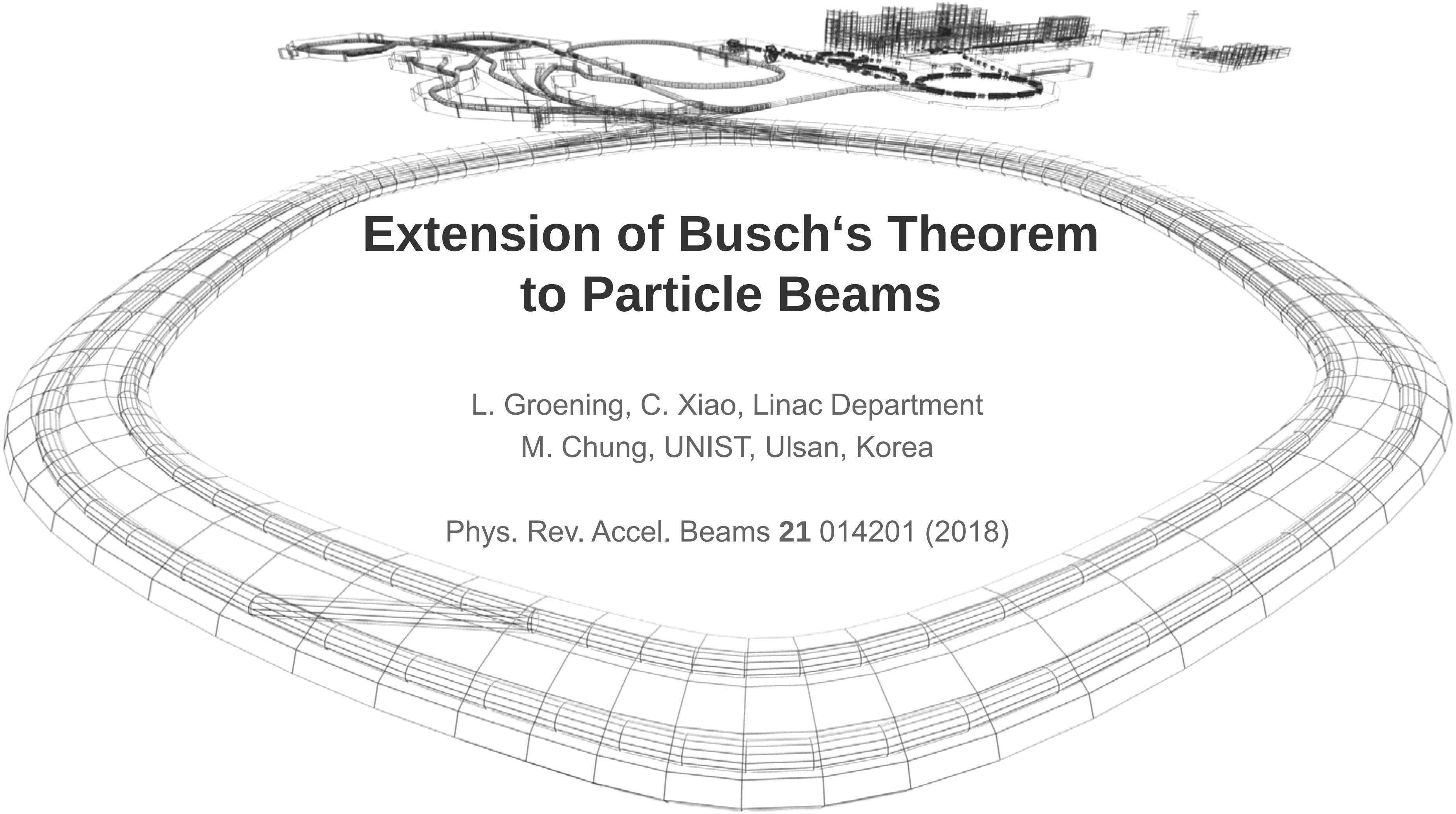

Extension of Busch‘s Theorem to Particle Beams

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Outline

- Hamiltonian, conjugate variables (short & basic)
- Busch theorem on single particle
- Applications:
  - electron cooling
  - magnetic bottle
- Projected rms-emittances, eigen emittances
- Extension of Busch theorem to beams
- Applications:
  - increase of FEL gain & collider luminosity
  - increase of injecion efficiency into ion rings

Hamiltonian, conjugate variables

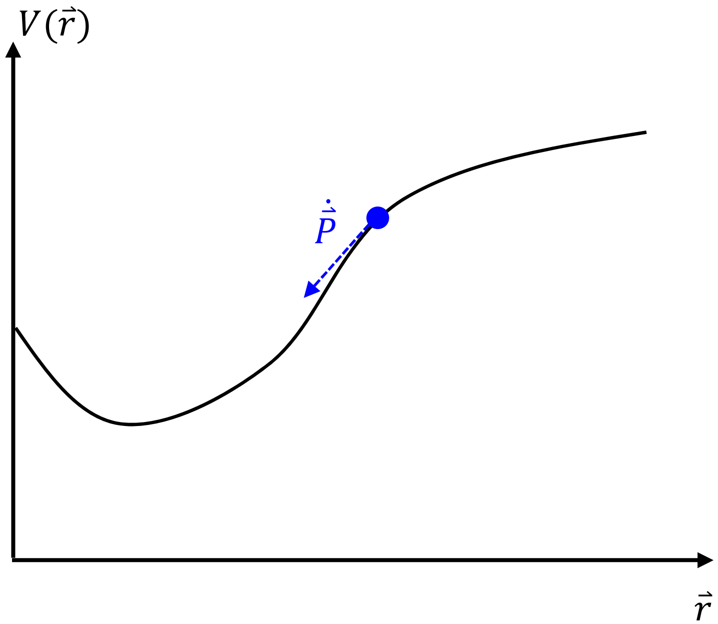
<u>Hamiltonian is total energy:</u>

$$H := E_{kin} + E_{pot}$$

$$H(\vec{p},\vec{r}) := \frac{\vec{P}^2}{2m} + V(\vec{r})$$

$$\vec{r} := \text{position}$$

$$\vec{P} := \text{mechanical momentum} = m\vec{v} := m\dot{\vec{r}}$$



<u>equations of motion through derivatives of *H* :</u>

$$\boxed{\begin{aligned} \dot{\vec{P}} &:= -\frac{\partial H}{d\vec{r}} = -\frac{\partial V}{d\vec{r}} = \vec{F} \\ \dot{\vec{r}} &:= \frac{\partial H}{\partial \vec{P}} = \frac{\vec{P}}{m} \end{aligned}}$$

$\vec{r}$ and $\vec{P}$ are conjugate variables

Force from magnetic field

force on particle by constant magnetic field $\vec{B} := \begin{bmatrix} 0 \\ 0 \\ B_s \end{bmatrix} = \vec{\nabla} \times \vec{A}$

$$\vec{F} = eq(\vec{v} \times \vec{B})$$

$$F_x = \dot{P}_x = \quad eqv_y B_s$$

$$F_y = \dot{P}_y = -\,eqv_x B_s$$

to obtain equations of motion from Hamiltonian mechanics → use „generalized" momentum

$\vec{p} := \vec{P} + eq\vec{A}$ in the Hamiltonian:

$H(\vec{r}, \vec{P}) = \tilde{H}(\vec{r}, \vec{p}) = \frac{(\vec{p} - eq\vec{A})^2}{2m}$ and apply previous formalism :

$\dot{\vec{p}} = \frac{\partial \tilde{H}}{\partial \vec{r}}$ together with $\vec{A} = \frac{1}{2} \begin{bmatrix} -yB_s \\ xB_s \\ 0 \end{bmatrix}$ and $\dot{\vec{p}} = \dot{\vec{P}} + eq\dot{\vec{A}}$

Generalized angular momentum

- analogue to generalized momentum, the generalized angular momentum is defined

  $\vec{L} \coloneqq \vec{r} \times \vec{p}$

- if there is just magnetic field, the generalized angular momentum is preserved

  $\vec{L} = \vec{r} \times \left(\vec{P} + eq\vec{A}\right) = const$

- Busch theorem is special case for :

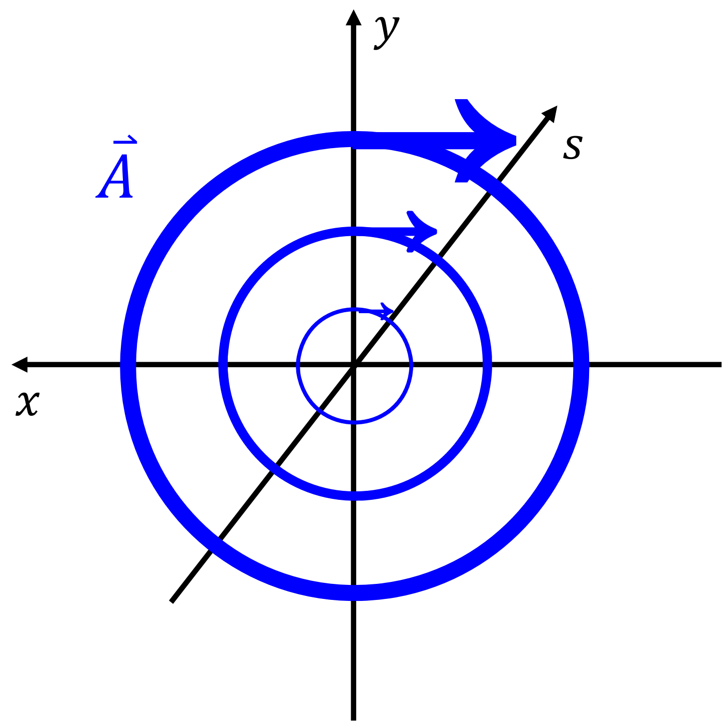
  - cylindrical symmetric magnetic field $\vec{B} = \vec{B}(s)$

  - s-component of $\vec{L}$: $L_s = const$

Busch Theorem

cylindrically symmetric magnetic field with $\vec{\nabla}\,\vec{B} = 0$

$$\vec{B} = \frac{1}{2}\begin{bmatrix} -xB_s{}' \\ -yB_s{}' \\ 2B_s \end{bmatrix} = \vec{\nabla} \times \vec{A} = \frac{1}{2}\vec{\nabla} \times \begin{bmatrix} -yB_s \\ xB_s \\ 0 \end{bmatrix}$$



$$L_s = \left[\vec{r} \times (\vec{P} + eq\vec{A})\right] \cdot \vec{e}_s = const$$

- particle entering into region with $\vec{B}$ (hence $\vec{A}$) acquires orbital angular momentum
- $\vec{r} \times \vec{A}$ has angular momentum as well
- sum of both acquired angular momenta is zero

Busch Theorem

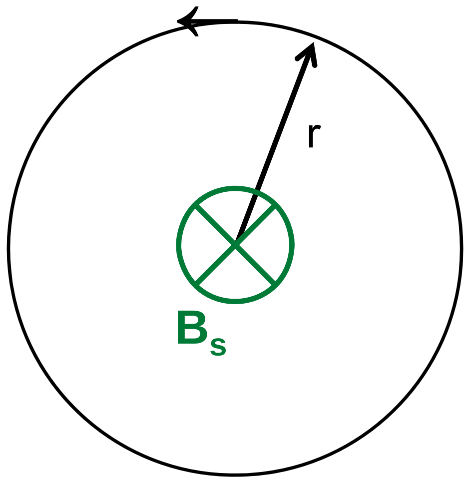
Busch theorem from 1926 states

$$L_s = \left[\vec{r} \times (\vec{P} + eq\vec{A})\right] \cdot \vec{e}_s = const$$

using cylindrical coordinates :

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$\vec{P} = m \begin{bmatrix} \dot{x} \\ \dot{y} \\ \beta c \end{bmatrix}$$



$$L_s = mr^2\dot{\theta} + \frac{1}{2}eqB_s r^2 = const$$

$$\boxed{L_s = mr^2\dot{\theta} + \frac{eq}{2\pi}\Psi = const}$$

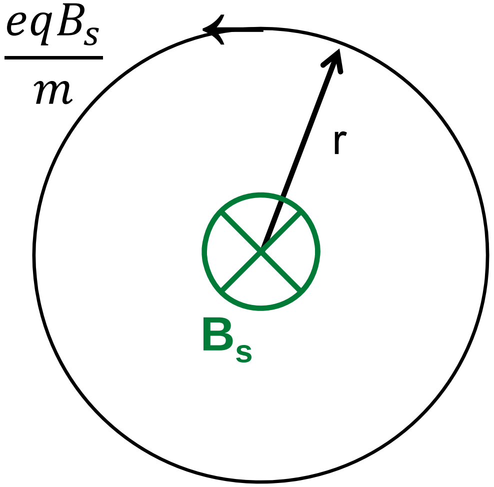
H. Busch, Z. Phys. **81** (5) 924 (1926)

L_s = orbital angular momentum + flux through area of cyclotron motion = const

Busch Theorem

$$m{r_0}^2\,\dot{\theta}_0 + \frac{eq}{2}B_{s0}{r_0}^2 = mr^2\,\dot{\theta} + \frac{eq}{2}B_s r^2 \quad = \quad const$$

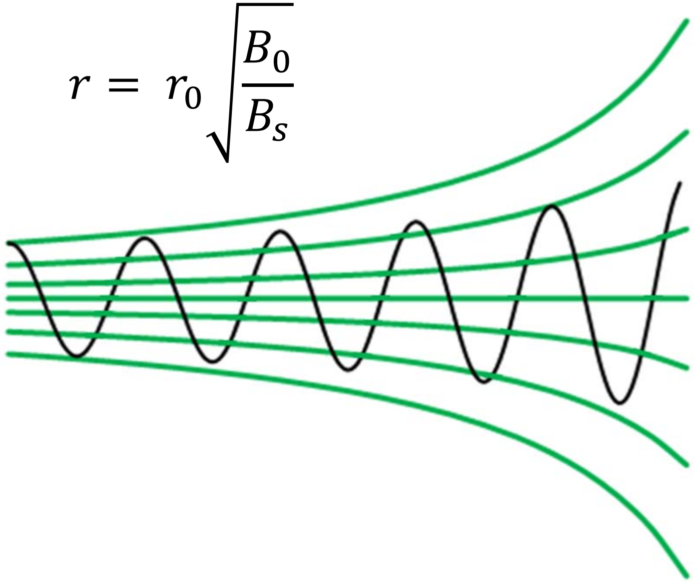
$$\dot{\theta} = \frac{eqB_s}{m}$$



preservation of magn. flux:

$$B_{s0}{r_0}^2 = B_s r^2 = const$$

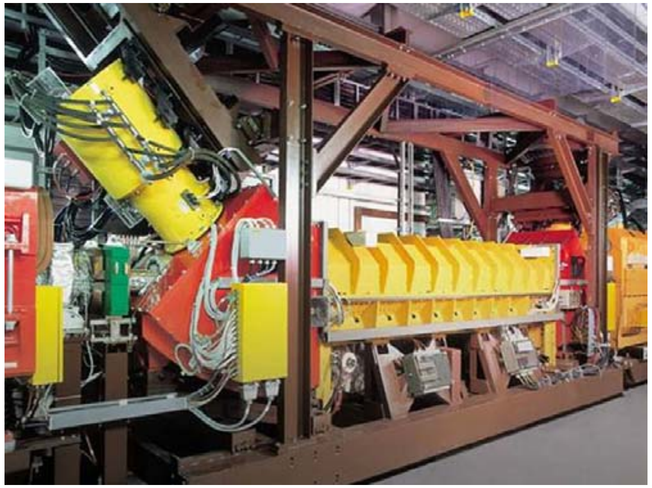
$$r = r_0\sqrt{\frac{B_0}{B_s}}$$

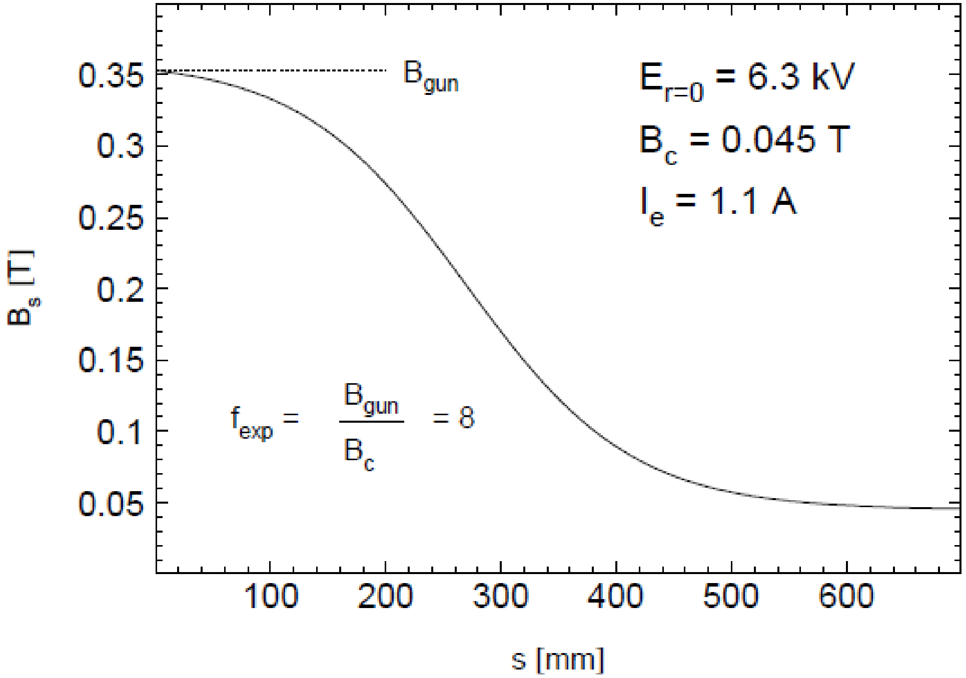


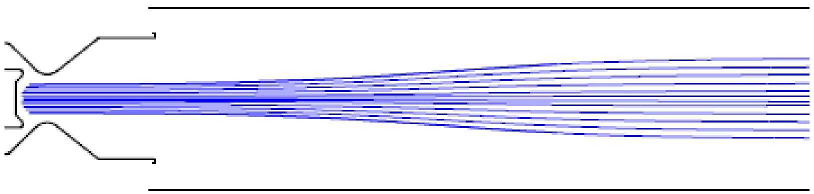
Applications:
Electron beam size shaping

$$r = r_0 \sqrt{\frac{B_{s0}}{B_s}}$$

- cyclotron radius much smaller than beam size → beam size shaped by magnetic field
- applied at low energy electron beams







Applications: Electron beam temperature reduction

mean transv. velocity is measure for transv. beam temperature

$T_\perp \sim < {v_\perp}^2 >$

using $v_\perp = \frac{reqB_s}{m}$ and $r = r_0 \sqrt{\frac{B_{s0}}{B_s}}$

results into $v_\perp = v_{\perp 0} \, \frac{B_s}{B_{s0}}$

electron beam expansion by magn. field expansion $B_s < \ B_{s0}$:

- increases beam radius
- lowers beam temperature
- lowers electron density

technique applied at many e-coolers: 2*IMP, LEIR, TSR, CRYRING, SIS-18

Applications: Magnetic bottle

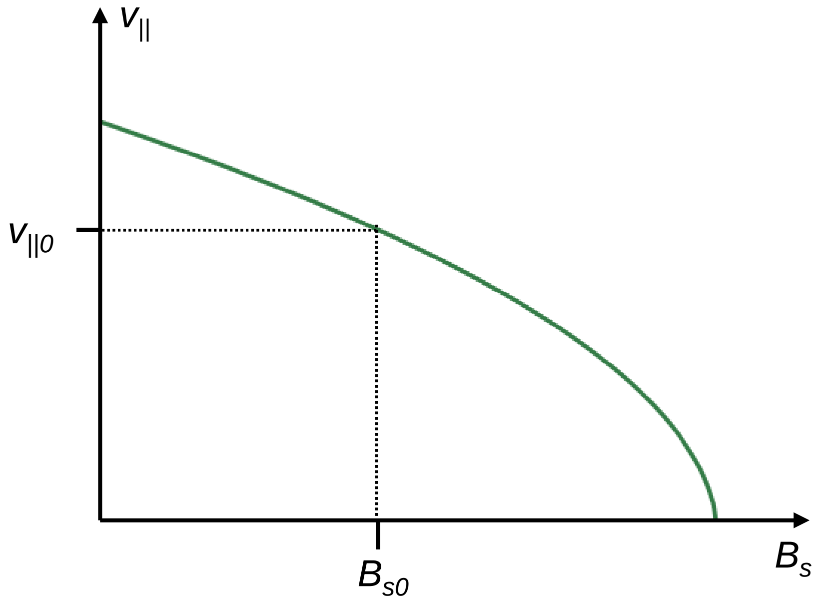
from Busch theorem:

$$B_{0s}{r_0}^2 = B_s r^2, \qquad \dot{\theta} = \frac{eqB_s}{m} = \frac{v_\perp}{r}$$

preservation of total kin. energy :

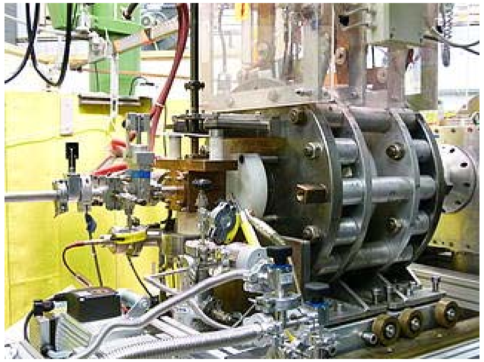
$${v_{||0}}^2 + {v_{\perp 0}}^2 = {v_{||}}^2 + {v_\perp}^2$$

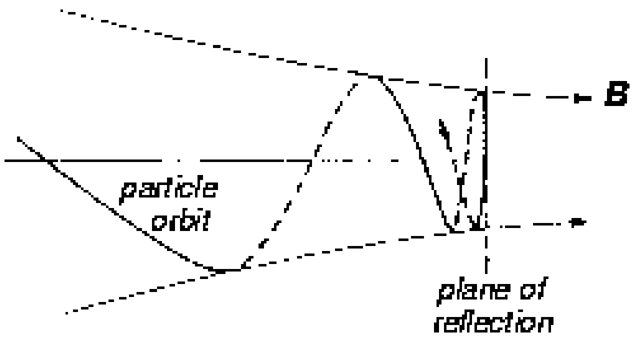
$$\rightarrow {v_{||}}^2(B_s) = {v_{||0}}^2 + {v_{\perp 0}}^2 - \left[\frac{eqr_0}{m}\right]^2 B_{s0} B_s$$



→ beam confinement by strong B_s in magn. bottles :

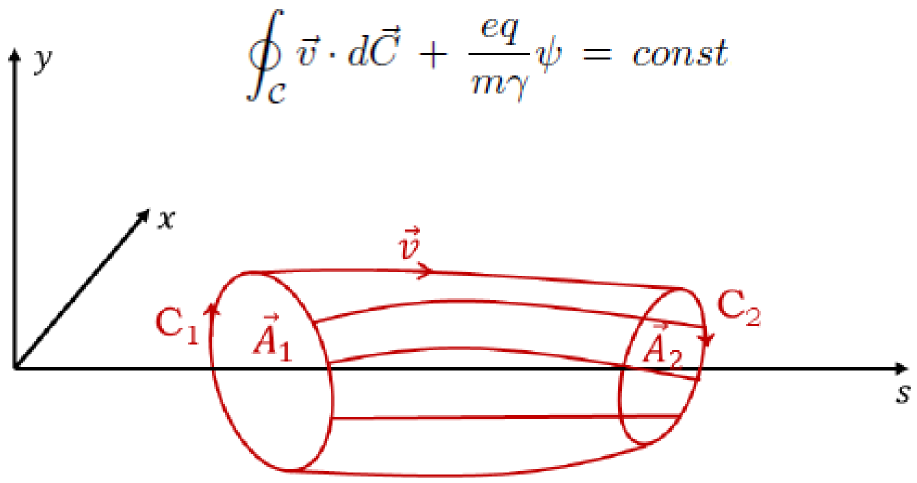
- traps
- ECR sources





Generalized Busch Theorem

Busch theorem can be further generalized to :



C_i enclose possible single particle trajectories

C_i := circles with constant r_i → $\boxed{m\gamma r^2\dot{\theta} + \frac{eq}{2\pi}\Psi \;=\; const}$

Projected rms emittance

rms emittances defined through beam's second moments :

- a_i, b_i : two coordinates of particle i
- <ab>: mean of product $a_i b_i$
- C is moment matrix (symmetric)

$$\varepsilon_x{}^2 = <xx><x'x'> - <xx'>^2$$

$$C_x = \begin{bmatrix} <xx> & <xx'> \\ <x'x> & <x'x'> \end{bmatrix}, \quad \varepsilon_x{}^2 = \det C_x$$

$$C_y = \begin{bmatrix} <yy> & <yy'> \\ <y'y> & <y'y'> \end{bmatrix}, \quad \varepsilon_y{}^2 = \det C_y$$

(x,y,x‘,y‘) are laboratory coordinates which can be measured

Transport of moments

linear transport from point_1 → point_2 through matrices :

$$\begin{bmatrix} x \\ x' \end{bmatrix}_2 = M_x \begin{bmatrix} x \\ x' \end{bmatrix}_1 \qquad M_x = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \; det\, M_x = 1$$

beam moments transport by matrix equation :

$$C_{x2} = M_x\, C_{x1}\, M_x^T$$

analogue in y

4d linear beam dynamics

$$\varepsilon_{4d}{}^2 = det \begin{bmatrix} <xx> & <xx'> & <xy> & <xy'> \\ <x'x> & <x'x'> & <x'y> & <x'y'> \\ <yx> & <yx'> & <yy> & <yy'> \\ <y'x> & <y'x'> & <y'y> & <y'y'> \end{bmatrix}$$

transport of moments from 1 → 2 as usual :

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_2 = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_1 , \; det\, M = 1$$

$$C_2 = M\, C_1\, M^T$$

<u>if x & y planes are not coupled</u>

$$\varepsilon_{4d}{}^2 = det \begin{bmatrix} <xx> & <xx'> & 0 & 0 \\ <x'x> & <x'x'> & 0 & 0 \\ 0 & 0 & <yy> & <yy'> \\ 0 & 0 & <y'y> & <y'y'> \end{bmatrix} = (\varepsilon_x \cdot \varepsilon_y)^2$$

transport of moments from 1 → 2 as usual :

$$M = \begin{bmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{bmatrix} , \; det\, M = det\, M_x \cdot det\, M_y = 1 \cdot 1 = 1$$

$$C_2 = M\, C_1\, M^T$$

Eigen-emittances

- linear (4d), Hamiltonian beam line elements preserve :
  - rms emittance $\varepsilon_{4d}{}^2 = det \begin{bmatrix} <xx> & <xx'> & <xy> & <xy'> \\ <x'x> & <x'x'> & <x'y> & <x'y'> \\ <yx> & <yx'> & <yy> & <yy'> \\ <y'x> & <y'x'> & <y'y> & <y'y'> \end{bmatrix}$

A.J. Dragt, Phys. Rev. A **45** 4 (1992)

  - the two eigen-emittances

$$\varepsilon_1 = \frac{1}{2}\sqrt{-tr[(CJ)^2] + \sqrt{tr^2[(CJ)^2] - 16det(C)}}$$

$$\varepsilon_2 = \frac{1}{2}\sqrt{-tr[(CJ)^2] - \sqrt{tr^2[(CJ)^2] - 16det(C)}}$$

$$\varepsilon_{4d} = \varepsilon_1 \cdot \varepsilon_2$$

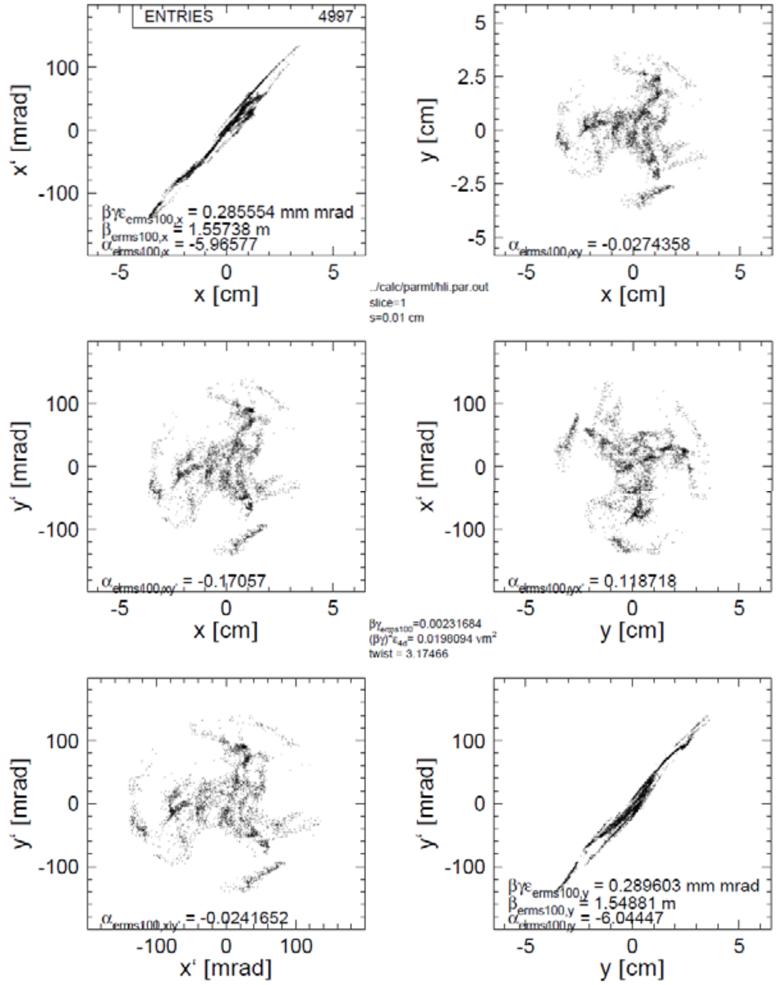
$$C = \begin{bmatrix} <xx> & <xx'> & <xy> & <xy'> \\ <x'x> & <x'x'> & <x'y> & <x'y'> \\ <yx> & <yx'> & <yy> & <yy'> \\ <y'x> & <y'x'> & <y'y> & <y'y'> \end{bmatrix} \qquad J := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

  - the formulas are pretty ugly and their application consumes much time & paper

eigen-emittances

- if, and <u>only</u> if there is no x↔y coupling, i.e. $C = \begin{bmatrix} <xx> & <xx'> & 0 & 0 \\ <x'x> & <x'x'> & 0 & 0 \\ 0 & 0 & <yy> & <yy'> \\ 0 & 0 & <y'y> & <y'y'> \end{bmatrix}$
  - rms emittances = eigen-emittances
- if there is any coupling
  - rms emittances ≠ eigen-emittances
  - coupling parameter $t = \frac{\varepsilon_x \varepsilon_y}{\varepsilon_1 \varepsilon_2} - 1 \geq 0$

term „eigen-emittance" is quite unknown, since generally coupling is just ignored



coutesy P. Spädtke

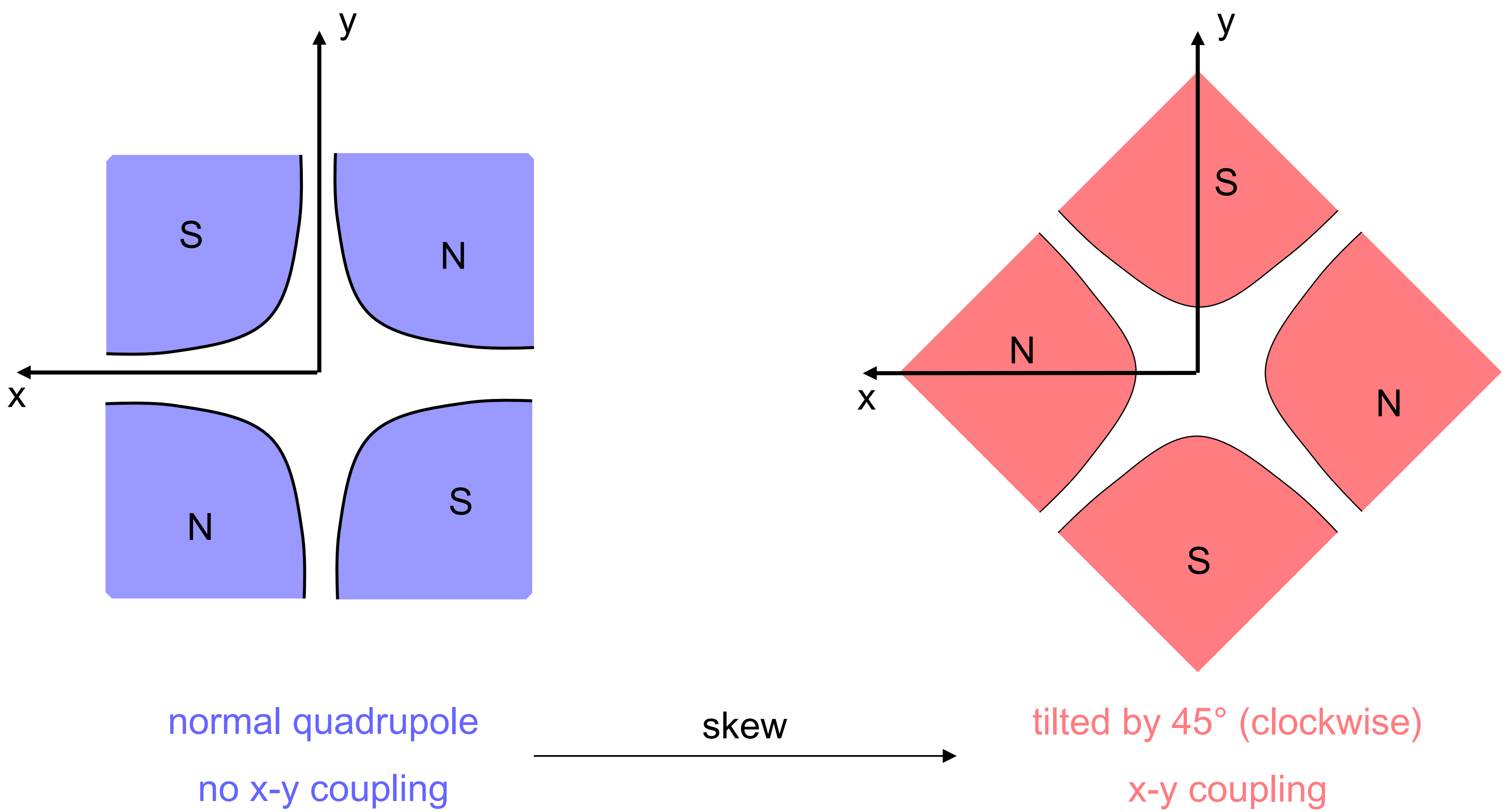
<u>4d distribution behind ECR source</u>

ε_x = 123 mm mrad
ε_y = 125 mm mrad

ε_1 = 17 mm mrad
ε_2 = 231 mm mrad

$\varepsilon_{4d} = \varepsilon_1 \cdot \varepsilon_2 = 3927\ (\text{mm mrad})^2$
$\varepsilon_x \cdot \varepsilon_y = 15375\ (\text{mm mrad})^2$
$\varepsilon_x \cdot \varepsilon_y = 3.9\ \varepsilon_{4d}$

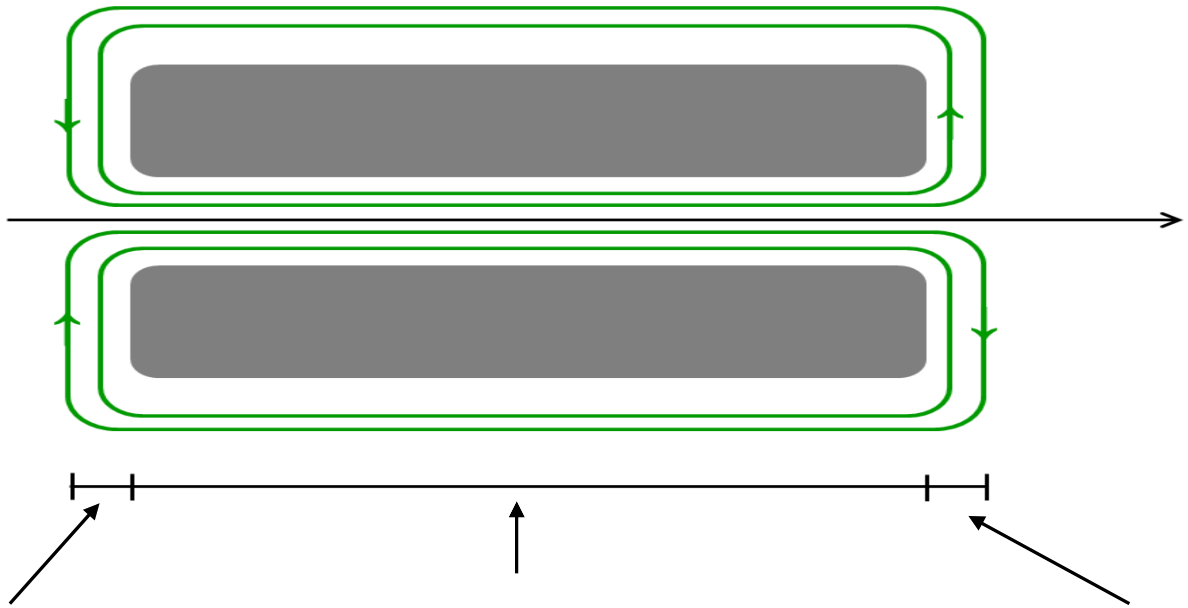
Coupling linear elements: Skew quadrupole



Coupling linear elements: Solenoid

$$\kappa := \frac{B}{2(B\rho)}$$

$$\alpha(L) = -2\kappa L$$



entrance fringe

$$M_{fi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \kappa & 0 \\ 0 & 0 & 1 & 0 \\ -\kappa & 0 & 0 & 1 \end{bmatrix}$$

axial field

$$M_{||} = \begin{bmatrix} 1 & -\frac{1}{2\kappa} sin(\alpha) & 0 & \frac{1-cos(\alpha)}{2\kappa} \\ 0 & cos(\alpha) & 0 & -sin(\alpha) \\ 0 & \frac{cos(\alpha)-1}{2\kappa} & 1 & -\frac{sin\alpha)}{2\kappa} \\ 0 & sin(\alpha) & 0 & cos(\alpha) \end{bmatrix}$$

exit fringe

$$M_{fo} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\kappa & 0 \\ 0 & 0 & 1 & 0 \\ \kappa & 0 & 0 & 1 \end{bmatrix}$$

complete solenoid matrix $M_{sol} = M_{fo} \cdot M_{||} \cdot M_{fi}$

How elements change emittances

applying ugly formulas →

element	$\mathrm{rms}_{x,y}$	4d rms	$\mathrm{eigen}_{1,2}$
drift	no	no	no
quadrupole	no	no	no
tilted quadrupole	yes	no	no
dipole	no	no	no
tilted dipole	yes	no	no
solenoid	yes	no	no
solenoid fringe	yes	no	yes
solenoid axial field	yes	no	yes

How elements change emittances

element	$rms_{x,y}$	4d rms	$eigen_{1,2}$
drift	no	no	no
quadrupole	no	no	no
tilted quadrupole	yes	no	no
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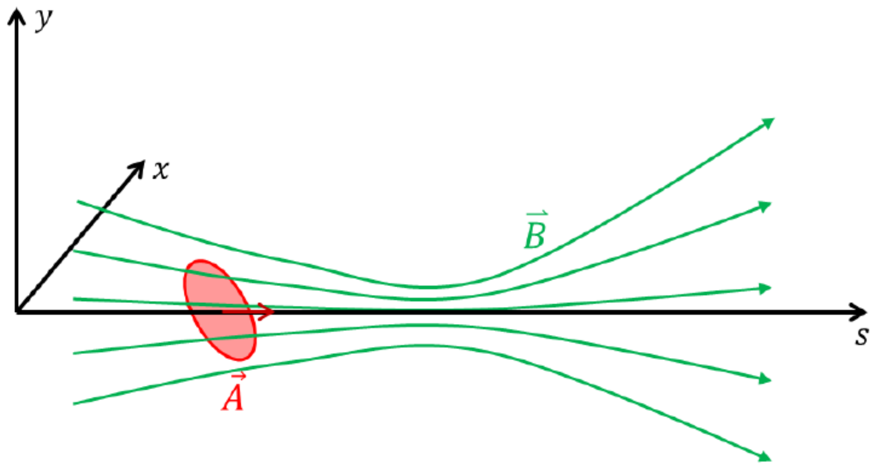
tilted dipole	yes	no	no
solenoid	yes	no	no
solenoid fringe	yes	no	yes
solenoid axial field	yes	no	yes



- eigen-emittances seem to change by magnetic flux through beam surface (note: in front of and behind solenoid flux is zero !)
- in the following this will be proven ...

Preservation of eigen emittances in conjugate coordinates

- reminder: generalized momentum $\vec{p} \coloneqq \vec{P} + eq\vec{A}$, i.e.,

$$p_x := x' + \frac{A_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)}$$

$$p_y := y' + \frac{A_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}$$

- original ansatz of Busch for single particle :
  - angular momentum including the contribution from $\vec{r} \times eq\vec{A}$ is preserved
  - „generalized angular momentum“ is preserved

- ansatz for extension to beams :
  - eigen-emittances including contribution from $\tilde{\vec{r}} \coloneqq \vec{A}/(B\rho)$ are preserved
  - „generalized“ eigen-emittances are preserved

Preservation of eigen emittances in conjugate coordinates

- calculation of „generalized“ eigen-emittance through replacing (x‘,y‘) by (p_x,p_y)

$$C = \begin{bmatrix} <xx> & <xx'> & <xy> & <xy'> \\ <x'x> & <x'x'> & <x'y> & <x'y'> \\ <yx> & <yx'> & <yy> & <yy'> \\ <y'x> & <y'x'> & <y'y> & <y'y'> \end{bmatrix} \longrightarrow \tilde{C} = \begin{bmatrix} \langle x^2\rangle & \langle xp_x\rangle & \langle xy\rangle & \langle xp_y\rangle \\ \langle xp_x\rangle & \langle p_x^2\rangle & \langle yp_x\rangle & \langle p_xp_y\rangle \\ \langle xy\rangle & \langle yp_x\rangle & \langle y^2\rangle & \langle yp_y\rangle \\ \langle xp_y\rangle & \langle p_xp_y\rangle & \langle yp_y\rangle & \langle p_y^2\rangle \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad \tilde{\varepsilon}_{1/2} = \frac{1}{2}\sqrt{-tr[(\tilde{C}J)^2] \pm \sqrt{tr^2[(\tilde{C}J)^2] - 16\,det(\tilde{C})}}$$

- if $\tilde{\varepsilon}_{1/2}$ are preserved, also $\tilde{\varepsilon}_1^2 + \tilde{\varepsilon}_2^2$ must be preserved

- state $\tilde{\varepsilon}_1{}^2 + \tilde{\varepsilon}_2{}^2 = const$ by substituting

$$p_x := x' + \frac{A_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)}$$
$$p_y := y' + \frac{A_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}$$

to obtain expression for useful „laboratory“ eigen-emittances

Preservation of eigen emittances in conjugate coordinates

- state $\tilde{\varepsilon}_1{}^2 + \tilde{\varepsilon}_2{}^2 = const$ by substituting

  $$p_x := x' + \frac{\mathcal{A}_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)}$$
  $$p_y := y' + \frac{\mathcal{A}_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}$$

  with $A := \sqrt{\langle x^2\rangle\langle y^2\rangle - \langle xy\rangle^2}$

- this delivers:

$$(\varepsilon_1 - \varepsilon_2)^2 + \left[\frac{AB_s}{(B\rho)}\right]^2 + 2\frac{B_s}{(B\rho)}\left[\langle y^2\rangle\langle xy'\rangle - \langle x^2\rangle\langle yx'\rangle + \langle xy\rangle(\langle xx'\rangle - \langle yy'\rangle)\right] = const$$

- confirmed: change of laboratory eigen-emittances just through long. magn. field B_s

- quadrupoles and dipoles (even) skewed: $B_s = 0 \;\rightarrow\; \varepsilon_{1/2} = const$

Sum of quantities is invariant

$$\boxed{(\varepsilon_1 - \varepsilon_2)^2} + \boxed{\left[\frac{AB_s}{(B\rho)}\right]}^2 + 2\frac{B_s}{(B\rho)}\boxed{[\langle y^2\rangle\langle xy'\rangle - \langle x^2\rangle\langle yx'\rangle + \langle xy\rangle(\langle xx'\rangle - \langle yy'\rangle)]} = const$$

- sum of three quantities forms an invariant

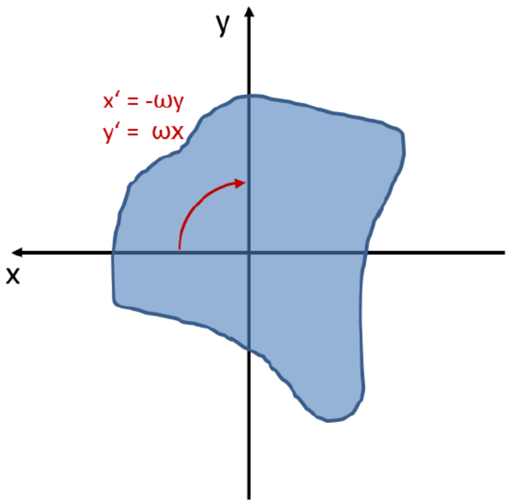
- difference of eigen-emittances, flux through beam area, ...

- what is the third ?
  - has dimension m^3
  - scales with beam rms area as for $y = ax$ it vanishes
  - vanishes for uncorrelated beams
  - invariant under rotation around beam axis
  - investigate term for some examples ...

Understanding third term: Example objects

$$\mathcal{W}_A := \langle y^2\rangle\langle xy'\rangle - \langle x^2\rangle\langle yx'\rangle + \langle xy\rangle(\langle xx'\rangle - \langle yy'\rangle)$$

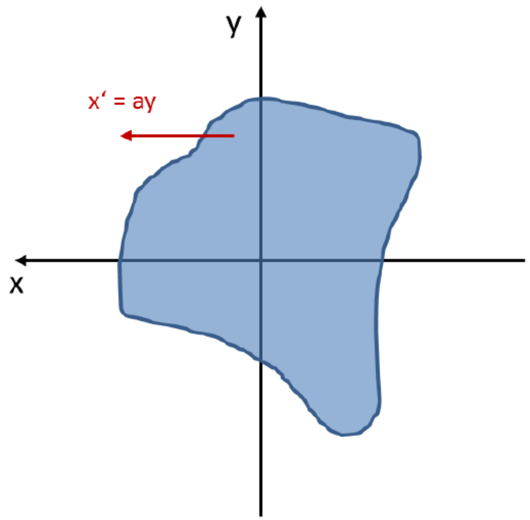
<u>rigid object rotating with ω :</u>



$$W_A = 2\omega A^2 = 2AL$$

solenoid fringe field performs rigid beam rotation

<u>object under shear :</u>



$$W_A = -aA^2 \neq 2AL$$

thin skew quad performs shear

Understanding W_A: Transformations

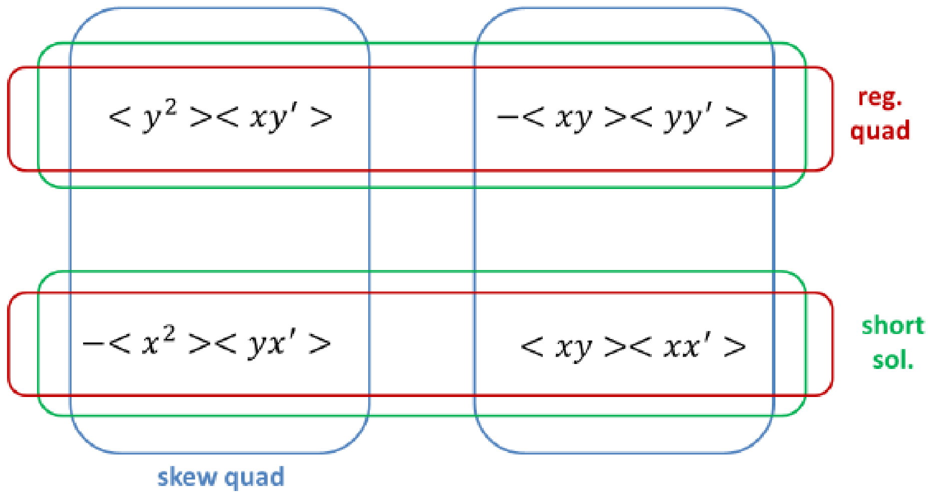
$$W_A := \langle y^2\rangle\langle xy'\rangle - \langle x^2\rangle\langle yx'\rangle + \langle xy\rangle(\langle xx'\rangle - \langle yy'\rangle)$$

check, how W_A is changed under transformation through :

- thin reg. quadrupole: $x' \to x' - qx$ and $y' \to y' + qy$
- thin skew quadrupole: ...
- short solenoid: ...

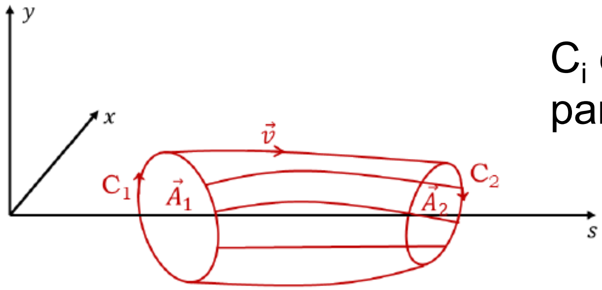
- $\rightarrow$ non of them changes W_A

scheme of mutual cancellation of constituents of W_A



Understanding W_A:
Pick idea from gen. Busch theorem

$$\oint_C \vec{v} \cdot d\vec{C} + \frac{eq}{m\gamma}\psi = const$$



C_i enclose possible single particle trajectories

sort of mean rotation around beam area

<u>try ansatz :</u>

$$\mathcal{W}_A = 2A \oint_C \vec{\bar{r}}'(x,y,s) \cdot d\vec{C}$$

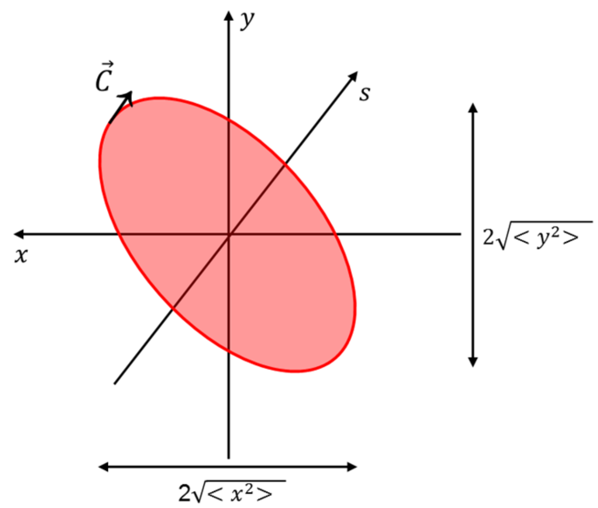
mean angle integrated along curve enclosing beam area ... multiplied with beam area

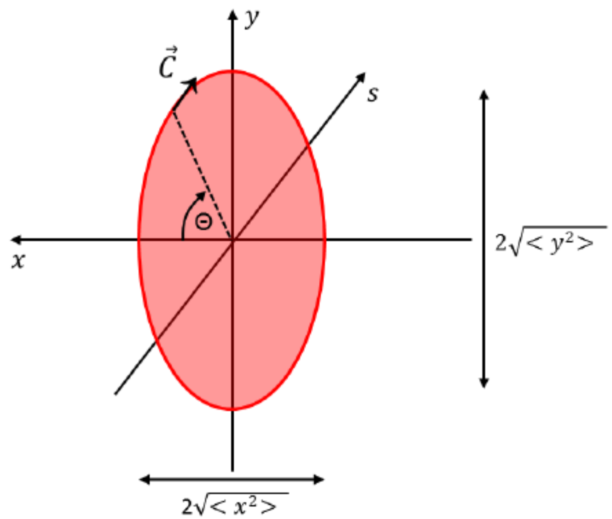
$$\vec{\bar{r}}'(x,y,s) := [\bar{x}'(x,y,s), \bar{y}'(x,y,s), 1]$$

mean: average (x‘,y‘) at given (x,y)

Calculating W_A

$$W_A = 2A \oint_C \vec{r}'(x, y, s) \cdot d\vec{C}$$

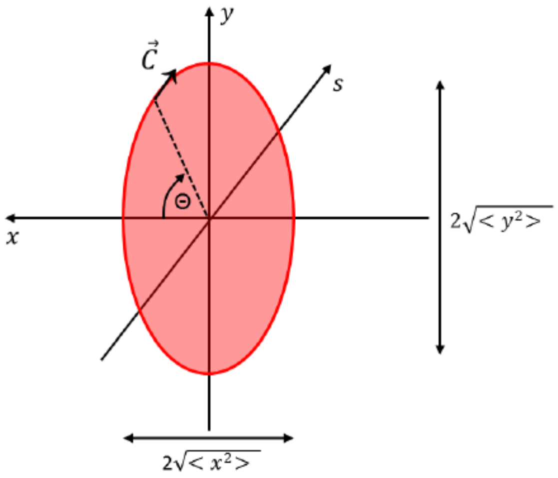




as W_A is invariant under rotation
→ calculated for ellipse being turned upright
→ <xy> = 0 !

Calculating W_A

$$\mathcal{W}_A = 2A \oint_C \vec{\bar{r}}'(x,y,s) \cdot d\vec{C}$$



$$x = \sqrt{\langle x^2 \rangle}\cos\theta,$$

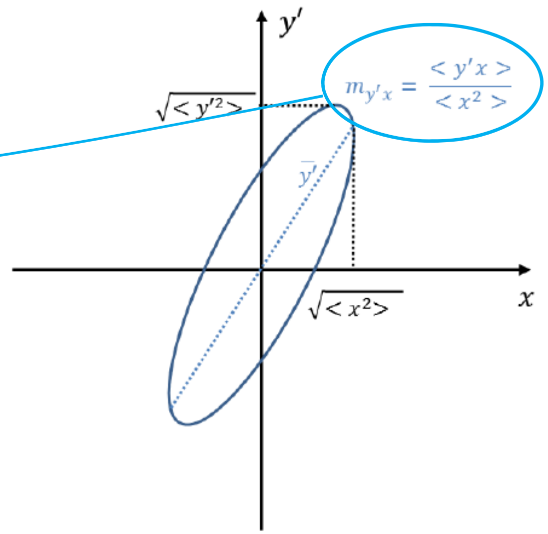
$$y = \sqrt{\langle y^2 \rangle}\sin\theta,$$

$$\mathrm{d}\vec{C} = \begin{bmatrix} -x \\ y \end{bmatrix} d\theta$$

<u>Tylor expansion of $\vec{\bar{r}}'$ to first order :</u>

$$\bar{x}'(x,y) := \bar{x}'(0,0) + \frac{\partial \bar{x}'}{\partial x} \cdot x + \frac{\partial \bar{x}'}{\partial y} \cdot y$$

$$\bar{y}'(x,y) := \bar{y}'(0,0) + \frac{\partial \bar{y}'}{\partial x} \cdot x + \frac{\partial \bar{y}'}{\partial y} \cdot y$$



<u>finally confirms:</u>

$$\mathcal{W}_A := \langle y^2 \rangle \langle x y' \rangle - \langle x^2 \rangle \langle y x' \rangle + \langle \cancel{xy} \rangle (\langle x x' \rangle - \langle y y' \rangle)$$

Busch theorem for particle beams

using expression for W_A finally delivers :

$$(\varepsilon_1 - \varepsilon_2)^2 + \left[\frac{AB_s}{(B\rho)}\right]^2 + \frac{4AB_s}{(B\rho)} \oint_C \vec{r}' d\vec{C} = const$$

acceleration can be included by initially multiplying

$$p_x := x' + \frac{A_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)}$$
$$p_y := y' + \frac{A_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}$$

with $m\gamma\beta c$ at both sides

... resulting in Busch's theorem extended to accelerated particle beams :

$$\boxed{(\varepsilon_{n1} - \varepsilon_{n2})^2 + \left[\frac{eq\psi}{mc\pi}\right]^2 + \frac{4eq\psi\beta\gamma}{mc\pi} \oint_C \vec{r}' \cdot d\vec{C} = const}$$

Busch theorem for particle beams

original Busch theorem for single particle :

$$\frac{eq}{m\gamma}\psi + \oint_C \vec{v} \cdot d\vec{C} = const$$

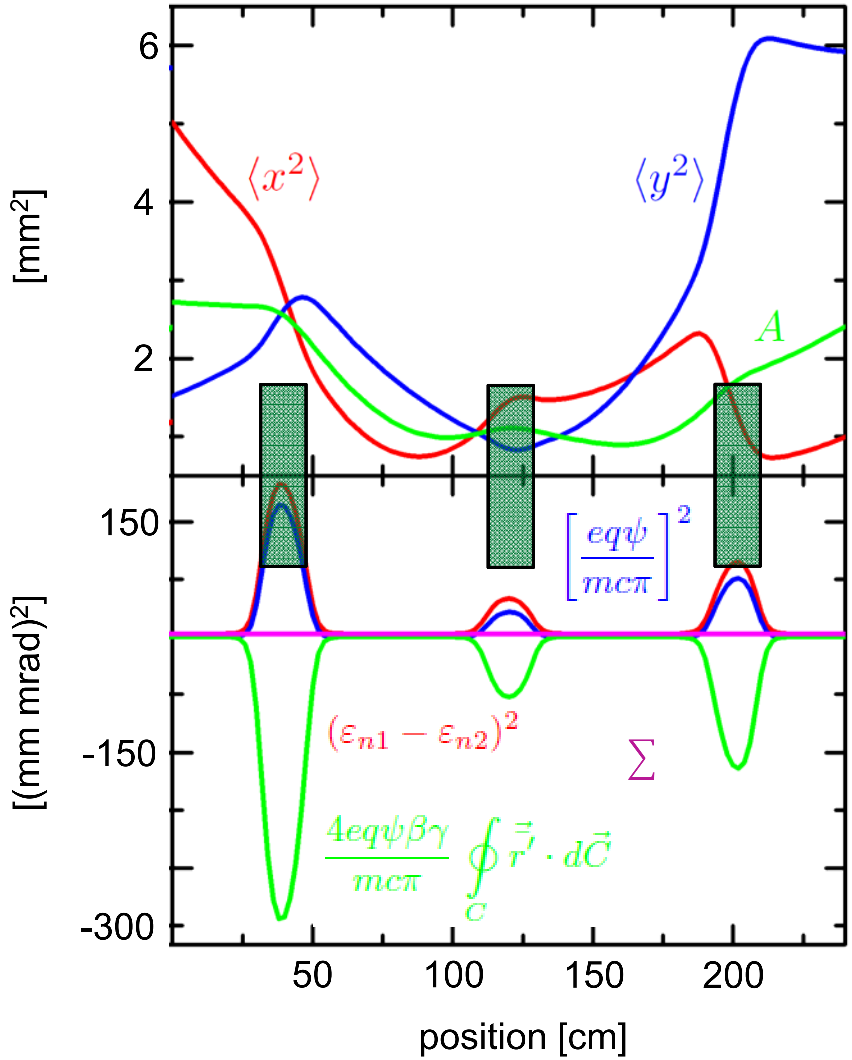
theorem extended to accelerated particle beams :

$$(\varepsilon_{n1} - \varepsilon_{n2})^2 + \left[\frac{eq\psi}{mc\pi}\right]^2 + \frac{4eq\psi\beta\gamma}{mc\pi} \oint_C \vec{\vec{r}}' \cdot d\vec{C} = const$$

- both expressions include flux and „vorticity“ $\vec{v} \cdot d\vec{C} \cong (\vec{\nabla} \times \vec{v})\, d\vec{A}$ (Stoke‘s law)
- the extended theorem additionally includes eigen-emittances
- theorem allows very fast modelling of setups for emittance gymnastics

Varification through simulations

- beam tracking through three solenoids
- extended fringe fields from $\vec{B}$ -maps
- invariance confirmed



Application to electron beams: Flat beam creation at FERMILAB

<u>flat electron beams, i.e., $\varepsilon_x << \varepsilon_y$ useful for :</u>

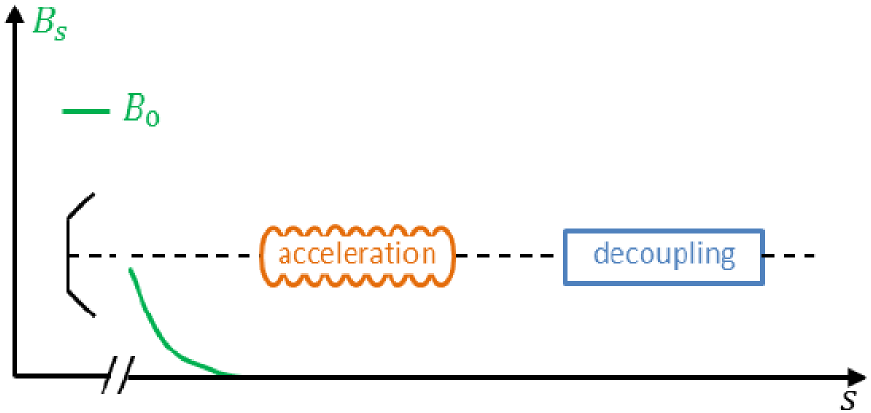
- increase of luminosity in e^- / e^+ colliders
- production of X-ray pulses with femto seconds in duration

<u>test accelerator at FERMILAB demonstrated $\varepsilon_y / \varepsilon_x = 100$:</u>

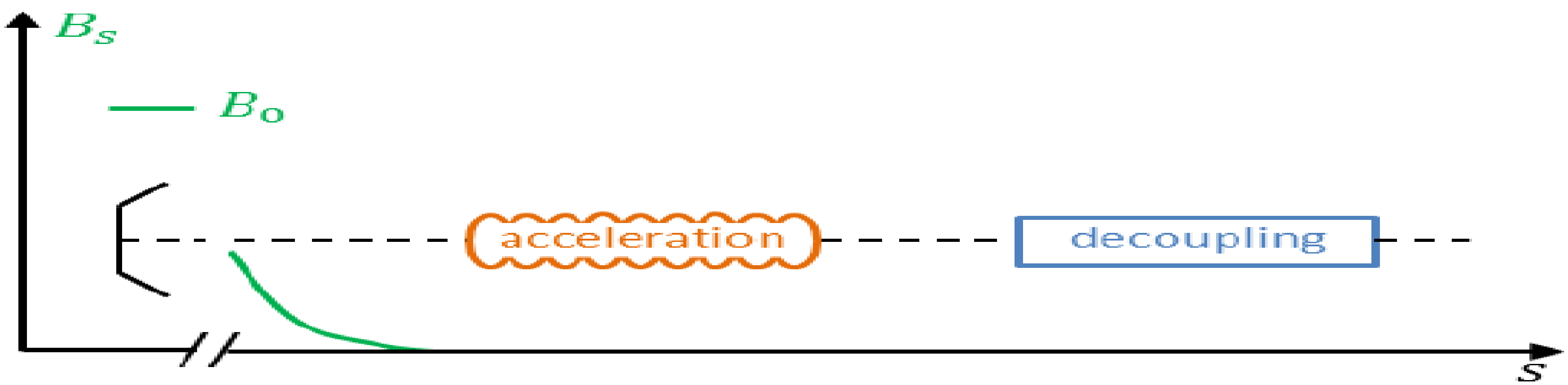
- create beam at photo cathode being immersed into $B_s = B_0$
- reduce B_s to zero, accelerate, and decouple x/y - planes

Northern Illinois Center for
Accelerator and Detector Development (NICADD)

P. Piot et al., Phys. Rev. ST Accel. Beams **9** 031001 (2006)



Application to electron beams:
Flat beam creation at FERMILAB



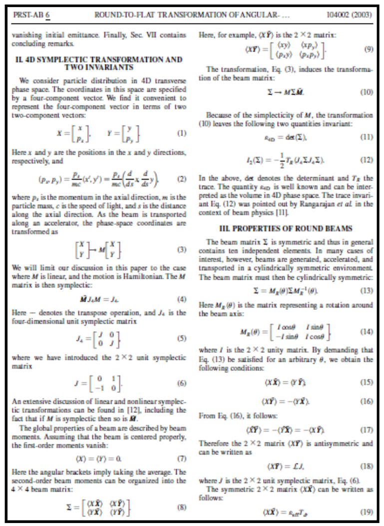
prior to formulation of extended Busch theorem, the deviaton of final beam emittances took several pages ...

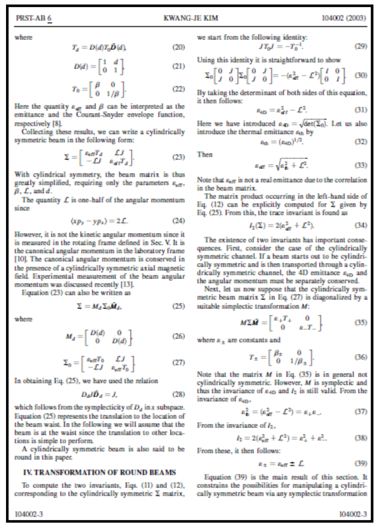
$$\varepsilon_{nfx/y} = \pm L\beta\gamma + \sqrt{(L\beta\gamma)^2 + {\varepsilon_{4d}}^2}$$

with $\varepsilon_{4d} = \varepsilon_{ni1} \cdot \varepsilon_{ni2}$

and $L := (eB_0A_0)/(2m\gamma\beta c)$

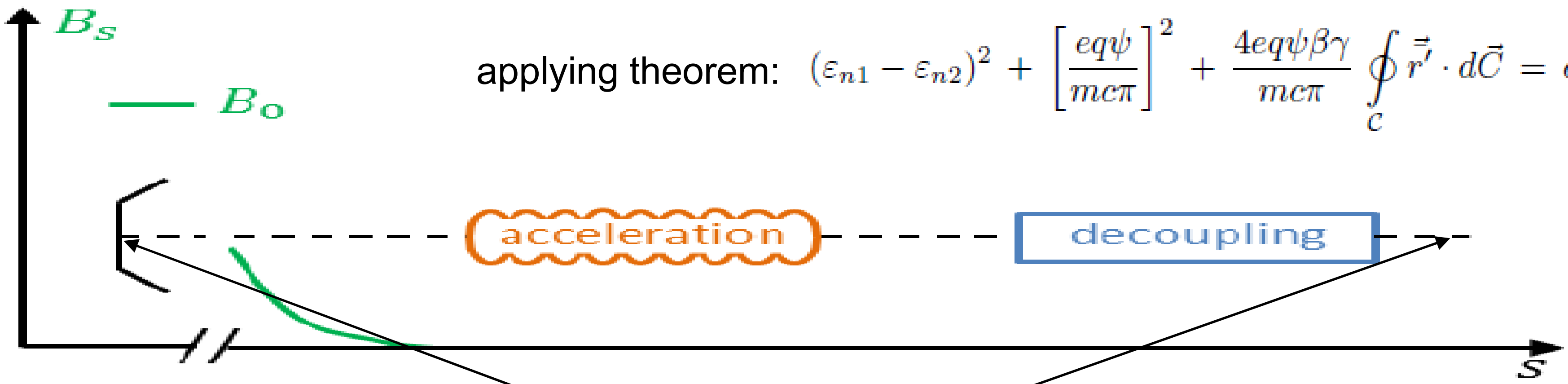
Kwang-Je Kim, Phys. Rev. ST Accel. Beams **6** 104002 (2006)





Application to electron beams: Flat beam creation at FERMILAB

applying theorem: $(\varepsilon_{n1}-\varepsilon_{n2})^2 + \left[\frac{eq\psi}{mc\pi}\right]^2 + \frac{4eq\psi\beta\gamma}{mc\pi}\oint_C \vec{r}\,'\cdot d\vec{C} = const$



$$\boxed{0 + \left[\frac{eB_0A_0}{mc}\right]^2 + 0} = \boxed{(\varepsilon_{nfx}-\varepsilon_{nfy})^2 + 0 + 0}$$

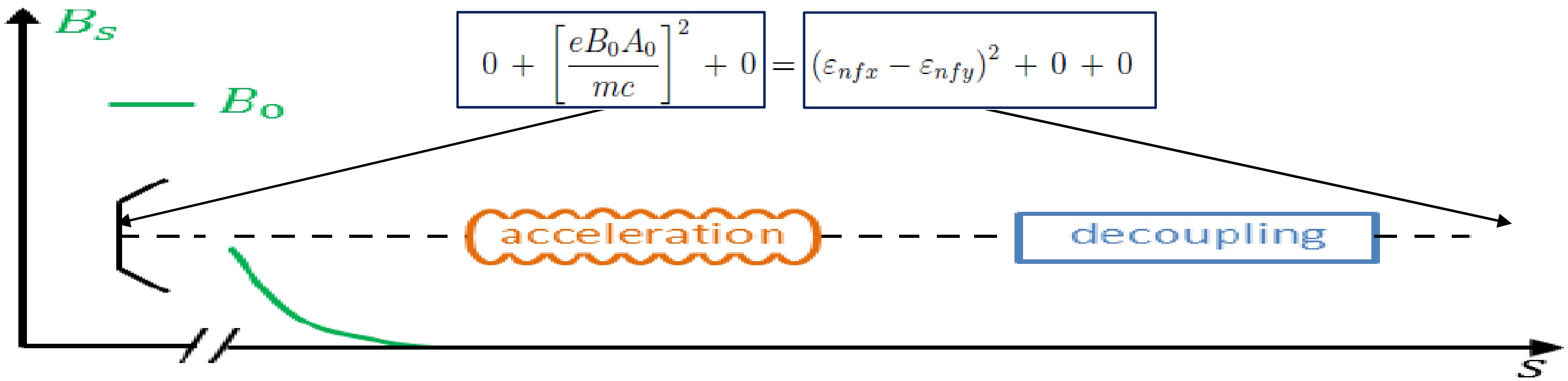
<ins>at cathode beam is symmetric</ins>:

- eigen = rms emittances: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- eigen (rms) emittances are equal: $\varepsilon_{1/x} = \varepsilon_{2/y}$
- immersed into B-flux
- no x/y coupling → $W_A = 0$

<ins>final beam</ins>:

- eigen = rms emittances: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- eigen (rms) emittances differ: $\varepsilon_{1/x} \neq \varepsilon_{2/y}$
- no B-flux
- x/y coupling removed → $W_A = 0$

Application to electron beams: Flat beam creation at FERMILAB



replacing $\varepsilon_{nfy} = \varepsilon_{4d}/\varepsilon_{nfx}$

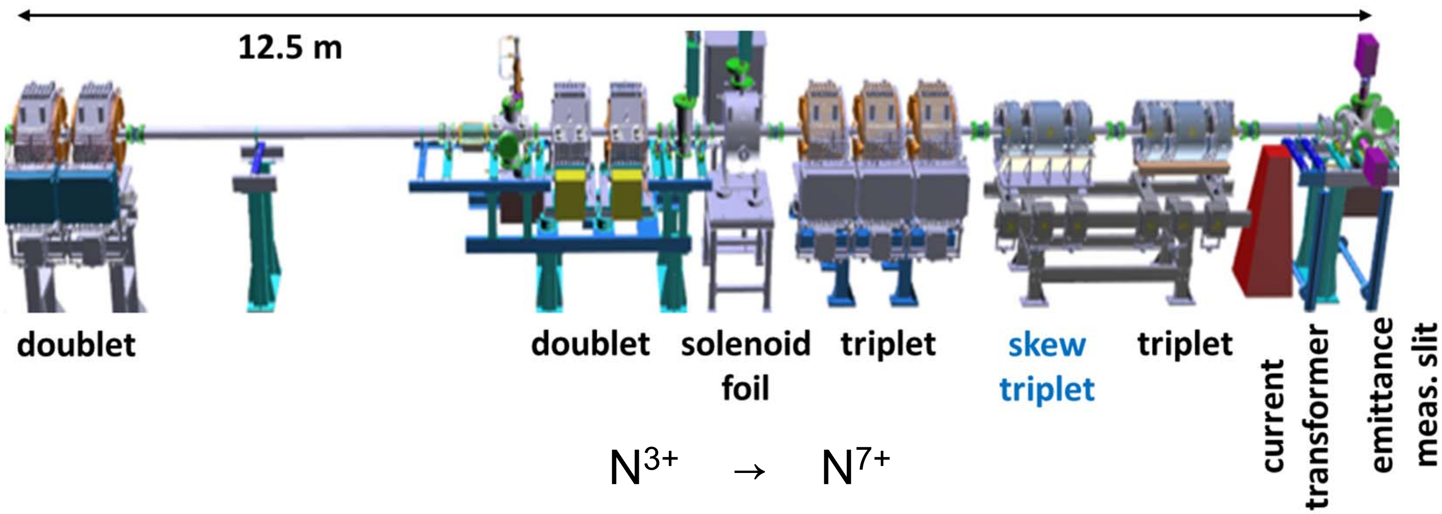
and $\quad eB_0A_0 = 2m\gamma\beta c \cdot L$

results into $\varepsilon_{nfx} = L\beta\gamma \pm \sqrt{(L\beta\gamma)^2 + {\varepsilon_{4d}}^2}$

using upper sign gives $\varepsilon_{nfx/y} = \pm L\beta\gamma + \sqrt{(L\beta\gamma)^2 + {\varepsilon_{4d}}^2}$

Application to ion beams:
EmTEx at GSI (Emitt. Transf. Exp.)

- beams from linacs: $\varepsilon_x \approx \varepsilon_y$
- hor. multi-turn injection into rings profits from $\varepsilon_x < \varepsilon_y$
- EmTEx @ transfer channel :
  - place charge state stripper inside short solenoid
  - x/y-decoupling afterwards

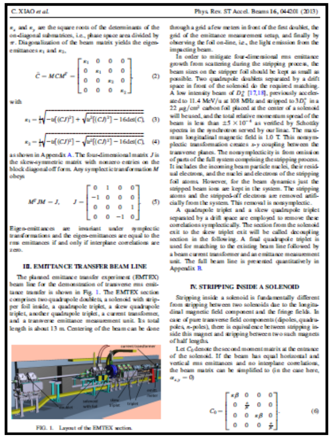


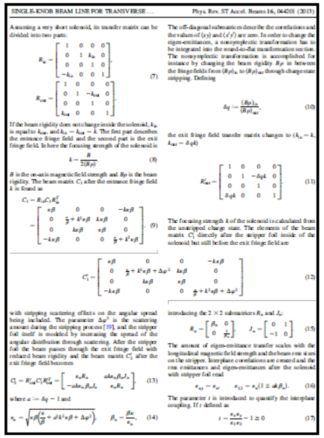
$N^{3+} \rightarrow N^{7+}$

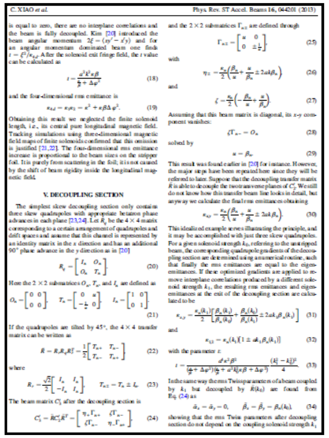
Application to ion beams:
EmTEx at GSI (Emitt. Transf. Exp.)

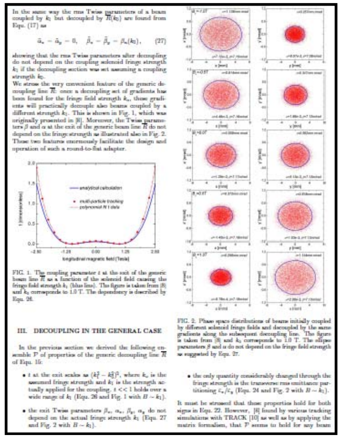
prior to formulation of extended Busch theorem, the deviaton of final beam emittances took several pages ...

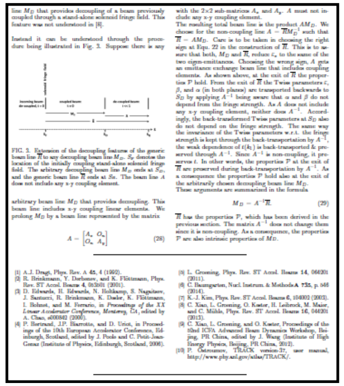
$$(\varepsilon_{x,7+} - \varepsilon_{y,7+})^2 \;=\; (\varepsilon_{x,3+} - \varepsilon_{y,3+})^2 + \,(A_f B_0)^2 \left[\frac{1}{(B\rho)_{7+}} - \frac{1}{(B\rho)_{3+}}\right]^2$$







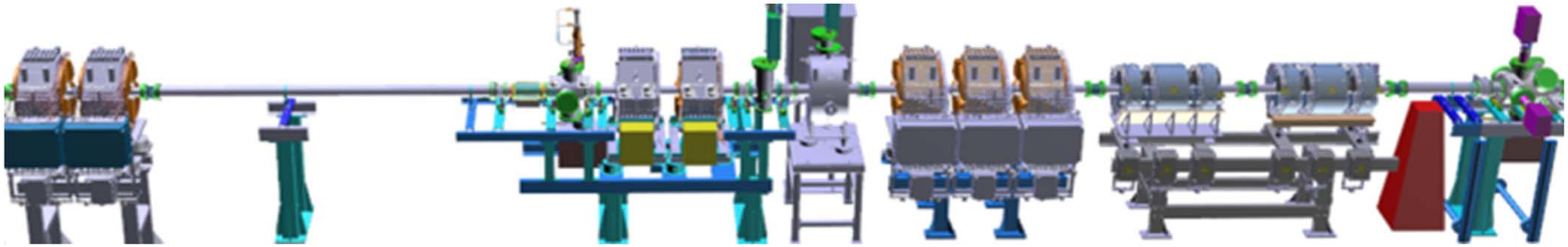




C. Xiao et. al, Phys. Rev. ST Accel. Beams **16** 044201 (2013)

L. Groening, arXiv 1403.6962 (2014)

Application to ion beams:
EmTEx at GSI (Emitt. Transf. Exp.)



change of q is a non-Hamiltonian action → „splitting“ into two Hamiltonian actions

extended theorem before stripping (q=3+) :

$$(\varepsilon_{1f}-\varepsilon_{2f})^2+\left[\frac{A_f B_0}{(B\rho)_{3+}}\right]^2+\frac{2B_0}{(B\rho)_{3+}}\mathcal{W}_{Af}=const$$

extended theorem after stripping (q=7+) :

$$(\varepsilon_{1f}-\varepsilon_{2f})^2+\left[\frac{A_f B_0}{(B\rho)_{7+}}\right]^2+\frac{2B_0}{(B\rho)_{7+}}\mathcal{W}_{Af}=\overline{const}$$

Application to ion beams: EmTEx up to stripping foil

A_f ≔ beam area at foil,
from measurements,
≈ const along short solenoid

$$(\varepsilon_{x,3+} - \varepsilon_{y,3+})^2 + 0 + 0 = (\varepsilon_{1f} - \varepsilon_{2f})^2 + \left[\frac{A_f B_0}{(B\rho)_{3+}}\right]^2 + \frac{2B_0}{(B\rho)_{3+}} \mathcal{W}_{Af}$$

beam line entrance :

- eigen = rms emittances: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- measured: $\varepsilon_{x,3+}$ and $\varepsilon_{y,3+}$
- no B-flux
- no x/y coupling → $W_A = 0$

inside solenoid & just before foil :

- eigen ≠ rms emittances: $\varepsilon_{1/2} \neq \varepsilon_{x/y}$
- eigen emittances differ: $\varepsilon_1 \neq \varepsilon_2$
- B-flux
- x/y coupling → $W_{Af} = -(B_0 {A_f}^2)/(B\rho)_{3+}$

Application to ion beams: EmTEx up to stripping foil

$$\boxed{(\varepsilon_{x,3+} - \varepsilon_{y,3+})^2 + 0 + 0} = \boxed{(\varepsilon_{1f} - \varepsilon_{2f})^2 + \left[\frac{A_f B_0}{(B\rho)_{3+}}\right]^2 + \frac{2B_0}{(B\rho)_{3+}}\mathcal{W}_{Af}}$$

plugging in $W_{Af} = -(B_0 {A_f}^2)/(B\rho)_{3+}$:

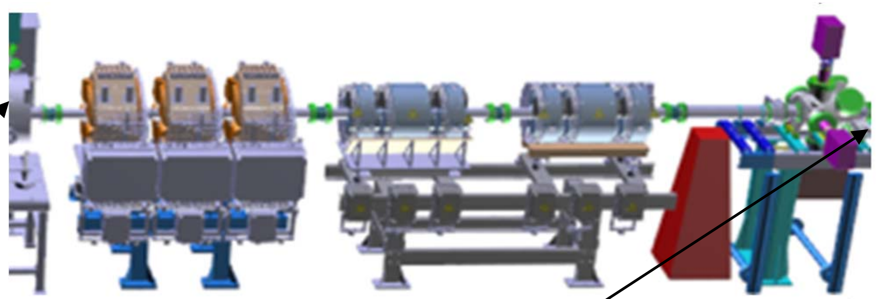
$$(\varepsilon_{x,3+} - \varepsilon_{y,3+})^2 + 0 + 0 = (\varepsilon_{1f} - \varepsilon_{2f})^2 - \left[\frac{A_f B_0}{(B\rho)_{3+}}\right]^2$$

using experiment's parameters :

$$(\varepsilon_{1f} - \varepsilon_{2f})^2 = 2.755\ (mm\ mrad)^2$$

Application to ion beams: EmTEx behind the stripping foil

- foil changes just q, i.e., $(B\rho)$
- $\varepsilon_{1,2,x,y}$, B-flux, and W_{Af} do not change



$$\boxed{(\varepsilon_{1f}-\varepsilon_{2f})^2+\left[\frac{A_f B_0}{(B\rho)_{7+}}\right]^2+\frac{2B_0}{(B\rho)_{7+}}\mathcal{W}_{Af}}=\boxed{(\varepsilon_{x,7+}-\varepsilon_{y,7+})^2+0+0}$$

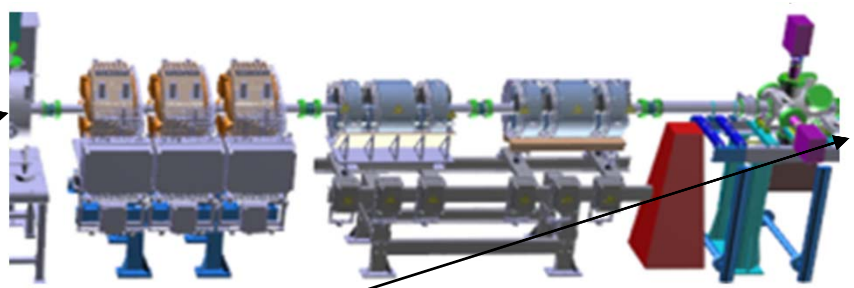
inside solenoid & just after foil :

- eigen ≠ rms emittances: $\varepsilon_{1/2} \neq \varepsilon_{x/y}$
- eigen emittances differ: $\varepsilon_1 \neq \varepsilon_2$
- B-flux
- x/y coupling → $W_{Af} = -(B_0 {A_f}^2)/(B\rho)_{3+}$

beam line exit :

- eigen = rms emittances: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- to be calculated/measured: $\varepsilon_{x,7+}$ and $\varepsilon_{y,7+}$
- no B-flux
- no x/y coupling → $W_A = 0$

Application to ion beams: EmTEx behind the stripping foil



$$\boxed{(\varepsilon_{1f}-\varepsilon_{2f})^2+\left[\frac{A_f B_0}{(B\rho)_{7+}}\right]^2+\frac{2B_0}{(B\rho)_{7+}}\mathcal{W}_{Af}} = \boxed{(\varepsilon_{x,7+}-\varepsilon_{y,7+})^2+0+0}$$

plugging in $W_{Af}=-(B_0 A_f{}^2)/(B\rho)_{3+}$ and $(\varepsilon_{1f}-\varepsilon_{2f})^2 = \ 2.755\ (mm\ mrad)^2$

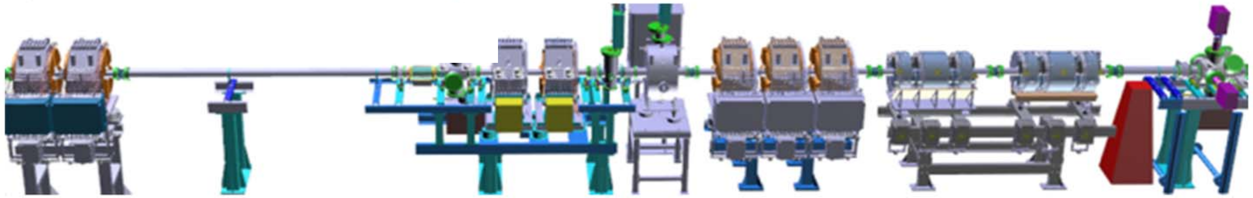
finally delivers $|\varepsilon_{x,7+}-\varepsilon_{y,7+}| = 2.21\ mm\ mrad$

the measured values are :

- $\varepsilon_{x,7+}=2.76(14)\ mm\ mrad$
- $\varepsilon_{y,7+}=0.72(4)\ mm\ mrad$
- $|\varepsilon_{x,7+}-\ \varepsilon_{y,7+}| = 2.04(14)\ mm\ mrad$ $\quad\rightarrow\quad$ very good agreement to extended Busch theorem

L. Groening et. al, Phys. Rev. Lett. **113** 044201 (2014)

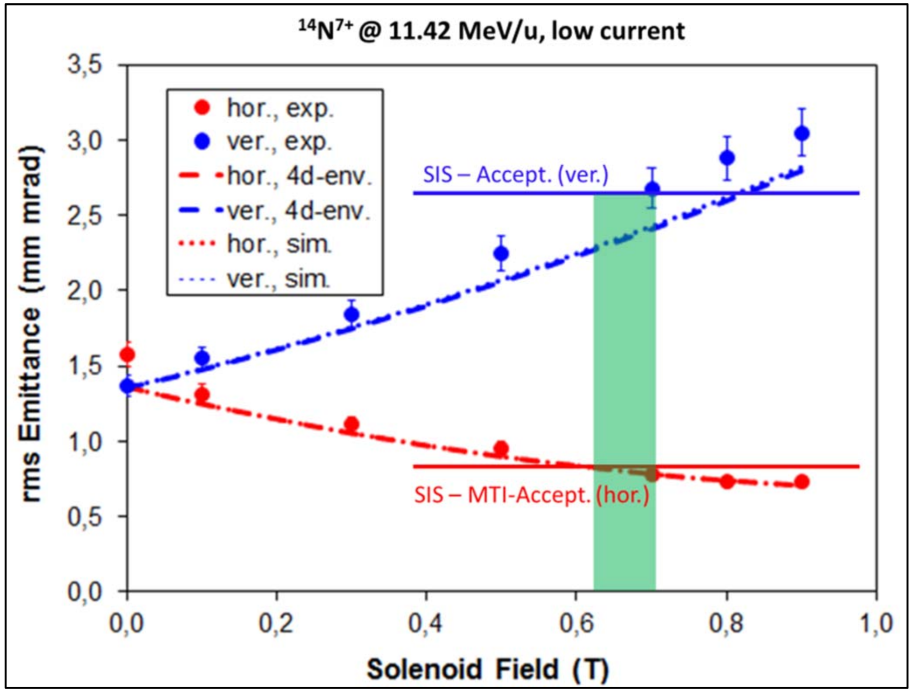
EmTEx increases injection efficiency into SIS18

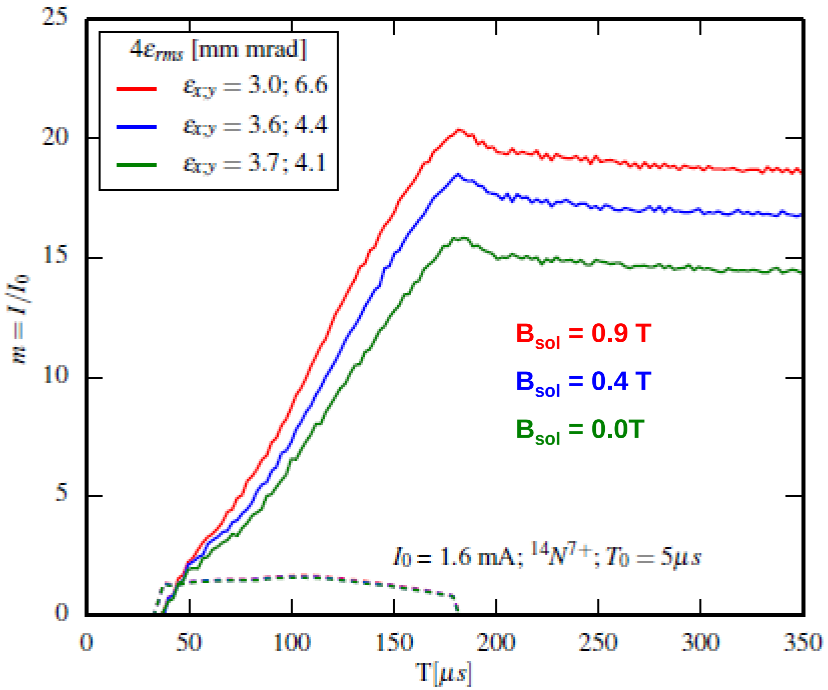


$$(\varepsilon_{x,7+} - \varepsilon_{y,7+})^2 = (\varepsilon_{x,3+} - \varepsilon_{y,3+})^2 + (\boldsymbol{A_f B_0})^2 \left[\frac{1}{(B\rho)_{7+}} - \frac{1}{(B\rho)_{3+}}\right]^2$$

emittance shaping

L. Groening et al. IPAC2015 MOPWA026





Summary

- Busch‘s original theorem for single particle was extended to particle beams
- Original and extended theorem look very similar

$$\frac{eq}{m\gamma}\psi + \oint_C \vec{v}\cdot d\vec{C} = const$$

$$(\varepsilon_{n1}-\varepsilon_{n2})^2 + \left[\frac{eq\psi}{mc\pi}\right]^2 + \frac{4eq\psi\beta\gamma}{mc\pi}\oint_C \vec{\vec{r}}\,' \cdot d\vec{C} = const$$

- Extended theorem for very fast modelling of emittance gymnastic exp. as
  - flat electron beams at FERMILAB
  - flat ion beams at GSI
- Its power is through provision of an invariant
- Using invariants is much more convenient than solving equs. of motion