Statistical Modeling of SiPM Noise

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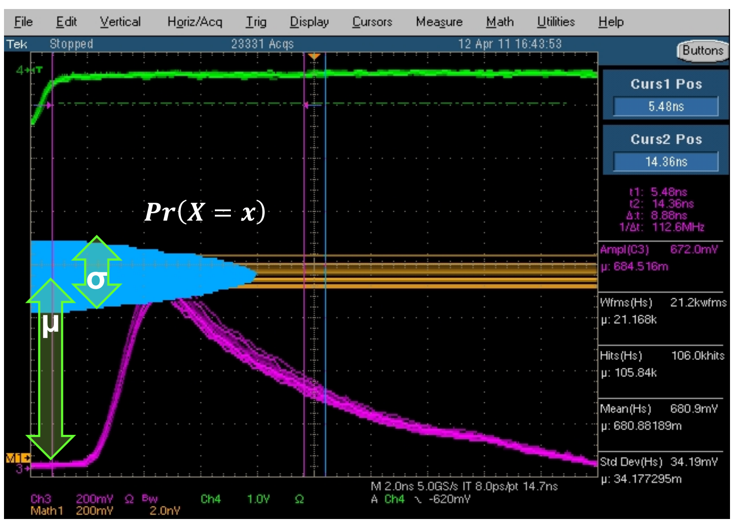
Scope & outline

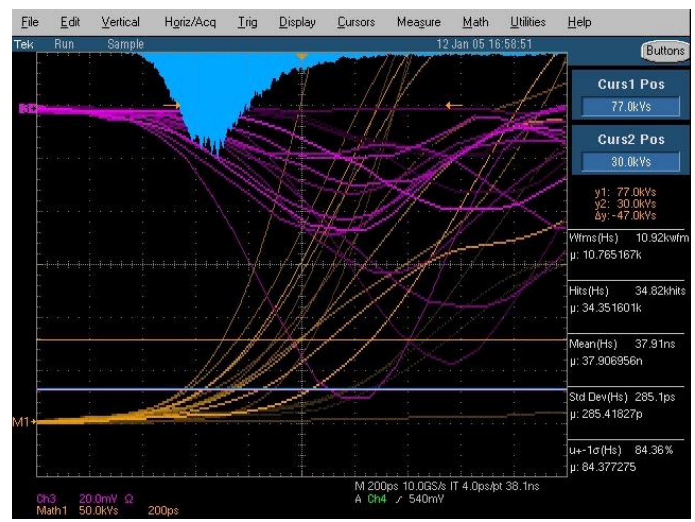
- Photon detection – a series of stochastic processes described by statistics
  - SiPM response – a result of the stochastic processes – a random variable
  - SiPM "noise" has a double meaning: as well as a "signal"
    - Physics point of view – specific nuisance contributions to the response
      - Dark counts - DCR
      - Crosstalk - CT
      - Afterpulsing – AP
    - Statistics point of view – standard deviation of the response
      - All above +…
      - Multiplication – Gain
      - Photo-conversion!!! – PDE

- Statistics of the response times
- Statistics of the response quantities
- Statistics of the response transients
-
- Excess noise factor – ENF – as a Figure of Merit for all noise contributions

Definitions of statistics

- Statistics – wide sense – math related to random variables
- Random variable is fully described by its probability distribution Pr(X)
- Statistic – narrow sense – any function on probability distribution
  - Statistical moments n^th order m_n of random variable X:
    - $m_n(X) = \sum_{i=0}^{\infty} i^n \Pr(X = i)$ X – discrete r.v.
    - $m_n(X) = \int_{-\infty}^{\infty} x^n \Pr(X = x)\, dx$ X – continuous r.v.
    - first moment Mean (μ) = m_1 ;
    - second central moment – Variance (σ^2) = $m_2 - m_1^2$





Statistics of random times

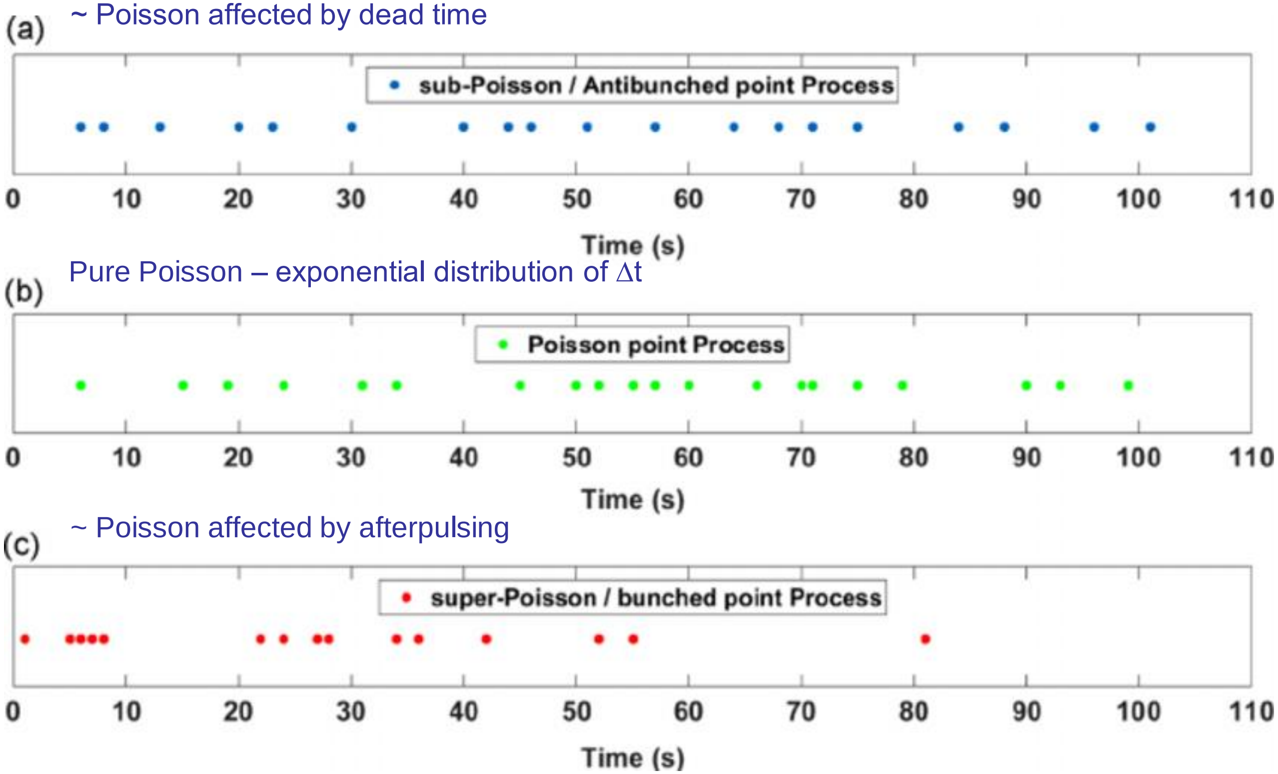
- **How probability distribution of signal and noise event times are transformed by photon detection processes**

Statistics of random times

~ Poisson affected by dead time

Pure Poisson – exponential distribution of Δt

~ Poisson affected by afterpulsing



https://doi.org/10.1016/j.nima.2015.07.009

Pr(Δt) or PDF– full characterization
Mean and Variance – not so obvious because affected by recovery losses and correlated events

Poisson process:

No events in $\Delta t: \Pr(\Delta t \mid 0) = e^{-DCR \cdot \Delta t}$

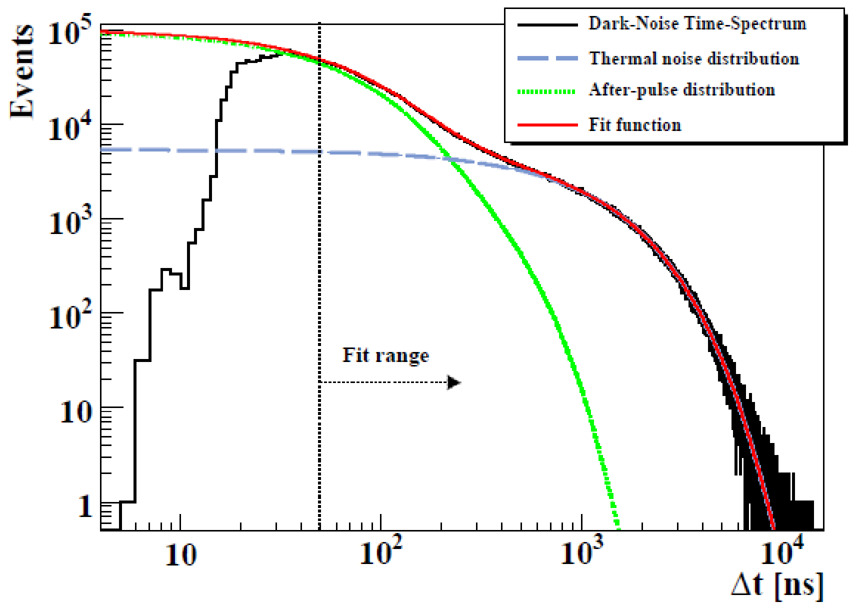
Interval between events $: \Pr(\Delta t \mid > 0) = 1 - e^{-DCR \cdot \Delta t}$

Probability density function: $PDF(\Delta t) = DCR \cdot e^{-DCR \cdot \Delta t}$

Mean time $\overline{\Delta t} = 1/DCR$ Standard deviation $\sigma(\Delta t) = \overline{\Delta t} = 1/DCR$

Analysis based on probability density function (PDF) = histogram of interarrival times

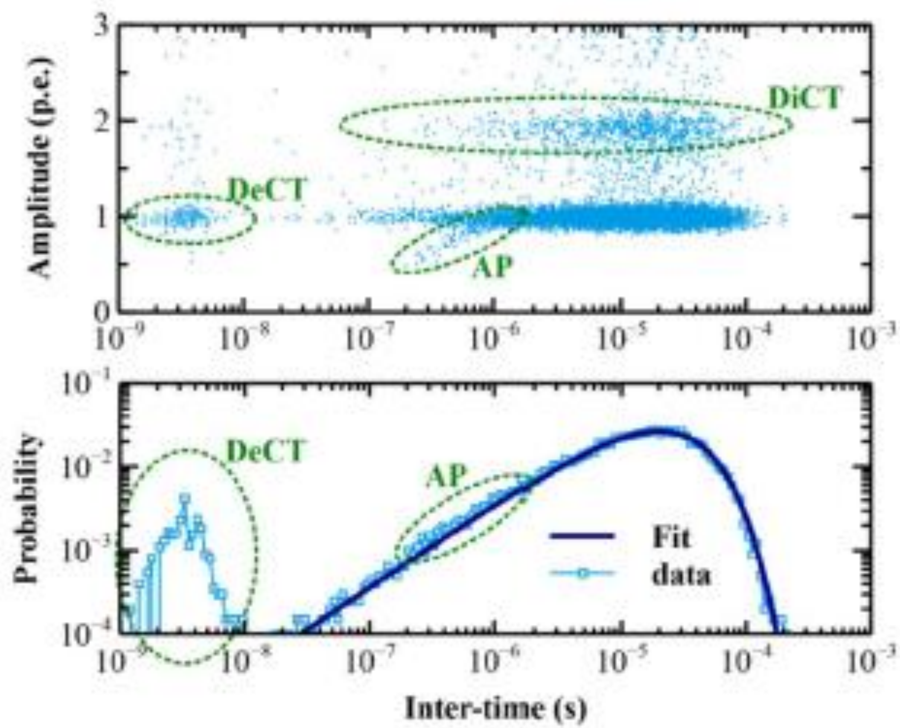
- Common-sense analysis of DCR and correlated events
- Contributions are fitted separately in specific time frames



P. Eckert et al, NIMA 2010

$$n_{tp}(\Delta t) = N_{tp}/\tau_{tp} \cdot e^{-\frac{\Delta t}{\tau_{tp}}}$$

$$n_{ap}(\Delta t) = N_{apf}/\tau_{apf} \cdot e^{-\frac{\Delta t}{\tau_{apf}}} + N_{aps}/\tau_{aps} \cdot e^{\frac{-\Delta t}{\tau_{aps}}}$$



F. Acerbi et al, TNS 2016

Analysis based on cumulative distribution function (CDF) = list of interarrival times

- Complimentary CDF approach: probabilities of independent “zero” dark and “zero” correlated events in Δt are multiplied:

$$CCDF_{total} = CCDF_{corr} \cdot CCDF_{dark} \qquad CCDF = 1 - CDF \quad \text{complimentary CDF}$$

$$1 - F_{total}(t, P_{corr}, DCR) = \left(1 - F_{corr}(t, P_{corr})\right) \cdot \left(1 - F_{dark}(t, DCR)\right) \Big| \, F_{dark}(t) = 1 - \exp(-DCR \cdot t)$$

So, now we've got all to extract

pure distribution of correlated events $F_{corr}(t, P_{corr})$:

$$F_{corr}(t, P_{corr}) = 1 - \left(1 - F_{total}(t, N_{ph}, DCR)\right) \cdot \exp(DCR \cdot t)$$

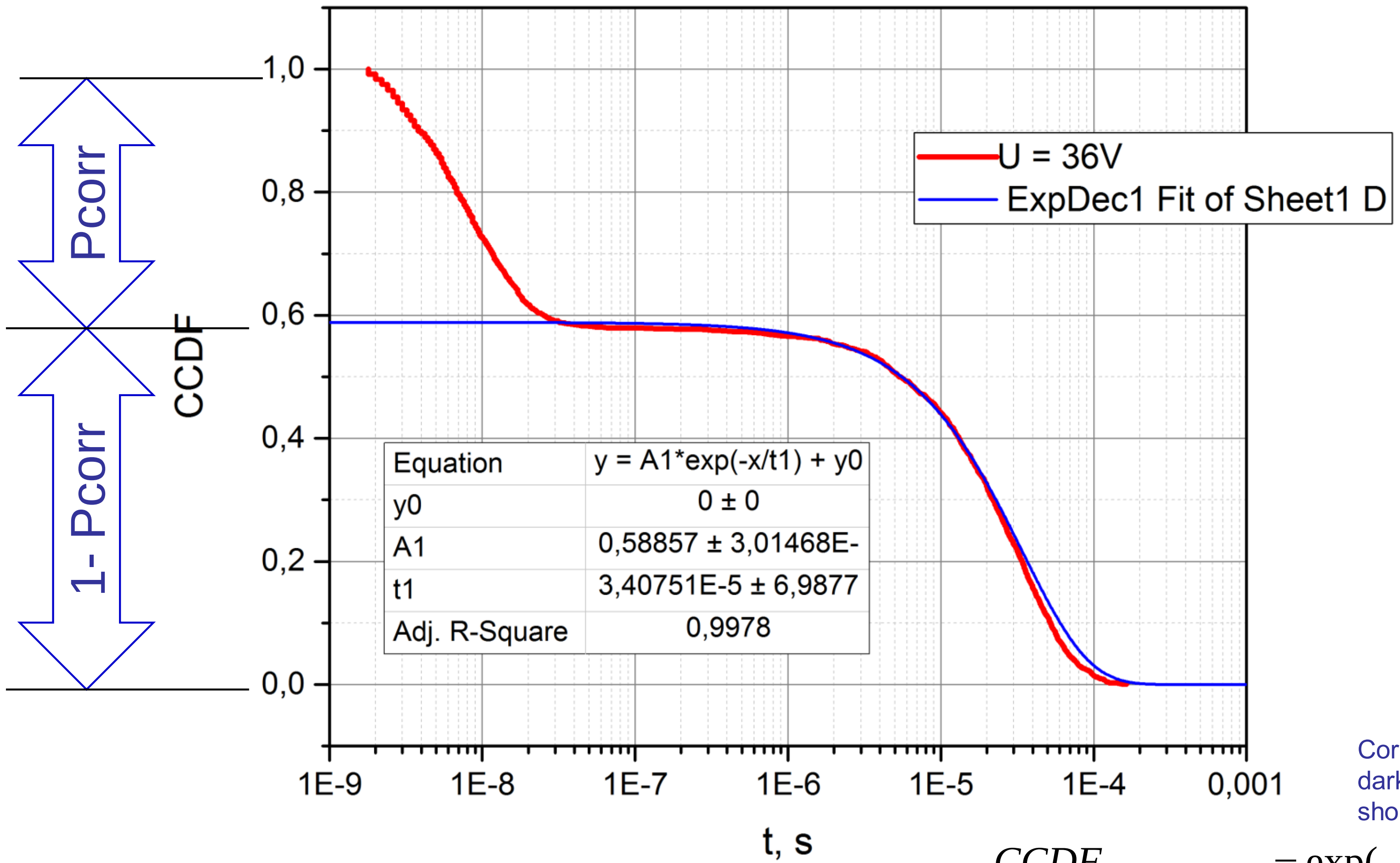
And analyse $F_{corr}(t, P_{corr})$ or $f_{corr}(t, P_{corr})$ without any assumptions on its shape

Moreover, we can further extend (C)CDF approach

for separation of crosstalk and afterpulses

$$CCDF_{corr} = CCDF_{CT} \cdot CCDF_{AP} \ldots\ldots$$

CCDF time distribution: dark and correlated events



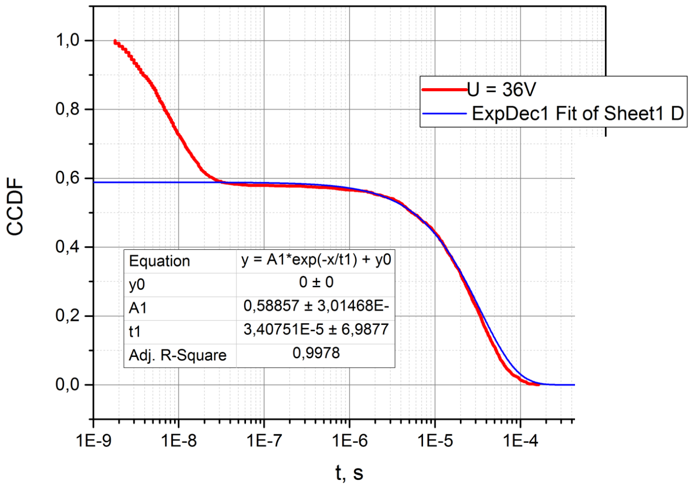
Correction on lost dark counts at Tmax should be applied:

$$CCDF_{dark\_correction} = \exp(-DCR \cdot T_{\max})$$

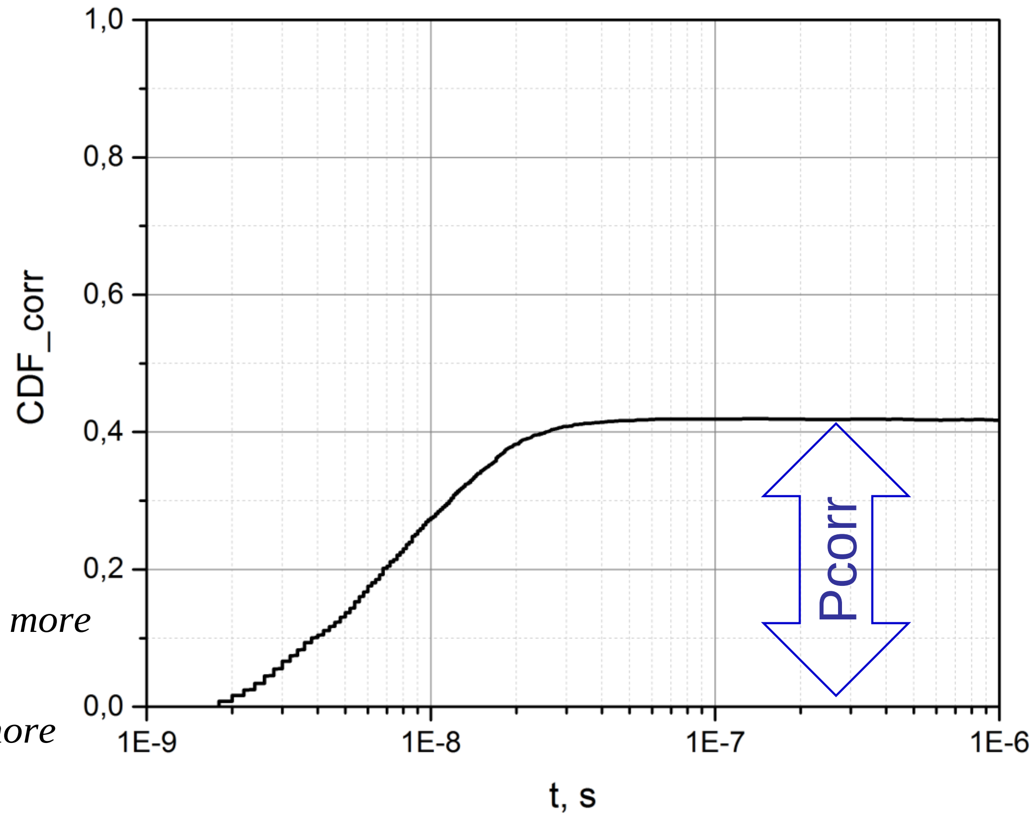
Experimental example – courtesy of E. Popova, D. Philippov… , NSS/MIC 2016

CCDF time distribution: reconstruction of correlated event CDF

More details – next talk by Eugen Engelmann @ ICASiPM



$$F_{corr}(t,P_{corr})=1-\left(1-F_{total}(t,N_{ph},DCR)\right)\cdot\exp(DCR\cdot t)$$



Now we are ready to play with correlated event CDF or PDF applying any models and analysis.

For example, typical assumption on exponential time distribution, but take care about multiple exponents (see backup):

$$F_{corr}(t,P_{corr},\tau_{corr})=1-P_{corr}e^{\frac{-t}{\tau_{corr}}} \quad ...and\ \ more$$

$$f_{corr}(t,P_{corr},\tau_{corr})=\frac{P_{corr}}{\tau_{corr}}e^{\frac{-t}{\tau_{corr}}} \quad ...and\ \ more$$

Experimental example – courtesy of E. Popova, D. Philippov , NSS/MIC 2016

Empirical CDF (list mode) vs PDF (histogram):
highest precision due to full information from all data points

- Histograming:
  - Problem of appropriate selection of bin size / number *Nbins*
  - Lost of measured data *Npoints* → *Nbins*

  - Kolmogorov-Smirnov test for "goodness-of-fit" problem is based on Kolmogorov metrics on distribution **distance**:

$$K(a,b) = \max|CDFa(x) - CDFb(x)| \quad -\infty < x < +\infty$$

Application to measurements of the same distribution [1]:

self-distance of two Empirical CDFs (ECDF): $= \sqrt{\pi}\ln(2) \cdot \sqrt{\frac{1}{Npoints}}$

self-distance of two Empirical PDFs (histogram): $= \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{Nbins}{Npoints}}$

[1] Y. Cao, L. Petzold, "Accuracy limitations and the measurement of errors in the stochastic simulation…", J. Comput. Phys. 212, 2006

Statistics of random quantities

- **How probability distributions of signal and noise quantities are transformed from input to output**

Statistics of random quantities

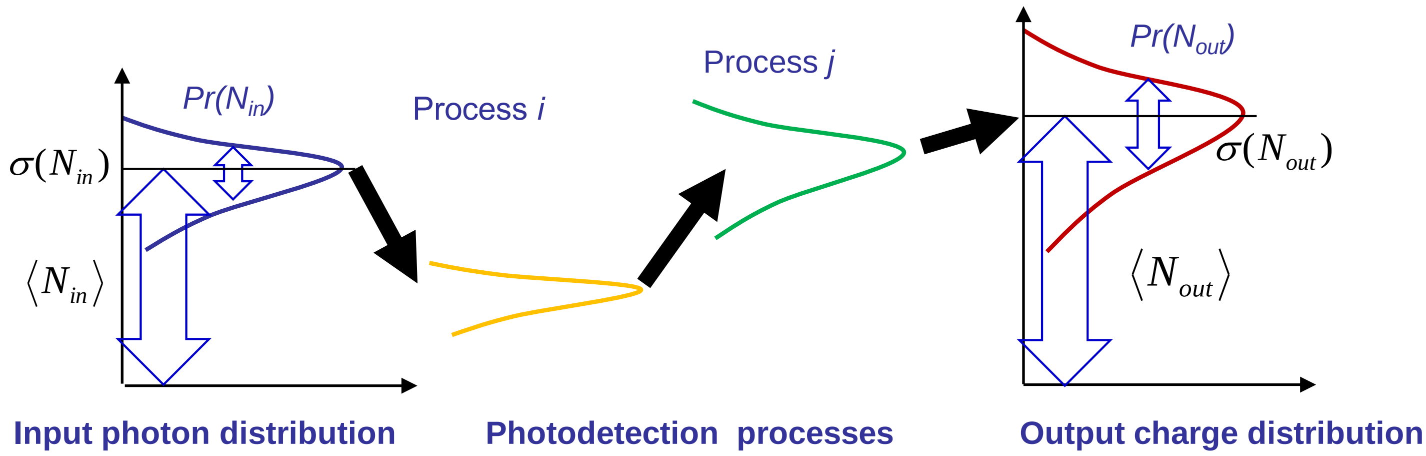
- Pr(*Nout*|*Nin*) – full characterization
- Mean and Variance – partial characterization

Photons (Poisson) ⇒ photoconversion (Bernulli) ⇒ photoelectrons (Poisson)

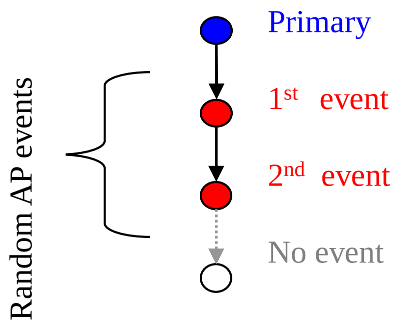
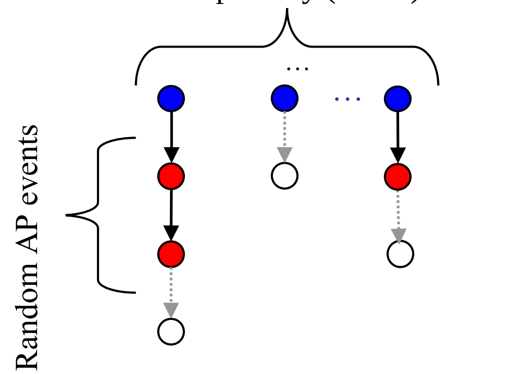
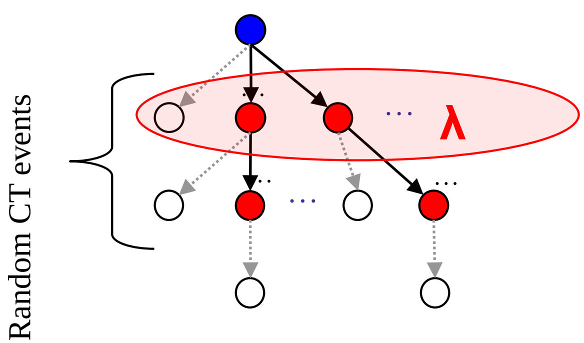
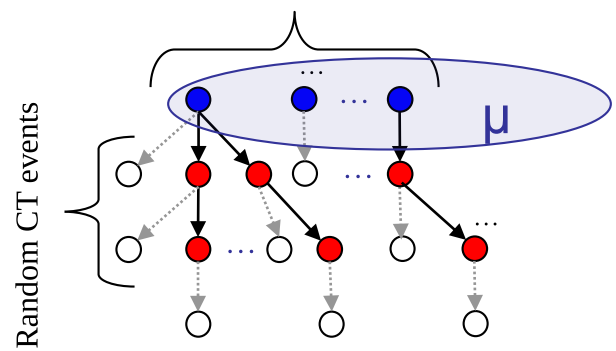
Dark+photoelectrons (Poisson) ⇒ triggering (Bernulli) ⇒ primary avalanches (Poisson)

Primary avalanches (Poisson) ⇒ *CT*, *AP* (TBD) ⇒ secondary avalanches (TBD)

All avalanches(TBD) ⇒ multiplication (TBD) ⇒ output electrons (TBD)



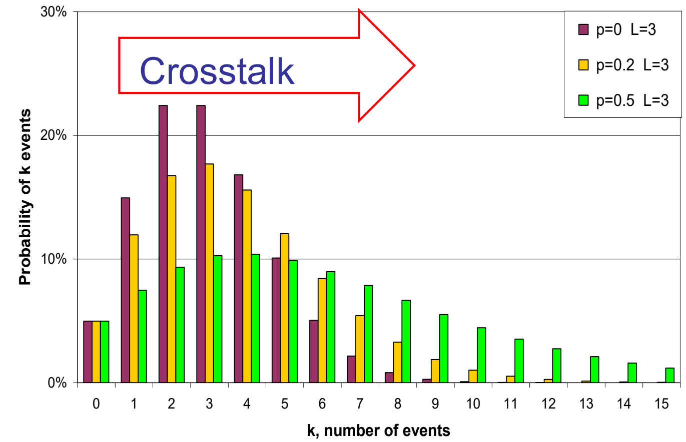
Correlated stochastic processes of CT & AP

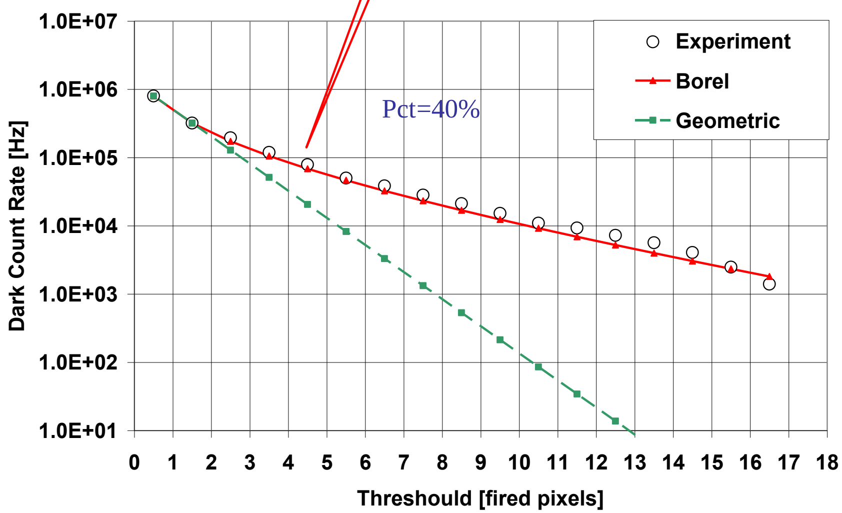
Process Models	Single primary event N≡1 e.g. SiPM dark electron spectrum	Poisson number of primaries <N>=μ e.g. SiPM photoelectron spectrum
Geometric Chain Process	Non-random (Dark) event 	Random primary (Photo) events 
Branching Poisson Process	Non-random (Dark) event 	Random primary (Photo) events 

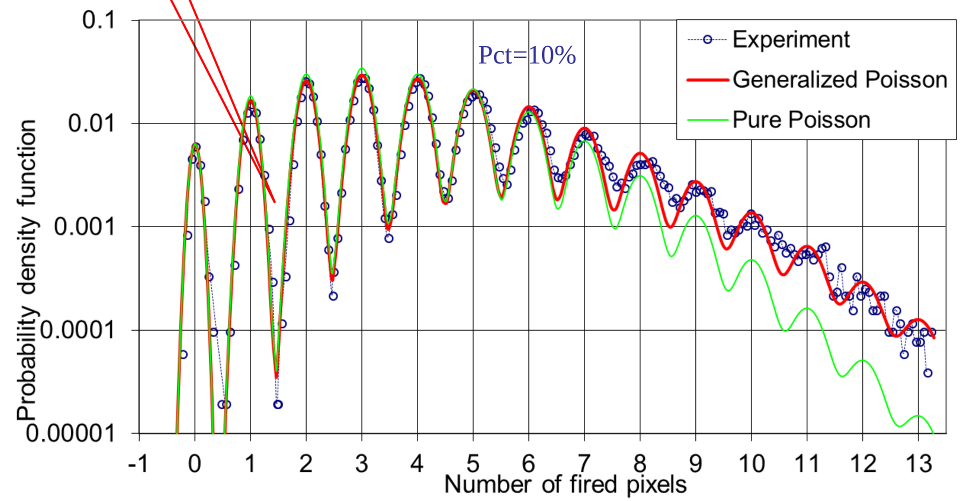
S. Vinogradov, IEEE NSS/MIC 2009, TNS 2011, NDIP 2011

Analytical results for CT & AP statistics

Models	Geometric chain process		Branching Poisson process	
Primary event distribution	Non-random single ($N\equiv1$)	Poisson (μ)	Non-random single ($N\equiv1$)	Poisson (μ)
Total event distribution	Geometric (p)	Compound Poisson (μ, p)	Borel (λ)	Generalized Poisson (μ, λ)
$P(X=k)$	$p^{k-1}\cdot(1-p)$	$\tilde{L}\left\{e^{\mu\frac{s-1}{1-p\times s}}\right\}$	$\frac{(\lambda\cdot k)^{k-1}\cdot\exp(-k\cdot\lambda)}{k!}$	$\frac{\mu\cdot(\mu+\lambda\cdot k)^{k-1}\cdot\exp(-\mu-k\cdot\lambda)}{k!}$
$E[X]$	$\frac{1}{1-p}$	$\frac{\mu}{1-p}$	$\frac{1}{1-\lambda}$	$\frac{\mu}{1-\lambda}$
$Var[X]$	$\frac{p}{(1-p)^2}$	$\frac{\mu\cdot(1+p)}{(1-p)^2}$	$\frac{\lambda}{(1-\lambda)^3}$	$\frac{\mu}{(1-\lambda)^3}$
ENF	$1+p$		$\frac{1}{1-\lambda}=\frac{1}{1+\ln(1-p)}\approx1+p+\frac{3}{2}p^2+o(p^3)...$	







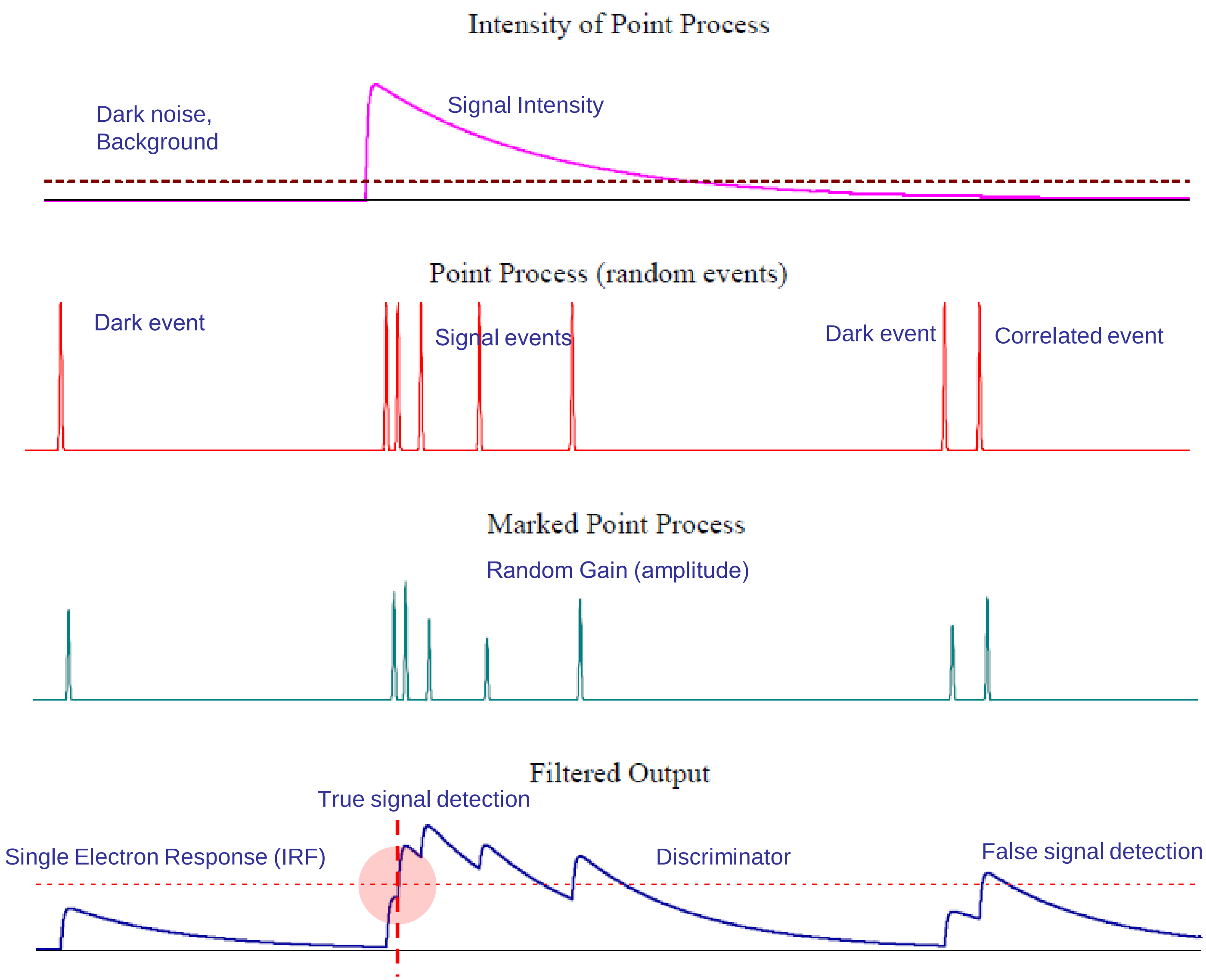
S. Vinogradov, NDIP 2011 (experiment – R. Mirzoyan, 2008 (MEPhI SiPM))

S. Vinogradov, NDIP 2011 (experiment – LPI, Hamamatsu MPPC)

Statistics of transient signals - stochastic functions

- **How random quantities are developed in time**

Statistics of filtered marked correlated point processes



Mean and Variance of filtered marked Poisson point process

$$X_{in}(t) = \sum_{i=1}^{N} \delta(t - t_i) \qquad N - Poissonian \qquad t_i - iidrv(i = 1...N)$$

$$Y_{out}(t) = \sum_{i=1}^{N} IRF(t - t_i) \qquad IRF(t) = A_i \cdot h(t) \qquad A_i - iidrv(i = 1...N)$$

$$E[Y_{out}(t)] = E[X_{in}(t)] * E[IRF(t)] = \overline{A} \cdot [\lambda * h](t) \qquad N = \int_0^t \lambda(t')dt'$$

$$Var[Y_{out}(t)] = COV[X_{in}] * COV[IRF](t - t') \Big|_{t'=t} = \overline{A}^2 \cdot \left(1 + \frac{\sigma_A^2}{\overline{A}^2}\right) \cdot [\lambda * h^2](t)$$

$$REMARK : COV[X_{in}] = \lambda(t) \cdot \delta(t - t') \qquad Var[X_{in}(t)] \underset{t' \to t}{\to} \infty$$

SIPM specific

$$E[Vout(t)] = \overline{V}_{ser} \cdot [\lambda * h](t) \qquad \overline{V}_{ser} \approx \frac{q \cdot \overline{Gain} \cdot R_{load}}{\tau_{fall}} \qquad \lambda(t) = N_{ph} \cdot PDE \cdot [\rho_{ph} * \rho_{sptr}](t)$$

$$Var[V_{out}(t)] = \overline{V}_{ser}^2 \cdot ENF_{gain} \cdot [\lambda * h^2](t) + V_{noise}^2$$

Example: baseline fluctuation due to DCR

$$\lambda(t) = DCR = const \qquad h(t) = e^{-t/\tau_{fall}} - e^{-t/\tau_{rise}}$$

$$\mathrm{Var}[V_{out}] = \overline{\mathrm{V}}_{ser}^2 \cdot ENF \cdot DCR \cdot \frac{\tau_{fall} + \tau_{rise}}{2}$$

Figure of Merit for all noise contributions

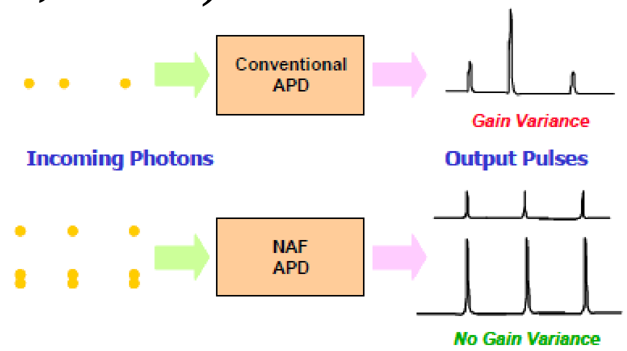
- **How to compare noises and optimize SiPM operations:**

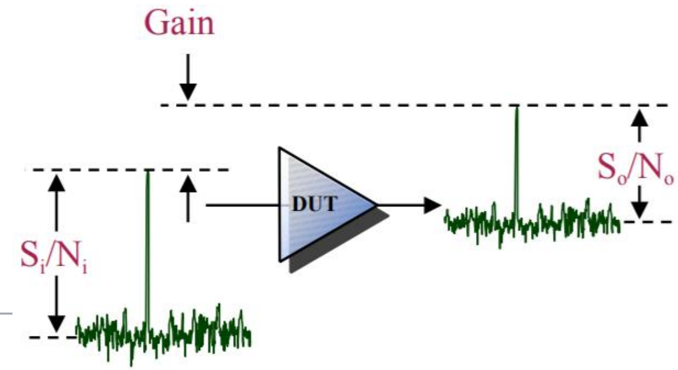
  Excess Noise Factor – ENF – approach

ENF: two definitions

- ENF in multiplying photodetectors (APD, PMT):

$$ENF = \frac{m_2(M)}{m_1^2(M)} = 1 + \frac{\sigma_M^2}{\overline{M}^2}$$





- ENF in amplifying electronics:

Definition [edit]

The **noise factor** F of a system is defined as[3]

$$F = \frac{\mathrm{SNR_{in}}}{\mathrm{SNR_{out}}},$$

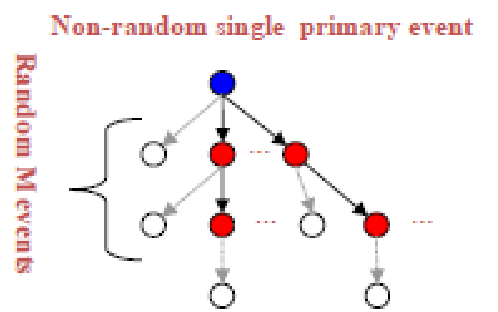
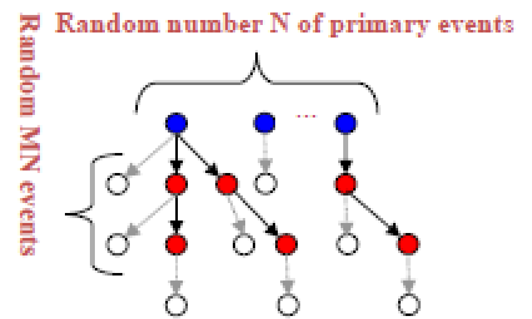
$$ENF = \frac{\mu_{in}^2}{\sigma_{in}^2} \Big/ \frac{\mu_{out}^2}{\sigma_{out}^2} = Res_{out}^2 / Res_{in}^2$$

Agilent Technologies

where $\mathrm{SNR_{in}}$ and $\mathrm{SNR_{out}}$ are the input and output signal-to-noise ratios respectively. The SNR quantities are power ratios.

Noise factor (F) measures degradation of the signal-to-noise ratio (SNR), caused by components in a signal chain.

ENF: two definitions are equal for Poisson input

Statistics of Random Amplification processes		
Input process:	**Single primary event N≡1**	**Random number of primaries N**
Output process: **M random events** **<M>, σ(M)**		

$$ENF(X_{in} \equiv 1) = 1 + \frac{\sigma_M^2}{\overline{M}^2} \qquad vs \qquad ENF(X_{in} = N) = 1 + \frac{\sigma_M^2}{\overline{M}^2} \cdot \frac{1}{Fano(N)}$$

$$\text{Input quantity N} \in \text{Poisson} \quad \Rightarrow \quad ENF(X_{in} = N) = ENF(X_{in} \equiv 1)$$

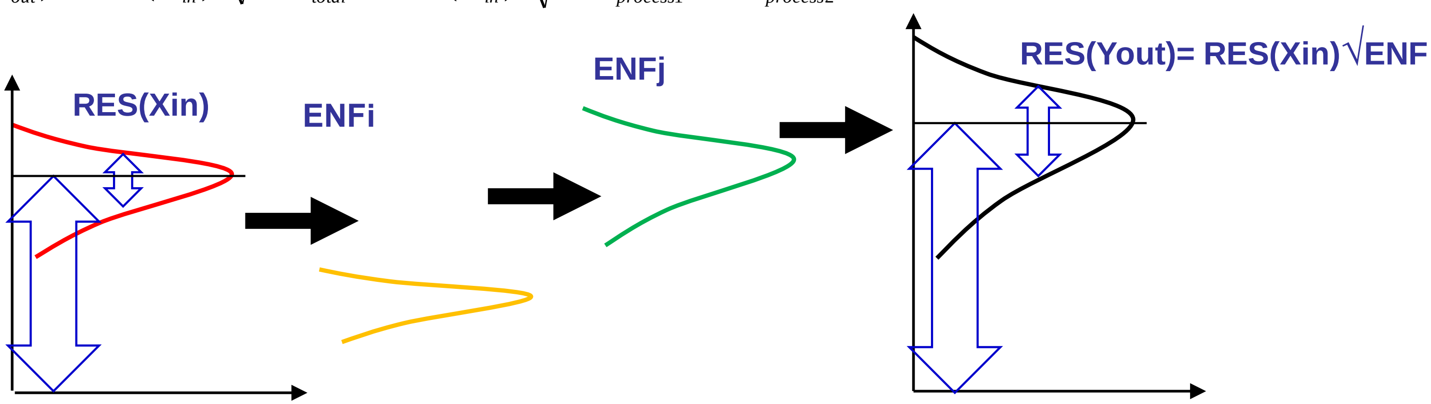
ENF as a measure of SNR (resolution) degradation

- Resolution at output is a product of Resolution at input and $\sqrt{ENF(X_{in}\equiv 1)}$

- Total ENF for a sequence of specific processes is approximately

$$RES(Y_{out}) = RES(X_{in})\cdot\sqrt{1+\frac{\sigma_{out}^2(1)}{\overline{Y_{out}(1)}^2}\cdot\frac{1}{Fano(X_{in})}}\Bigg|_{Fano(X_{in})=1} = RES(X_{in})\cdot\sqrt{ENF(X_{in}\equiv 1)}$$

$$RES(Y_{out}) = RES(X_{in})\cdot\sqrt{ENF_{total}} = RES(X_{in})\cdot\sqrt{ENF_{process1}\cdot ENF_{process2}\cdots}$$



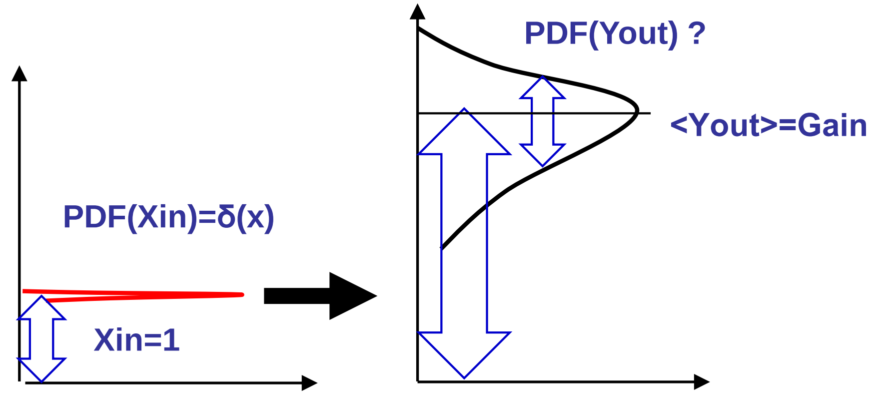
Input photon distribution **Photodetection processes** **Output charge distribution**

ENF related to Gain, PDE and DCR

- Multiplication (single electron)

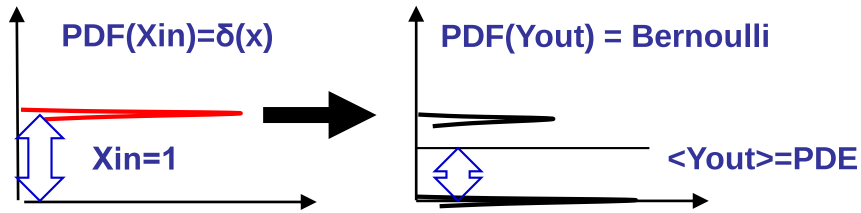
$$ENF_{gain} = 1 + \frac{\sigma_{gain}^2}{\overline{Gain}^2}$$

$$ENF_{gain} \approx 1.01 \ldots 1.05 \quad \text{for most of SiPM}$$



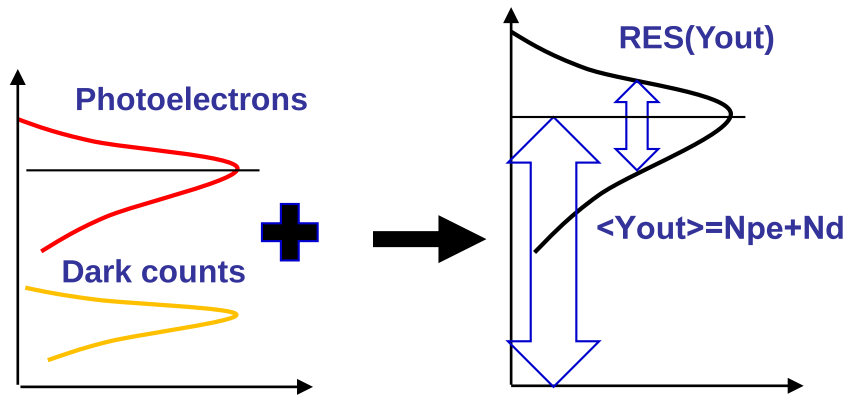
- Photon detection (single photon)

$$ENF_{pde} = 1 + \frac{PDE \cdot (1 - PDE)}{PDE^2} = \frac{1}{PDE}$$



- Dark counts (Poisson dark+pe)

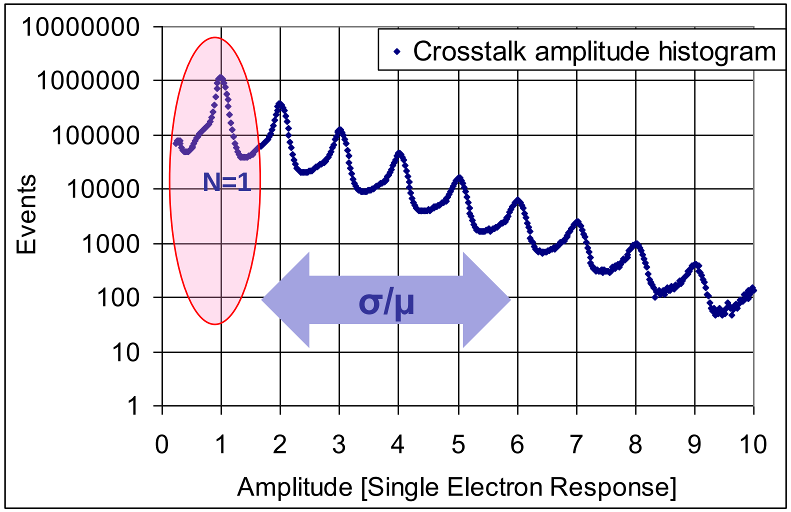
$$ENF_{dcr} = 1 + \frac{DCR \cdot t}{N_{pe}}$$



Methods of ENF measurements

N=1 non-random primary

$$ENF(X_{in} \equiv 1) = 1 + \frac{\sigma_{out}^2}{\mu_{out}^2}$$

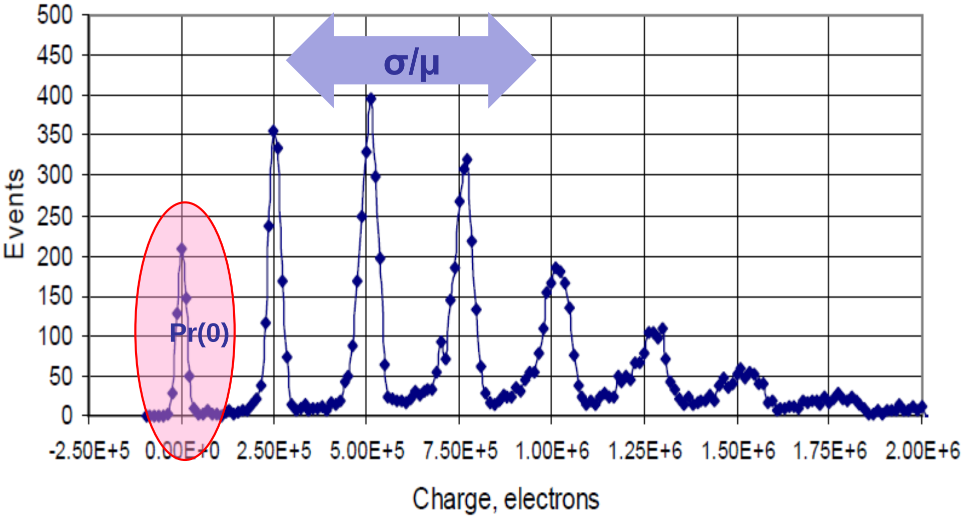


N ϵ Poisson random primaries

$$ENF(X_{in} = N) = \frac{\mu_{in}^2}{\sigma_{in}^2} \Big/ \frac{\mu_{out}^2}{\sigma_{out}^2} = N \cdot \frac{\sigma_{out}^2}{\mu_{out}^2}$$

$$N = -\ln(\Pr(0))$$

Nph could also be measured by reference PD



Remark: ENF measurements in electronics: noise generator + noise power meter

Key SiPM parameters in ENF metrics

$$PNR = \frac{\sigma(N_{out*})}{\mu(N_{out*})} = \frac{\sigma(N_{ph})}{\mu(N_{ph})}\sqrt{ENF_{total}} \qquad ENF_{total} \approx F_m \cdot F_{pde} \cdot F_{dcr} \cdot F_{ct} \cdot F_{ap} \cdot F_{nl} \qquad DQE = \frac{1}{ENF_{total}}$$

Noise source	Key parameter	Typical values for SiPM	ENF expression	SiPM ENF	PMT ENF
Fluctuation of multiplication	Mean *gain* and standard deviation of *gain*	μ(gain) ~ 10^6 σ(gain) ~ 10^5	$Fm = 1 + \frac{\sigma^2(gain)}{\mu^2(gain)}$	1.01	1.2
Crosstalk	Probability of crosstalk event *Pct*	5% – 40%	$Fct = \frac{1}{1 + \ln(1 - Pct)}$	1.05 – 2	1
Afterpulsing	Probability of afterpulse *Pap*	5% – 20%	$Fap = 1 + Pap$	1.05 – 1.2	1.01
Shot noise of dark counts	Dark count rate *DCR* Signal events in *Tpulse*	10^5 – 10^6 cps	$Fdcr = 1 + \frac{DCR \cdot Tpulse}{Nph \cdot PDE}$	1.01	1
Noise / losses of photon detections	Photon detection efficiency *PDE*	20% – 50%	$Fpde = \frac{1}{PDE}$	2 – 5	3 – 5
Total noise in linear dynamic range	Total *ENF_linear*	$ENF\_linear = Fm \cdot Fct \cdot Fap \cdot Fdcr \cdot Fpde$		3 – 12	4 – 6
Binomial nonlinearity	Mean number of possible single pixel firings *n*	$n = \frac{Nph \cdot PDE}{Npix}$; $ENFpix = \frac{\exp(n) - 1}{n}$; binomial distribution			
Dead time nonlinearity	Mean rate of possible single pixel firings *λ*	$\lambda = \frac{n}{Tpulse}$; $ENFrec = 1 + \lambda \cdot Trec$; nonparalizible model			

The end

Thank you for your attention!

Questions?

Objections?

Opinions?

…

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FBK results on Generalized Poisson model

$$\mathrm{ENF} = \frac{1}{(1-\gamma)} = \mathrm{ECF}. \quad (7)$$

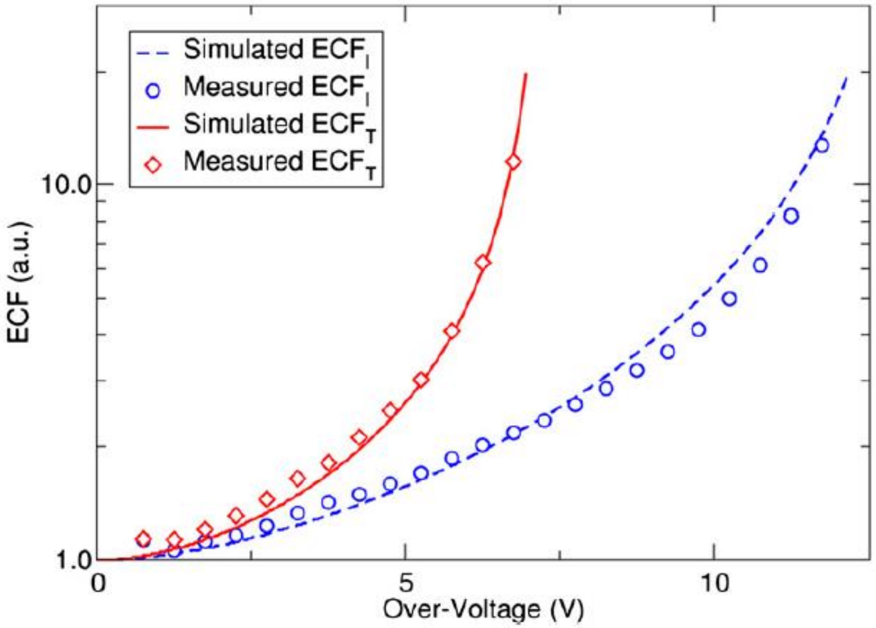

Figure 5. Comparison between the simulated and measured values of ECF for the internal and for the total (internal + external) crosstalk.

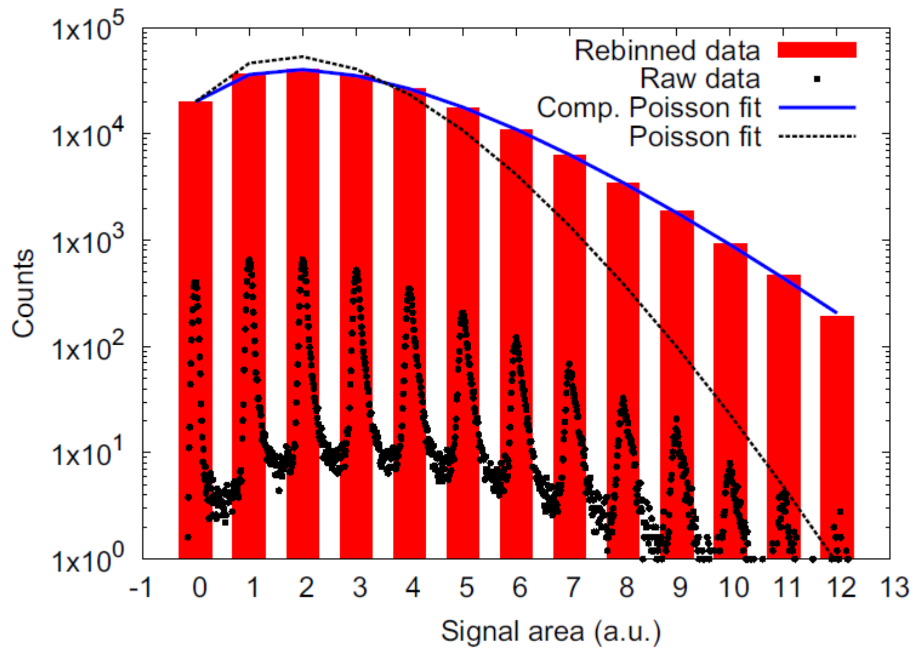

Fig. 8. Histogram of the signal area of a 50 × 50–μm^2 RGB SiPM excited with short and faint light pulses. Measurement conditions: $T = 20$ °C; $V_{OV} = 3$ V.

A. Gola, A. Ferri, A. Tarolli, N. Zorzi, and C. Piemonte, "SiPM optical crosstalk amplification due to scintillator crystal: effects on timing performance," *Phys. Med. Biol.*, vol. 59, no. 13, p. 3615, 2014.

DESY results on Generalized Poisson

- E. Garutti group @ DESY: V. Chmill et al., NIMA 2017

V. Chmill et al. Nuclear Instruments and Methods in Physics Research A 854 (2017)