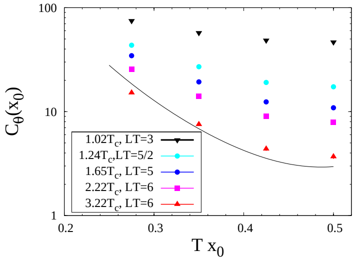


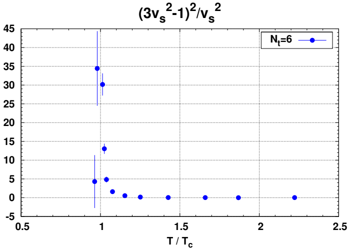
- "all theories with a classical gravity dual description satisfy $\eta/s = 1/(4\pi)$"
- most substances have much larger values

Kovtun, Son, Starinets PRL 94:111601, 2005

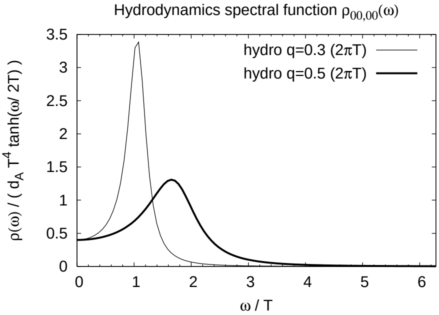
The contribution of the $\omega\delta(\omega)$ term to the Euclidean correlator

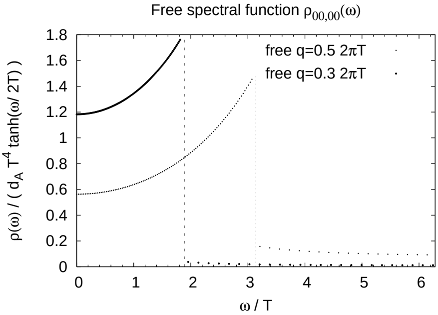
$$\frac{\rho_{\mu\mu,\nu\nu}(\omega,\mathbf{0},T)}{e+p} = \frac{\rho^{\text{smooth}}_{\mu\mu,\nu\nu}(\omega,\mathbf{0},T)}{e+p} + \frac{(3c_s^2-1)^2}{c_s^2}\omega\delta(\omega), \quad \frac{\rho^{\text{smooth}}_{\mu\mu,\nu\nu}(\omega,\mathbf{0},T)}{\omega} \overset{\omega=0}{=} \frac{9\zeta}{\pi}.$$





Sound channel spectral function $\rho_{00,00}(\omega, \mathbf{q})$ at weak coupling





Hydrodynamics spectral function

$v_s^2 = \frac{1}{3}$, $s = \frac{3}{4} s_{SB}$, $T\Gamma_s = \frac{1}{3\pi}$

SU(N_c) gauge theory
[HM, 08]

$$\rho_{\rm snd}(\omega, q, T) = \frac{d_A}{4\,(4\pi)^2}\, q^4\ \mathcal{I}([(1-z^2)^2], \omega, q, T)\,,$$

$$\mathcal{I}([P], \omega, q, T) = \theta(\omega - q)\int_0^1 dz \frac{P(z)\ \sinh\frac{\omega}{2T}}{\cosh\frac{\omega}{2T} - \cosh\frac{qz}{2T}} + \theta(q-\omega)\int_1^\infty dz \frac{P(z)\ \sinh\frac{\omega}{2T}}{\cosh\frac{qz}{2T} - \cosh\frac{\omega}{2T}}.$$

How to reconstruct the spectral function $\rho(\omega, \mathbf{q}, T)$?

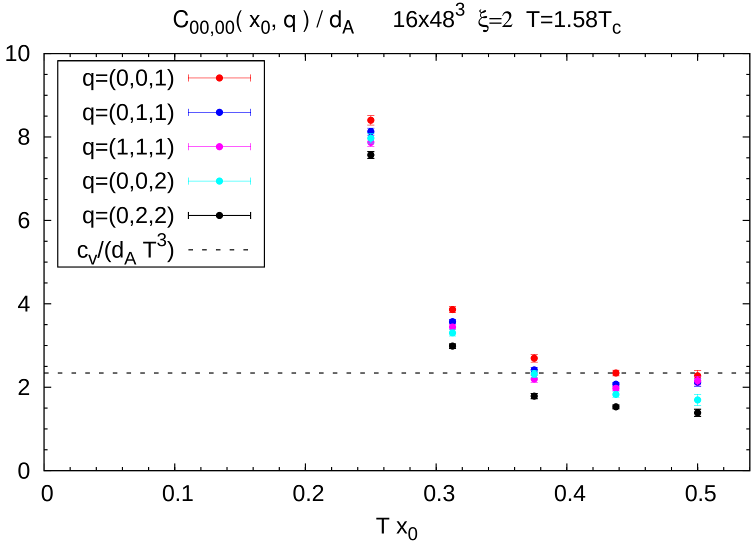
$$C_{33,33}(x_0, \mathbf{q}, T) = \int_0^\infty d\omega\, \rho(\omega, \mathbf{q}, T) \frac{\cosh \omega(1/2T - x_0)}{\sinh \omega/2T}$$

$$\frac{4}{3}\eta + \zeta = \pi \left. \frac{\rho(\omega, \mathbf{q} = \mathbf{0}, T)}{\omega} \right|_{\omega=0} .$$

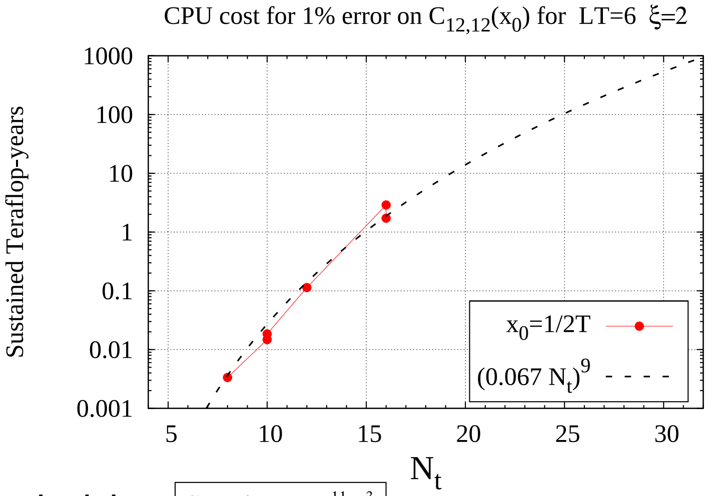
Hydrodynamics is a low-energy effective theory:
it predicts the functional form of ρ at small $(\omega, \mathbf{q})$

Perturbation theory + OPE predicts ρ at large ω

Euclidean correlator of the energy density T_{00} [200'000 PC hours, allocated by USQCD]



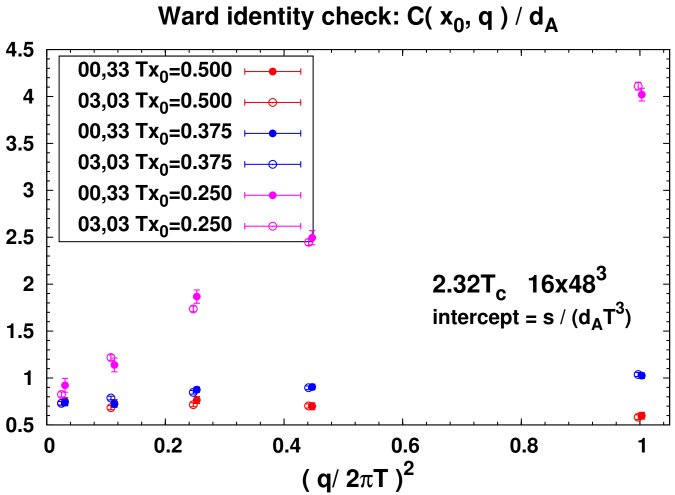
Cost formula for shear channel correlator



One-level algo: $\text{CPU-time} \propto N_\tau^{11} N_\sigma^3$

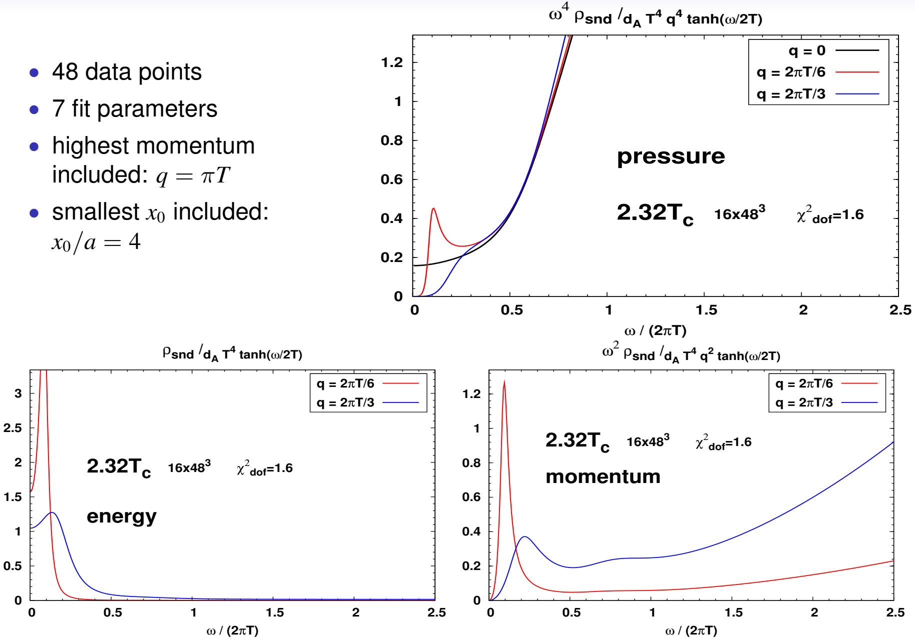
Two-level algo: $\text{CPU-time} \propto N_\tau^{6} N_\sigma^3$

Testing the restoration of translation invariance

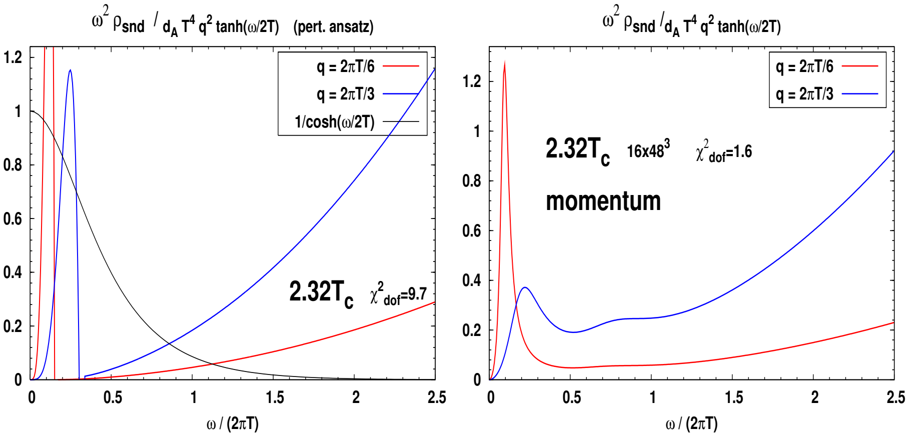


Sound channel spectral functions from 16×48^3 lattice

- 48 data points
- 7 fit parameters
- highest momentum included: $q = \pi T$
- smallest x_0 included: $x_0/a = 4$



Contrast with weak-coupling ansatz



Results at 1.6 and 2.3 T_c PRELIMINARY

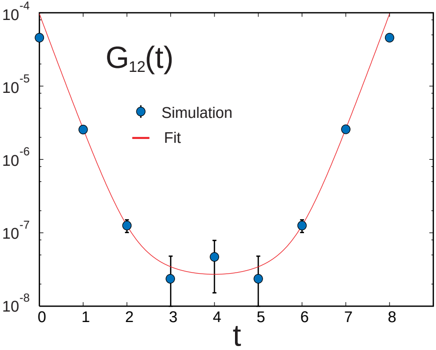
Quantity	1.575T$_c$	2.32T$_c$	free gluons	$\lambda = \infty$ SUSY
$[\eta + \frac{3}{4}\zeta]/s$	$0.20(3)_{\rm stat}$	$0.26(3)_{\rm stat}$	∞	$1/4\pi = 0.080$
$2\pi T\tau_\Pi$	$3.1(3)_{\rm stat}$	$3.2(3)_{\rm stat}$	∞	$2 - \log 2 = 1.31$
$[\eta + \frac{3}{4}\zeta]/[T\tau_\Pi s]$	$0.40(5)_{\rm stat}$	$0.51(5)_{\rm stat}$	0.17	0.38
			York,Moore	Son,Starinets

$\rightsquigarrow$ my current best guess for viscosity at LHC:

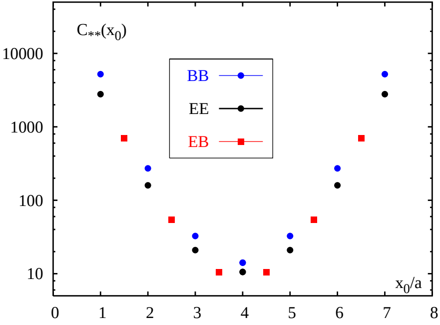
$$[\eta/s]_{QGP} \approx [\eta/s]_{GP,lattice} \cdot \left[\frac{[\eta/s]_{QGP}}{[\eta/s]_{GP}}\right]_{AMY} \approx 0.40$$

Progress in accuracy of Euclidean correlators (8×28^3, shear channel)

Nakamura, Sakai '04



HM '07



$$C(x_0) = \frac{1}{4}(C_{BB} + C_{EE} + 2C_{EB})$$

multi-level algorithm [HM 02]: gain $\approx$ 20 in the error bar on $C(L_0/2)$

Fit parameters

$$\rho_{\rm snd} = \rho_{\rm low} + \rho_{\rm med} + \rho_{\rm high} \; ,$$

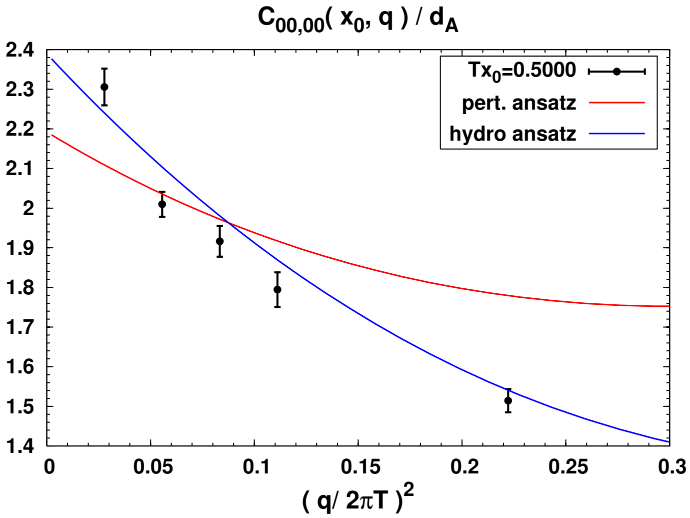
$$\frac{\rho_{\rm low}(\omega,q,T)}{\tanh(\omega/2T)} = \frac{2\widehat{\Gamma}_s}{\pi} \frac{(e+P)\,q^4}{(\omega^2 - v_s^2(q)q^2)^2 + (\widehat{\Gamma}_s\omega q^2)^2} \frac{1+\sigma_1\omega^2}{1+\sigma_2\omega^2}$$

$$\frac{\rho_{\rm med}(\omega,q,T)}{\tanh(\omega/2T)} = q^4 \tanh^2(\frac{\omega}{2T}) \frac{\ell\,\sigma}{\sigma^2 + (\omega^2 - q^2 - M^2)^2}$$

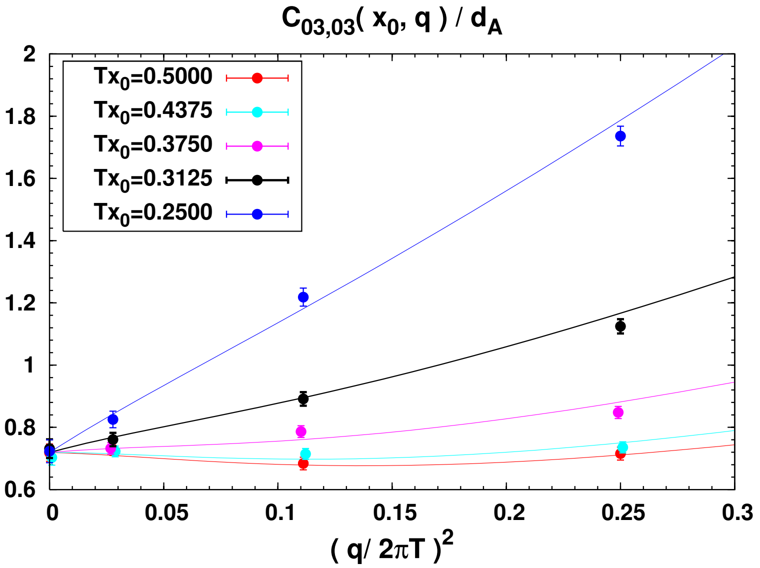
$$\frac{\rho_{\rm high}(\omega,q,T)}{\tanh(\omega/2T)} = q^4 \tanh^2(\frac{\omega}{2T}) \frac{2d_A}{15(4\pi)^2}$$

- perturbation theory and the operator product expansion $\Rightarrow$ systematically improve the knowledge of $\rho_{\rm high}$
- fully matching hydrodynamics & linear response theory to $2^{\rm nd}$ order $\Rightarrow$ systematically improve functional form of $\rho_{\rm low}$ (free parameters = viscosity, relaxation time, . . .)
- eventually, treat $\rho_{\rm med}$ with Maximum Entropy Method [Asakawa et al.] or expand it in a suitable orthogonal basis [HM 07].

Perturbative vs. hydrodynamics ansatz (II) ($2.32T_c,\ 16 \times 48^3$)



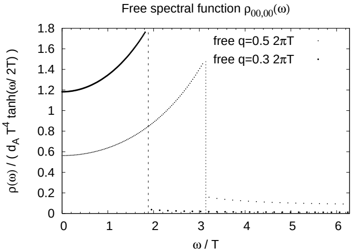
Fit at $2.32T_c$, 16×48^3 lattice



An alternative fit ansatz inspired by weak-coupling

Attempt to replace the low-frequency term by

$$\frac{\rho^{\rm low}_{\rm snd}(\omega,q,T)}{\tanh(\omega/2T)} \quad = \quad \frac{\ell_1(e+P)T}{q}\,\frac{a_0 + a_1(\omega/q)^2 + a_2(\omega/q)^4}{1+\exp\{[(\omega/q)^2-1]/f^2\}}$$



To get the correct $q \to 0$ limit, we must require

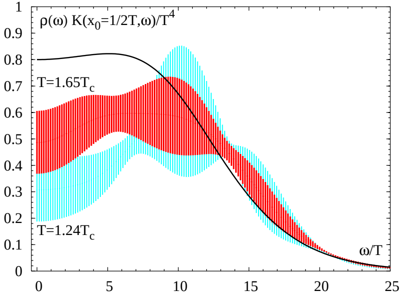
$$a_1 \quad = \quad \frac{-105}{4} + \frac{75}{4v_s^2} - 10a_0\,,$$

$$a_2 \quad = \quad \frac{175}{4} - \frac{105}{4v_s^2} + \frac{35}{3}a_0\,.$$

The fit is driven to $a_0 \simeq 0$ and leads to $\chi^2_{\rm dof} \simeq 9$.

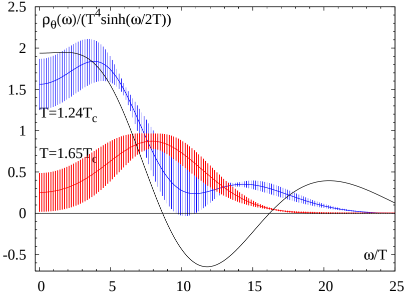
The reconstructed spectral functions

$\langle T_{12} T_{12} \rangle_c \ \ (N_\tau = 8)$



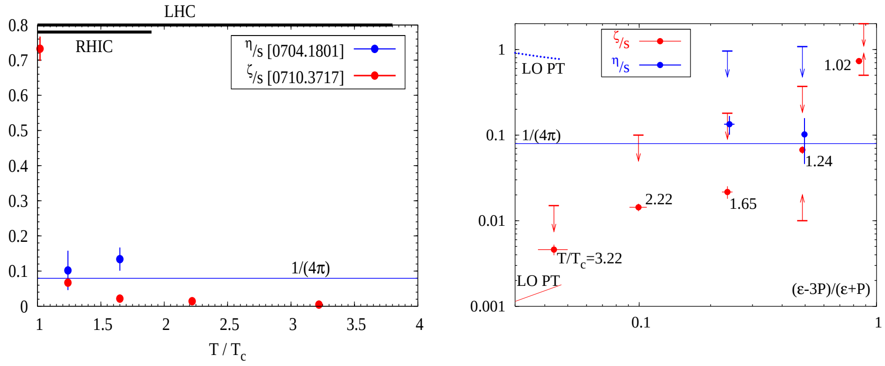
- $\eta/T^3 = \frac{\pi}{2} \ \times \ intercept$
- black curve $=$ normalized $\mathcal{N} = 4$ SYM spectral function.

$\langle \theta\ \theta \rangle_c \ \ (N_\tau = 12)$



- $\zeta/T^3 = (\frac{\pi}{18} \ \times \ intercept)$ increasing for $T \to T_c$
- black curve $= \hat{\delta}(0, \omega)$

2007 Results: η/s upper bounds, and estimates under smoothness assumption



Perturbative and AdS/CFT calculations:

$$\eta/s,\ \zeta/s = \begin{cases} \frac{0.484}{\pi^2 \alpha_s^2 \log(0.608/\alpha_s)}, & \frac{1.25\alpha_s^2}{\pi^2 \log(4.06/\alpha_s)} & N_f = 0\ PT \\ 1/(4\pi) \quad , & 0 & \mathcal{N} = 4\ SYM. \end{cases}$$

Arnold, Moore, Yaffe ’03; Arnold, Dogan, Moore ’06;

Policastro, Son, Starinets ’01; Kovtun, Son, Starinets ’04

The QCD energy-momentum tensor

- separating the traceless part $\theta_{\mu\nu}$ from the trace part θ
- gluons, denoted 'g', and quarks denoted 'q'

$$
\begin{aligned}
T_{\mu\nu} &\equiv \theta^{\rm g}_{\mu\nu} + \theta^{\rm q}_{\mu\nu} + \tfrac{1}{4}\delta_{\mu\nu}(\theta^{\rm g} + \theta^{\rm q}), \\
\theta^{\rm g}_{\mu\nu} &= \tfrac{1}{4}\delta_{\mu\nu} F^a_{\rho\sigma} F^a_{\rho\sigma} - F^a_{\mu\alpha} F^a_{\nu\alpha}, \\
\theta^{\rm q}_{\mu\nu} &= \tfrac{1}{4}\textstyle\sum_f \bar\psi_f \overleftrightarrow{D}_\mu \gamma_\nu \psi_f + \bar\psi_f \overleftrightarrow{D}_\nu \gamma_\mu \psi_f - \tfrac{1}{2}\delta_{\mu\nu} \bar\psi_f \overleftrightarrow{D}_\rho \gamma_\rho \psi_f, \\
\theta^{\rm g} &= \beta(g)/(2g)\, F^a_{\rho\sigma} F^a_{\rho\sigma}, \quad \theta^{\rm q} = -\textstyle\sum_f m_f\, \bar\psi_f \psi_f
\end{aligned}
$$

- $\overleftrightarrow{D}_\mu = \overrightarrow{D}_\mu - \overleftarrow{D}_\mu$
- $\beta(g)$ is the beta-function
- all expressions are written in Euclidean space.

- the Euclidean correlator $C_E(x_0) = \frac{1}{Z} \sum_{n,m} |\mathcal{O}_{nm}|^2 \, e^{-L_0 E_n} \, e^{E_{nm} x_0}$ (where $\mathcal{O}_{nm} = \langle n|\mathcal{O}|m\rangle$, $E_{nm} = E_n - E_m$, $L_0 = 1/T$) and the spectral function $\rho(L_0, \omega) = \frac{2}{Z} \, \sinh(\omega L_0/2) \, \sum_{n,m} \delta(\omega - E_{nm}) \, e^{-(E_n+E_m)L_0/2} \, |\mathcal{O}_{nm}|^2$ are related by $C_E(x_0) = \int_0^\infty d\omega \, \frac{\cosh \omega(L_0/2 - x_0)}{\sinh \omega L_0/2} \, \rho(L_0, \omega)$.

- the Minkovsky-space retarded correlator $iG_R(t) = \theta(t) \, \langle [\mathcal{O}(t), \mathcal{O}(0)] \rangle$ has the spectral representation $iG_R(t) = \frac{\theta(t)}{Z} \sum_{n,m} |\mathcal{O}_{nm}|^2 e^{-L_0 E_n} \, (e^{iE_{nm}t} - e^{-iE_{nm}t})$.

- Minkovsky and Euclidean correlator are related for $t > 0$ by

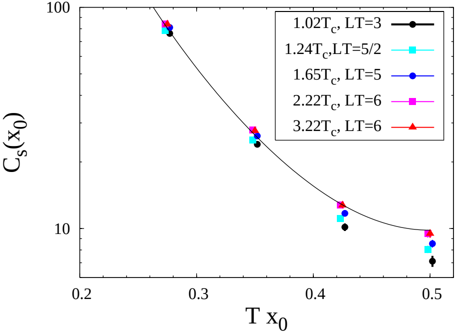
$$iG_R(t) = \lim_{\epsilon \to 0} (C_E(it + \epsilon) - C_E(-it + \epsilon)) = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} d\omega \, \rho(L_0, \omega) e^{-i\omega t} e^{-|\omega|\epsilon}$$

- the corresponding relation in frequency space is

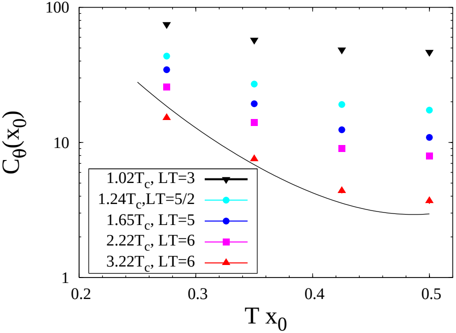
$$\begin{aligned} G_R(\omega + i\delta) &= -\int_{-\infty}^{\infty} d\omega' \, \frac{\rho(L_0, \omega') e^{-|\omega'|\epsilon}}{\omega' - \omega - i\delta}, \quad \omega \text{ real} \qquad \Rightarrow \\ \mathrm{Im}\, G_R(\omega + i\delta) &= \int_{-\infty}^{\infty} d\omega' \, (-\pi \rho(L_0, \omega') \, e^{-|\omega'|\epsilon}) \, \frac{1}{\pi} \frac{\delta}{(\omega' - \omega)^2 + \delta^2} = -\pi \rho(L_0, \omega). \end{aligned}$$

The Euclidean correlators ($N_\tau = 8,\ \xi = 1;\ N_f = 0$) HM 07

$\langle T_{12} T_{12} \rangle_c$ shear



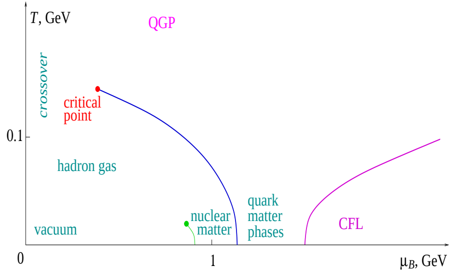
$\langle \theta\ \theta \rangle_c$ bulk



- near-conformal behavior in shear channel
- conformal anomaly correlator $\langle \theta\theta \rangle$ falls off rapidly with $T \nearrow$
- small errors thanks to multi-level algorithm (HM '03): cost scales as $\boxed{N_\tau^6 N_\sigma^3}$ instead of $\boxed{N_\tau^{11} N_\sigma^3}$

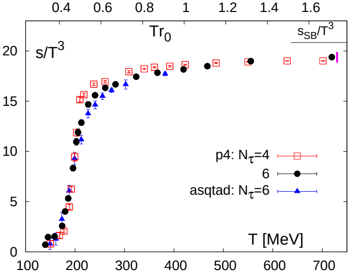
Phase diagram of QCD and equation of state at $\mu_B = 0$

Phase diagram



M. Stephanov, *Lattice* '06

Equation of state



Cheng et al., PRD 77 (2008) 014511

- entropy as a fct. of temperature on the vertical axis of the phase diagram
- entropy$/T^3 \sim$ #(d.o.f.) grows: color is liberated
- but no discontinuity.