Anomaly at Finite Density & Chiral Fermions on Lattice

*Rajiv V. Gavai & Sayantan Sharma**
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* *arXiv : 0906.5188, submitted to Phys. Rev. D.*

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Introduction

Anomaly for $\mu \neq 0$: Continuum

Two simple ideas for Lattice

Summary

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Introduction

♠ A fundamental aspect of the QCD Phase Diagram is the Critical Point in the T-μ_B plane expected on the basis of symmetries and models.

♡ In particular, two light flavours of quarks are crucial for it, as is the exact chiral symmetry on the lattice when the quark mass is tuned to zero.

♡ Temperature dependence of the Chiral Anomaly may be important as well; No CEP if instanton density is small enough below T_{ch} (Pisarski-Wilczek, 1984).

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♡ Temperature dependence of the Chiral Anomaly may be important as well; No CEP if instanton density is small enough below T_{ch} (Pisarski-Wilczek, 1984).

♠ The oft-used staggered fermions have (some) chiral symmetry but a not so well defined flavour number and no chiral anomaly.

♡ Use of Overlap fermions seems desirable. They even have an index theorem as well. (Hasenfratz, Laliena & Niedermeyer, PLB 1998; Luscher PLB 1998.)

♡ Note that chemical potential, μ_B, has to be introduced without violating the symmetries in order to investigate the entire T-μ_B plane.

Introducing Chemical Potential

- Ideally, one should construct the conserved charge, N, as a first step. Adding simply μN leads to a^{-2} divergences in the continuum limit.

- Multiply gauge links in positive/negative time direction by $\exp(a\mu)$ and $\exp(-a\mu)$ respectively. No change in chiral invariance as a result. (Hasenfratz-Karsch 1982; Kogut et al. 1982; Bilic-Gavai 1983).

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- Non-locality makes the construction of N difficult for the Overlap case, even non-unique (Mandula, 2007).

- Bloch-Wettig (PRL 2006; PRD 2007) proposal : Use the same prescription as above, i.e., $D_W(0) \rightarrow D_W(a\mu)$ in the sign function.

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- We (Banerjee, Gavai, Sharma, PRD 2008; PoS Lattice 2008) showed that the resultant overlap fermion action has i) no a^{-2} divergences but ii) unfortunately no chiral invariance for nonzero μ.

What if ...

♠ the chiral transformations were $\delta\psi = \alpha\gamma_5(1 - \frac{a}{2}D(a\mu))\psi$ and $\delta\bar{\psi} = \alpha\bar{\psi}(1 - \frac{a}{2}D(a\mu))\gamma_5$?

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- Symmetry transformations should not depend on "external" parameter μ. Chemical potential is introduced for charges N_i with $[H, N_i] = 0$. At least the symmetry should not change as μ does.

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- Symmetry transformations should not depend on "external" parameter μ. Chemical potential is introduced for charges N_i with $[H, N_i] = 0$. At least the symmetry should not change as μ does.
- Moreover, symmetry groups *different* at each μ. Recall we wish to investigate $\langle\bar{\psi}\psi\rangle(a\mu)$ to explore if chiral symmetry is restored.
- The symmetry group remains *same* at each T with $\mu = 0$ $\Longrightarrow \langle\bar{\psi}\psi\rangle(am = 0, T)$ is an order parameter for the chiral transition.

- Accepting the μ-dependent chiral transformations, on the other hand, leads to an index theorem for nonzero μ as well (Bloch-Wettig PRD 2007).

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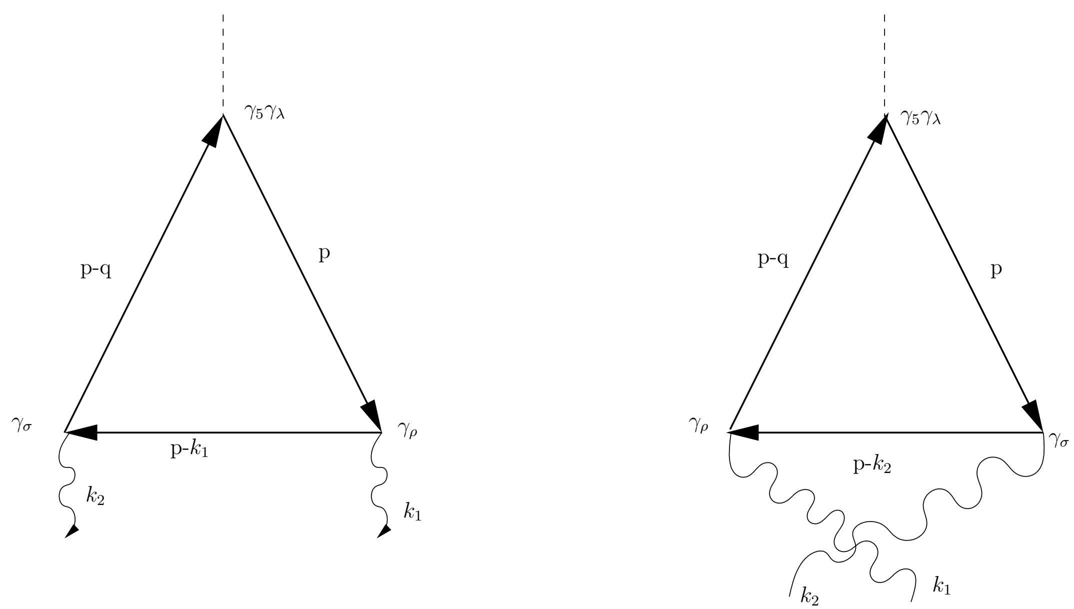
- What does nonzero μ do to the usual chiral anomaly in the continuum ?

- Calculations in real time (Qian, Su & Yu ZPC 1994) and Minkowski space-time (Hsu, Sannino & Schwetz MPLA 2001) formalisms show chiral anomaly is *unaffected* by nonzero μ.

- Conventional wisdom is that chiral anomaly is an ultra-violet effect. So expect no change due to nonzero μ.

- We re-visit the issue using a) Euclidean perturbation theory and b) the nonperturbative Fujikawa method.

Anomaly for $\mu \neq 0$: Continuum results



- Perturbatively we need to compute $\langle \partial_\mu j^5_\mu \rangle$, i.e., the triangle diagrams for $\mu \neq 0$.

- Denoting by $\Delta^{\lambda\rho\sigma}(k_1, k_2)$ the total amplitude and contracting it with q_λ,

$$
\begin{aligned}
q_\lambda \Delta^{\lambda\rho\sigma} &= -i\ g^2 \mathrm{tr}[T^a T^b] \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\ \left[\gamma^5 \frac{1}{\not{p} - \not{q} - i\mu\gamma^4} \gamma^\sigma \frac{1}{\not{p} - \not{k}_1 - i\mu\gamma^4} \gamma^\rho \right. \\
&- \gamma^5 \frac{1}{\not{p} - i\mu\gamma^4} \gamma^\sigma \frac{1}{\not{p} - \not{k}_1 - i\mu\gamma^4} \gamma^\rho + \gamma^5 \frac{1}{\not{p} - \not{q} - i\mu\gamma^4} \gamma^\rho \frac{1}{\not{p} - \not{k}_2 - i\mu\gamma^4} \gamma^\sigma \\
&- \left. \gamma^5 \frac{1}{\not{p} - i\mu\gamma^4} \gamma^\rho \frac{1}{\not{p} - \not{k}_2 - i\mu\gamma^4} \gamma^\sigma \right] .
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- Quadratic Divergent integrals; need cut-off which should be gauge invariant since

$$k_{1\rho}\Delta^{\lambda\rho\sigma}(k_1,k_2) = k_{2\sigma}\Delta^{\lambda\rho\sigma}(k_1,k_2) = 0.$$

- Can be done as for $\mu = 0$ by writing $q_\lambda \Delta^{\lambda\rho\sigma} =$
$(-i)\ \ \mathrm{tr}\ [T^aT^b]g^2 \int \frac{d^4p}{(2\pi)^4} [f(p-k_1,k_2) - f(p,k_2) + f(p-k_2,k_1) - f(p,k_1)]$.

- The final result is $\propto f$ due to the structure above.

- Due to nonzero μ, the function f has $(p_4^2 + \vec{p}^2) \rightarrow ((p_4 - i\mu)^2 + \vec{p}^2)$ in the denominator and terms proportional to μ and μ^2 in the numerator.

- Since the μ^2 terms have Tr $[\gamma^5 \gamma^4 \gamma^\sigma \gamma^4 \gamma^\rho] \sim \epsilon^{4\sigma 4\rho}$, they vanish.

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- Scaling the integration variable by the cut-off Λ, the μ-dependent terms appear with Λ^{-1}, leading to μ independence as $\Lambda \rightarrow \infty$: The same anomaly relation as for $\mu = 0$.
- In agreement with earlier calculations in real time (Qian, Su & Yu ZPC 1994) or Minkowski space-time (Hsu, Sannino & Schwetz MPLA 2001) , which are more involved.

Anomaly for $\mu \neq 0$: Fujikawa method

- Under the chiral transformation of the fermion fields, given by,

$$\psi' = \exp(i\alpha\gamma_5)\psi \quad \text{and} \quad \bar{\psi}' = \bar{\psi}\exp(i\alpha\gamma_5) \ , \tag{1}$$

the measure changes as

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \mathrm{Det}|\frac{\partial(\bar{\psi}', \psi')}{\partial(\bar{\psi}, \psi)}| = \exp(-2i\alpha \int d^4x \ \mathrm{Tr}\gamma_5) \ . \tag{2}$$

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- Evaluate the trace using the eigenvectors of the operator $\not{D}$ for $\mu = 0$.

- Since $\{\gamma_5, \not{D}\} = 0$, for each finite λ_n , $\phi_n^\pm = \phi_n \pm \gamma_5\phi_n$, eigenvectors of γ_5 with ± 1 eigenvalues, can be used, leading to zero.

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- Define two new vectors, ζ_m and υ_m

$$\zeta_m(\mathbf{x},\tau) = \mathrm{e}^{\mu\tau}\phi_{\mathrm{m}}(\mathbf{x},\tau) \quad , \quad \upsilon^\dagger_{\mathrm{m}}(\mathbf{x},\tau) = \phi^\dagger_{\mathrm{m}}(\mathbf{x},\tau)\mathrm{e}^{-\mu\tau} \ . \tag{3}$$

- Easy to show that ζ_m ($\upsilon^\dagger_m$) is the eigenvector of $\not{D}(\mu)$ ($\not{D}(\mu)^\dagger$) with the same (purely imaginary) eigenvalue λ_m (-λ_m).

- Further, one can show $\sum_m \int \zeta_m(\mathbf{x},\tau) \upsilon_m^\dagger(\mathbf{x},\tau)\ d^4x = \mathbf{I}$ and $\int \upsilon_m^\dagger(\mathbf{x},\tau)\zeta_m(\mathbf{x},\tau)\ d^4x = 1$.

- Using these eigenvector spaces of $\not{D}(\mu)$, trace of γ_5 can again be shown to be zero for all non-zero λ_m, leading to Tr $\gamma_5 = n_+ - n_-$ for $\mu \neq 0$ as well.

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- Both perturbatively, and nonperturbatively, we have shown that the anomaly does not change at finite density, as may have been expected naıvely.

- If chiral transformation on lattice is chosen to depend on μ, so that Bloch-Wettig proposal has chiral invariance for $\mu \neq 0$, then the resulting index theorem has μ-*dependent* zero modes which determine the anomaly, unlike in the continuum.

- It is undesirable for other reasons we pointed out earlier (Banerjee, Gavai, Sharma, PoS Lattice 2008).

A "Gauge-like" Transformation

- A non-unitary transformation of the fermion fields of the QCD action in the presence of μ, given by $\psi^{'}(\mathbf{x},\tau) = \mathrm{e}^{\mu\tau}\psi(\mathbf{x},\tau)$, $\bar{\psi}^{'}(\mathbf{x},\tau) = \bar{\psi}(\mathbf{x},\tau)\mathrm{e}^{-\mu\tau}$, makes the action μ-independent: $S_F(\mu) \rightarrow S'_F(0)$.

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- Easy to see that it works for any local fermion action, including for the lattice action, with $\mu\tau \rightarrow a_4\mu * n_4$ leading to the Hasenfratz-Karsch prescription or the more general $f(\mu a_4) * n_4$ to the others.

- Generalization for non-local cases, Overlap fermions, does not appear particularly useful. For the free case $\mathrm{e}^{\mu\tau}\mathrm{D}_{\mathrm{ov}}(\mu = 0)\mathrm{e}^{-\mu\tau}$ is free of a^{-2} divergence but does not appear to have chiral invariance. :'(

Two simple ideas for Lattice

- Only fermions confined to the domain wall are physical, so introduce a chemical potential only to count them:

$$D_{ov}(\hat{\mu})_{xy} = (D_{ov})_{xy} - \frac{a\hat{\mu}}{2a_4\ M}\left[(1-\gamma_4)U_4(y)\delta_{x,y-\hat{4}} + (\gamma_4+1)U_4^\dagger(x)\delta_{x,y+\hat{4}}\right]\ . \tag{4}$$

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- As the Bloch-Wettig proposal, this too breaks chiral invariance but D_{ov} is defined by the usual sign-function. But clearly Simpler !

- Expect a^{-2}-divergences as $a \to 0$. Follow the same prescription used for the Pressure computation (which diverges at zero temperature as Λ^4). Use Large N_τ and the same lattice spacing a for subtraction.

- Consider two Observables : $\Delta\epsilon(\mu, T) = \epsilon(\mu, T) - \epsilon(0, T)$ and
  Susceptibility, $\sim \partial^2 \ln \mathcal{Z} / \partial\mu^2$.

- Former computed for $r = \mu/T = 0.5$ while latter for $\mu = 0$.

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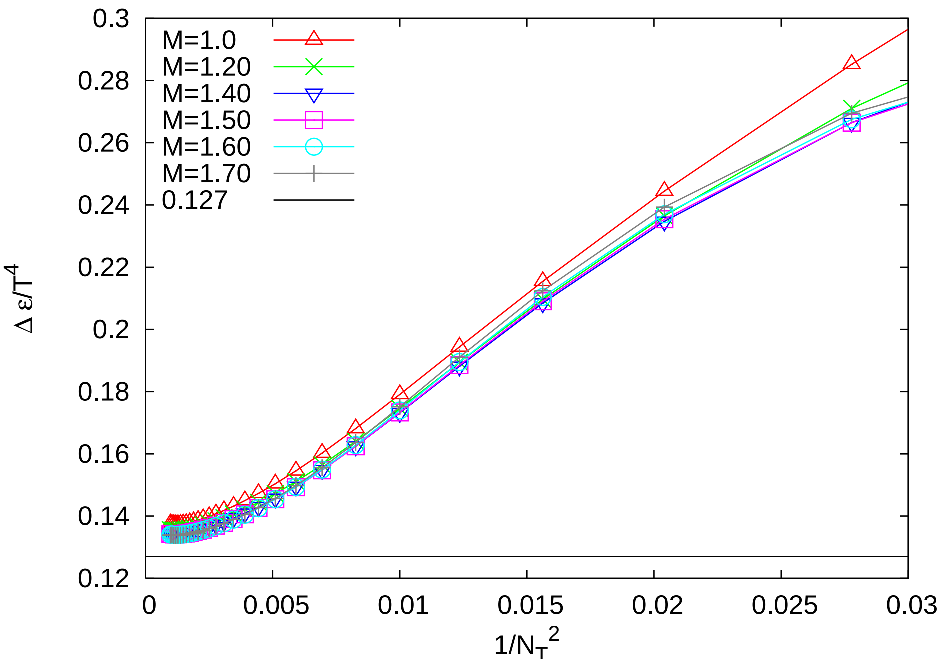
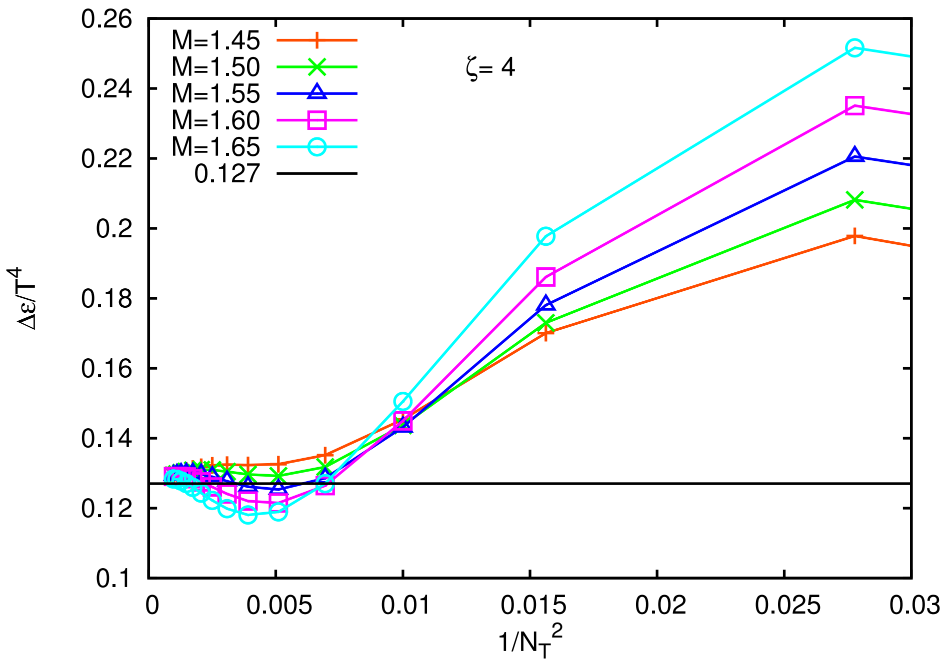
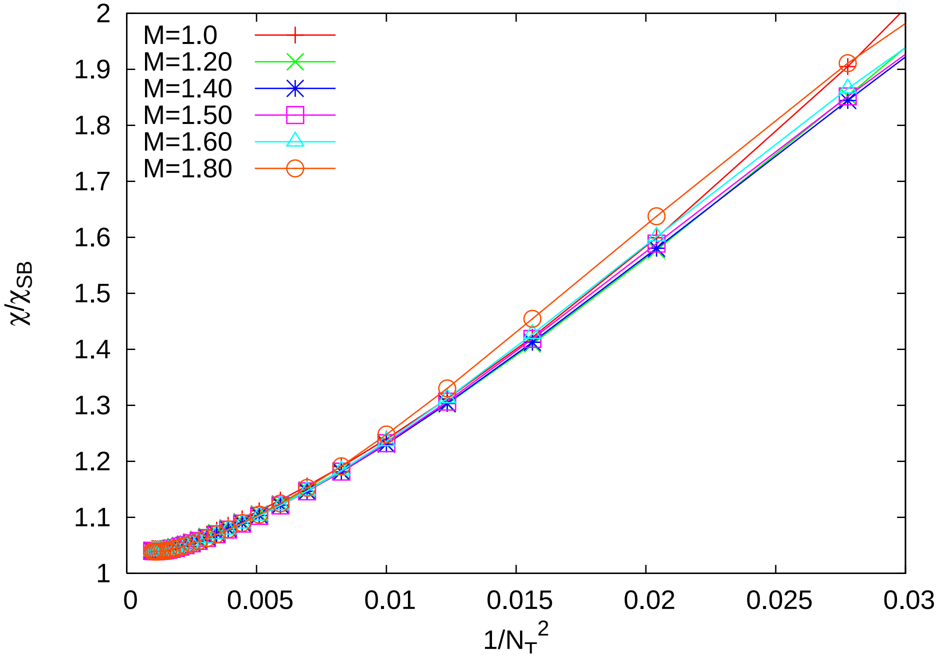
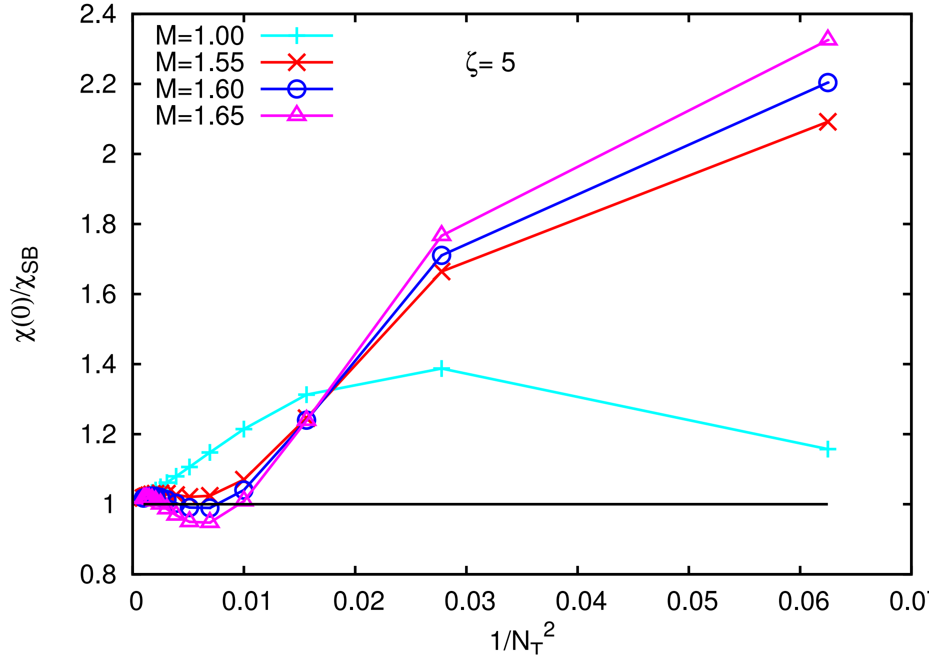


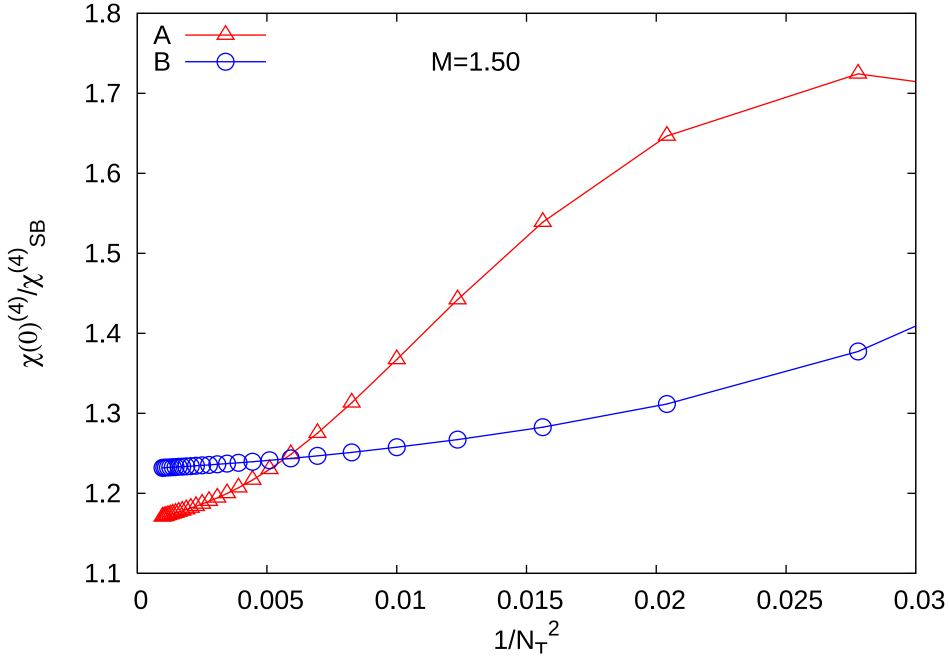




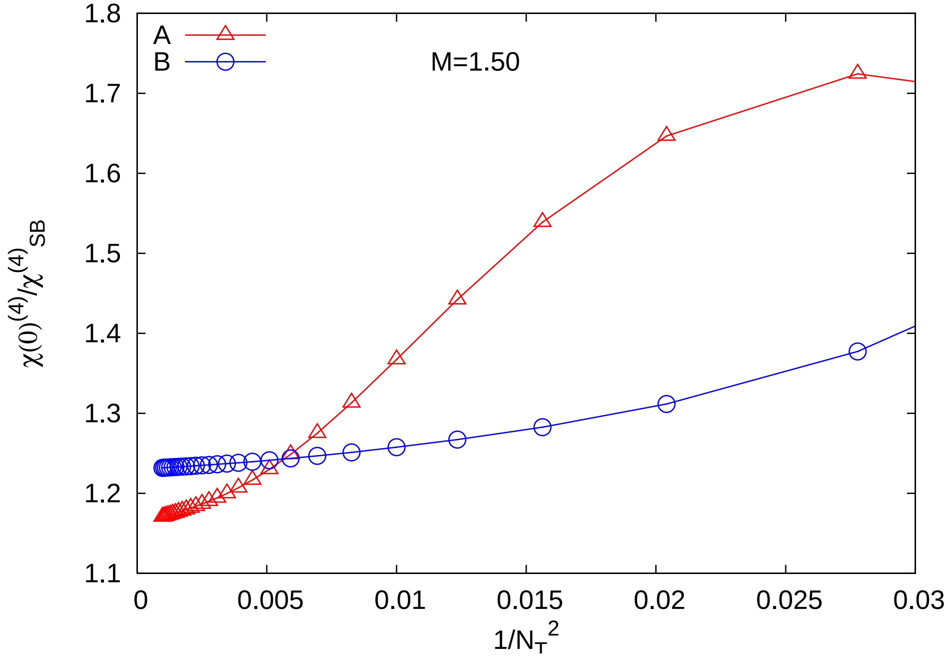


- Much less M-dependence for our proposal, and

- less oscillations compared to the Bloch-Wettig form.



- 4th Order Susceptibility : A) $\hat{\mu}/s$ and B) $\hat{\mu}/M$



- 4th Order Susceptibility : A) $\hat{\mu}/s$ and B) $\hat{\mu}/M$

- Divergences can be eliminated; M-dependence milder.

- Slow convergence to the expected continuum value; Can be improved by using higher-link derivatives, or even variations of the coefficients of the $\hat{\mu}$-term.

2nd Idea : Extend to Local Fermions

- We propose to introduce μ in general by

  $$S_F = \sum_{x,y} \bar{\Psi}(x) M(\mu; x, y) \Psi(y)$$
  $$= \sum_{x,y} \bar{\Psi}(x) D(x,y) \Psi(y) + \mu a \sum_{x,y} N(x,y).$$

  Here D can be the the staggered, overlap, the Wilson-Dirac or any other suitable fermion operator and $N(x,y)$ is the corresponding point-split and gauge invariant number density.

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- This leads to

  $M' = \sum_{x,y} N(x,y)$, and $M'' = M''' = M''''... = 0$,

  in contrast to the popular exp($\pm a\mu$)-prescription where *all* derivatives are nonzero:

  $M' = M'''... = \sum_{x,y} N(x,y)$ and $M'' = M'''' = M''''''... \neq 0$.

- Lot fewer terms in the Taylor coefficients, especially as the order increases. E.g., in the 4th order susceptibility, $\mathcal{O}_4 = -6 \ \mathrm{Tr} \ (M^{-1}M')^4$ in our proposal, compared to $\mathcal{O}_4 = -6 \ \mathrm{Tr} \ (M^{-1}M')^4 + 12 \ \mathrm{Tr} \ (M^{-1}M')^2 M^{-1}M'' - 3 \ \mathrm{Tr} \ (M^{-1}M'')^2 - 3 \ \mathrm{Tr} \ M^{-1}M'M^{-1}M''' + \mathrm{Tr} \ M^{-1}M''''$.

- $\mathcal{O}_8$ has one term in contrast to 18 in the usual case. $\Longrightarrow$ Number of M^{-1} computations needed are lesser.

Summary

- We showed, both perturbatively and non-perturbatively, that the introduction of nonzero μ leaves the anomaly unaffected. The zero modes of the Dirac operator for $\mu = 0$ govern it; nonzero μ simply scales the eigenvectors.
- A "gauge-like" symmetry in the continuum can be traced as the reason for anomaly preservation. Local lattice actions can easily adopt it. But how to ensure that for the overlap like non-local fermions remains unclear.

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- A "gauge-like" symmetry in the continuum can be traced as the reason for anomaly preservation. Local lattice actions can easily adopt it. But how to ensure that for the overlap like non-local fermions remains unclear.

- Overlap fermions at finite density could be studied by simply adding the μ-term linearly. The chiral symmetry breaking is similar but the inverse propagator simpler.

- Extending to staggered fermions, it may be less costly to implement this idea and may permit extensions to higher orders.