Extension of Busch’s Theorem to Particle Beams

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• Hamiltonian, conjugate variables (short & basic)
• Busch theorem on single particle
• Applications:
  • electron cooling
  • magnetic bottle
• Projected rms-emittances, eigen emittances
• Extension of Busch theorem to beams
• Applications:
  • increase of FEL gain & collider luminosity
  • increase of injection efficiency into ion rings
Hamiltonian, conjugate variables

Hamiltonian is total energy:

\[
H := E_{\text{kin}} + E_{\text{pot}}
\]

\[
H(\hat{p}, \hat{r}) := \frac{\hat{p}^2}{2m} + V(\hat{r})
\]

\[
\hat{r} := \text{position}
\]

\[
\hat{p} := \text{mechanical momentum} = m\dot{\hat{v}} := m\dot{\hat{r}}
\]

Equations of motion through derivatives of \( H \):

\[
\dot{\hat{p}} := -\frac{\partial H}{\partial \hat{r}} = -\frac{\partial V}{\partial \hat{r}} = \hat{F}
\]

\[
\dot{\hat{r}} := \frac{\partial H}{\partial \hat{p}} = \frac{\hat{p}}{m}
\]

\( \hat{r} \) and \( \hat{p} \) are conjugate variables.
Force from magnetic field

force on particle by constant magnetic field \( \vec{B} := \begin{bmatrix} 0 \\ 0 \\ B_s \end{bmatrix} = \vec{\nabla} \times \vec{A} \)

\( \vec{F} = eq(\vec{v} \times \vec{B}) \)

\( F_x = \dot{p}_x = eqv_yB_s \)

\( F_y = \dot{p}_y = -eqv_xB_s \)

to obtain equations of motion from Hamiltonian mechanics → use „generalized“ momentum

\( \vec{p} := \vec{\pi} + eq\vec{A} \) in the Hamiltonian:

\( H(\vec{r}, \vec{p}) = \tilde{H}(\vec{r}, \vec{p}) = \frac{(\vec{p} - eq\vec{A})^2}{2m} \) and apply previous formalism:

\( \dot{\vec{p}} = \frac{\partial \tilde{H}}{\partial \vec{r}} \) together with \( \tilde{A} = \frac{1}{2} \begin{bmatrix} -yB_s \\ xB_s \\ 0 \end{bmatrix} \) and \( \dot{\vec{p}} = \dot{\vec{\pi}} + eq\dot{\vec{A}} \)
Generalized angular momentum

• analogue to generalized momentum, the generalized angular momentum is defined
  \[ \vec{L} := \vec{r} \times \vec{p} \]

• if there is just magnetic field, the generalized angular momentum is preserved
  \[ \vec{L} = \vec{r} \times (\vec{P} + eq\vec{A}) = \text{const} \]

• Busch theorem is special case for:
  • cylindrical symmetric magnetic field \( \vec{B} = \vec{B}(s) \)
  • s-component of \( \vec{L} \): \( L_s = \text{const} \)
Busch Theorem

cylindrically symmetric magnetic field with $\vec{V} \cdot \vec{B} = 0$

$$\vec{B} = \frac{1}{2} \begin{bmatrix} -xB_s' \\ -yB_s' \\ 2B_s \end{bmatrix} = \vec{V} \times \vec{A} = \frac{1}{2} \vec{V} \times \begin{bmatrix} -yB_s \\ xB_s \\ 0 \end{bmatrix}$$

$$L_s = [\vec{r} \times (\vec{P} + eq\vec{A})] \cdot \vec{e}_s = \text{const}$$

• particle entering into region with $\vec{B}$ (hence $\vec{A}$) acquires orbital angular momentum

• $\vec{r} \times \vec{A}$ has angular momentum as well

• sum of both acquired angular momenta is zero
Busch Theorem

Busch theorem from 1926 states

\[ L_s = \left[ \vec{r} \times (\vec{P} + e\vec{q}\vec{A}) \right] \cdot \vec{e}_s = \text{const} \]

using cylindrical coordinates:

\[ x = r \cos \theta, \quad y = r \sin \theta \]

\[ \vec{P} = m \begin{bmatrix} \dot{x} \\ \dot{y} \\ \beta c \end{bmatrix} \]

\[ L_s = mr^2 \dot{\theta} + \frac{1}{2} e q B_s r^2 = \text{const} \]

\[ L_s = mr^2 \dot{\theta} + \frac{eq}{2\pi} \Psi = \text{const} \]

\[ L_s \text{ = orbital angular momentum + flux through area of cyclotron motion} = \text{const} \]

H. Busch, Z. Phys. 81 (5) 924 (1926)
Busch Theorem

\[ mr_0^2 \dot{\theta}_0 + \frac{eq}{2} B_{s0} r_0^2 = mr^2 \dot{\theta} + \frac{eq}{2} B_s r^2 = \text{const} \]

preservation of magn. flux:

\[ B_{s0} r_0^2 = B_s r^2 = \text{const} \]

\[ r = r_0 \sqrt{\frac{B_0}{B_s}} \]
Applications:

Electron beam size shaping

\[ r = r_0 \sqrt{\frac{B_{s0}}{B_s}} \]

- cyclotron radius much smaller than beam size → beam size shaped by magnetic field
- applied at low energy electron beams

![Diagram](image_url)
Applications:
Electron beam temperature reduction

mean transv. velocity is measure for transv. beam temperature

\[ T_\perp \sim < v_\perp^2 > \]

using \( v_\perp = \frac{reqB_s}{m} \) and \( r = r_0 \sqrt{\frac{B_{s0}}{B_s}} \)

results into \( v_\perp = v_{\perp0} \frac{B_s}{B_{s0}} \)

electron beam expansion by magn. field expansion \( B_s < B_{s0} \):

• increases beam radius
• lowers beam temperature
• lowers electron density

technique applied at many e-coolers: 2*IMP, LEIR, TSR, CRYRING, SIS-18
Applications: Magnetic bottle

from Busch theorem:

\[ B_{0s} r_0^2 = B_s r^2, \quad \dot{\theta} = \frac{eqB_s}{m} = \frac{v_\perp}{r} \]

preservation of total kin. energy:

\[ v_{||0}^2 + v_{\perp0}^2 = v_{||}^2 + v_{\perp}^2 \]

\[ \rightarrow v_{||}^2(B_s) = v_{||0}^2 + v_{\perp0}^2 - \left[ \frac{eqr_0}{m} \right]^2 B_{s0}B_s \]

→ beam confinement by strong \( B_s \) in magn. bottles:
  • traps
  • ECR sources
Busch theorem can be further generalized to:

\[
\oint_{\mathcal{C}} \vec{v} \cdot d\vec{C} + \frac{eq}{m\gamma} \Psi = \text{const}
\]

\(C_i\) enclose possible single particle trajectories

\[C_i := \text{circles with constant } r_i \rightarrow m\gamma r^2 \dot{\theta} + \frac{e\gamma}{2\pi} \Psi = \text{const}\]
rms emittances defined through beam's second moments:

• $a_i, b_i$: two coordinates of particle $i$
• $<ab>$: mean of product $a_i b_i$
• $C$ is moment matrix (symmetric)

Projected rms emittance $(x, y, x', y')$ are laboratory coordinates which can be measured:

$$
\varepsilon_x^2 = <xx><x'x'> - <xx'>^2
$$

$$
C_x = \begin{bmatrix}
<xx> & <xx'> \\
<x'x> & <x'x'>
\end{bmatrix}, \quad \varepsilon_x^2 = \det C_x
$$

$$
C_y = \begin{bmatrix}
<yy> & <yy'> \\
<y'y> & <y'y'>
\end{bmatrix}, \quad \varepsilon_y^2 = \det C_y
$$
Transport of moments

linear transport from point_1 → point_2 through matrices:

\[
\begin{bmatrix}
  x \\
  x'
\end{bmatrix}_2 = M_x 
\begin{bmatrix}
  x \\
  x'
\end{bmatrix}_1
\]

\[
M_x = \begin{bmatrix}
  m_{11} & m_{12} \\
  m_{21} & m_{22}
\end{bmatrix}, \quad \text{det } M_x = 1
\]

beam moments transport by matrix equation:

\[
C_{x2} = M_x C_{x1} M_x^T
\]

analogue in y
4d linear beam dynamics

\[ \varepsilon_{4d}^2 = \det \begin{bmatrix} \langle xx \rangle & \langle xx' \rangle & \langle xy \rangle & \langle xy' \rangle \\ \langle x'x \rangle & \langle x'x' \rangle & \langle x'y \rangle & \langle x'y' \rangle \\ \langle yx \rangle & \langle yx' \rangle & \langle yy \rangle & \langle yy' \rangle \\ \langle y'x \rangle & \langle y'x' \rangle & \langle y'y \rangle & \langle y'y' \rangle \end{bmatrix} \]

if \( x \) \& \( y \) planes are not coupled

\[ \varepsilon_{4d}^2 = \det \begin{bmatrix} \langle xx \rangle & \langle xx' \rangle & 0 & 0 \\ \langle x'x \rangle & \langle x'x' \rangle & 0 & 0 \\ 0 & 0 & \langle yy \rangle & \langle yy' \rangle \\ 0 & 0 & \langle y'y \rangle & \langle y'y' \rangle \end{bmatrix} = (\varepsilon_x \cdot \varepsilon_y)^2 \]

transport of moments from 1 \( \to \) 2 as usual:

\[
\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_2 = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_1, \ \det M = 1
\]

\[ C_2 = M C_1 M^T \]

transport of moments from 1 \( \to \) 2 as usual:

\[
M = \begin{bmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{bmatrix}, \ \det M = \det M_x \cdot \det M_y = 1 \cdot 1 = 1
\]

\[ C_2 = M C_1 M^T \]
linear (4d), Hamiltonian beam line elements preserve:

- rms emittance \( \varepsilon_{4d}^2 \)

\[
\begin{bmatrix}
<xx> & <xx'> & <xy> & <xy'> \\
<x'x> & <x'x'> & <x'y> & <x'y'> \\
<yx> & <yx'> & <yy> & <yy'> \\
y'x & <y'x'> & <y'y> & <y'y'>
\end{bmatrix}
\]

\( \varepsilon_{4d}^2 = \text{det} \)


the two eigen-emittances

\[
\varepsilon_1 = \frac{1}{2} \sqrt{-tr[(CJ)^2] + \sqrt{tr^2[(CJ)^2] - 16\text{det}(C)}}
\]

\[
\varepsilon_2 = \frac{1}{2} \sqrt{-tr[(CJ)^2] - \sqrt{tr^2[(CJ)^2] - 16\text{det}(C)}}
\]

\[ C = \begin{bmatrix}
<xx> & <xx'> & <xy> & <xy'> \\
<x'x> & <x'x'> & <x'y> & <x'y'> \\
<yx> & <yx'> & <yy> & <yy'> \\
y'x & <y'x'> & <y'y> & <y'y'>
\end{bmatrix} \]

\[ J := \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0
\end{bmatrix} \]

the formulas are pretty ugly and their application consumes much time & paper
eigen-emittances

- if, and only if there is no $x \leftrightarrow y$ coupling, i.e. $C = \begin{bmatrix}
<xx> & <xx'> & 0 & 0 \\
<xx'> & <xx''> & 0 & 0 \\
0 & 0 & <yy> & <yy'> \\
0 & 0 & <yy'> & <yy''>
\end{bmatrix}$
  - rms emittances = eigen-emittances

- if there is any coupling
  - rms emittances ≠ eigen-emittances
  - coupling parameter $t = \frac{\varepsilon_x \varepsilon_y}{\varepsilon_1 \varepsilon_2} - 1 \geq 0$

term „eigen-emittance“ is quite unknown, since generally coupling is just ignored
rms vs eigen-emittances: Example

4d distribution behind ECR source

\[ \varepsilon_x = 123 \text{ mm mrad} \]
\[ \varepsilon_y = 125 \text{ mm mrad} \]
\[ \varepsilon_1 = 17 \text{ mm mrad} \]
\[ \varepsilon_2 = 231 \text{ mm mrad} \]

\[ \varepsilon_{4d} = \varepsilon_1 \cdot \varepsilon_2 = 3927 \text{ (mm mrad)}^2 \]
\[ \varepsilon_x \cdot \varepsilon_y = 15375 \text{ (mm mrad)}^2 \]
\[ \varepsilon_x \cdot \varepsilon_y = 3.9 \varepsilon_{4d} \]
Coupling linear elements:
Skew quadrupole

normal quadrupole
no x-y coupling

skew

tilted by 45° (clockwise)
x-y coupling
Coupling linear elements: Solenoid

\[ \kappa := \frac{B}{2(B\rho)} \]

\[ \alpha(L) = -2\kappa L \]

\[
M_{fi} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & \kappa & 0 \\
0 & 0 & 1 & 0 \\
-\kappa & 0 & 0 & 1
\end{bmatrix}
\]

\[
M_{||} = \begin{bmatrix}
1 & -\frac{1}{2\kappa}\sin(\alpha) & 0 & \frac{1-\cos(\alpha)}{2\kappa} \\
0 & \cos(\alpha) & 0 & -\sin(\alpha) \\
0 & \frac{\cos(\alpha)-1}{2\kappa} & 1 & -\frac{\sin(\alpha)}{2\kappa} \\
0 & \sin(\alpha) & 0 & \cos(\alpha)
\end{bmatrix}
\]

\[
M_{fo} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -\kappa & 0 \\
0 & 0 & 1 & 0 \\
\kappa & 0 & 0 & 1
\end{bmatrix}
\]

complete solenoid matrix \[ M_{sol} = M_{fo} \cdot M_{||} \cdot M_{fi} \]
How elements change emittances

applying ugly formulas →

<table>
<thead>
<tr>
<th>element</th>
<th>( \text{rms}_{x,y} )</th>
<th>4d rms</th>
<th>( \text{eigen}_{1,2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>drift</td>
<td>no</td>
<td>no</td>
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</tr>
<tr>
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<td>no</td>
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<td>no</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>yes</td>
</tr>
<tr>
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<td>no</td>
<td>yes</td>
</tr>
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<td>yes</td>
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</tr>
<tr>
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<td>yes</td>
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- eigen-emittances seem to change by magnetic flux through beam surface (note: in front of and behind solenoid flux is zero !)

- in the following this will be proven ...
Preservation of eigen emittances in conjugate coordinates

- reminder: generalized momentum \( \tilde{p} := \tilde{P} + eq\tilde{A} \), i.e.,
  \[
  p_x := x' + \frac{A_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)} \\
  p_y := y' + \frac{A_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}
  \]

- original ansatz of Busch for single particle:
  - angular momentum including the contribution from \( \tilde{r} \times eq\tilde{A} \) is preserved
  - „generalized angular momentum“ is preserved

- ansatz for extension to beams:
  - eigen-emittances including contribution from \( \tilde{r} := \tilde{A}/(B\rho) \) are preserved
  - „generalized“ eigen-emittances are preserved
Preservation of eigen emittances in conjugate coordinates

- calculation of „generalized“ eigen-emittance through replacing \((x', y')\) by \((p_x, p_y)\)

\[
C = \begin{bmatrix}
<xx> & <xx'> & <xy> & <xy'> \\
<xx'> & <xx''> & <xy'> & <xy''> \\
<yx> & <yx'> & <yy> & <yy'> \\
<y'x> & <y'x'> & <y'y> & <y'y'>
\end{bmatrix}
\]

\[
\tilde{C} = \begin{bmatrix}
<x^2> & <xp_x> & <xy> & <xp_y> \\
<xp_x> & <p_x'^2> & <yp_x> & <p_xp_y> \\
<xy> & <yp_x> & <y^2> & <yp_y> \\
<xp_y> & <p_xp_y> & <yp_y> & <p_y'^2>
\end{bmatrix}
\]

\[
J = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}
\]

\[
\tilde{\varepsilon}_{1/2} = \frac{1}{2} \sqrt{-\text{tr}[(\tilde{C}J)^2] \pm \sqrt{\text{tr}^2[(\tilde{C}J)^2] - 16 \det(\tilde{C})}}
\]

- if \(\tilde{\varepsilon}_{1/2}\) are preserved, also \(\tilde{\varepsilon}_1^2 + \tilde{\varepsilon}_2^2\) must be preserved

- state \(\tilde{\varepsilon}_1^2 + \tilde{\varepsilon}_2^2 = \text{const}\) by substituting

\[
p_x := x' + \frac{A_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)}
\]

\[
p_y := y' + \frac{A_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}
\]

...to obtain expression for useful „laboratory“ eigen-emittances...
Preservation of eigen emittances in conjugate coordinates

- state $\tilde{\varepsilon}_1^2 + \tilde{\varepsilon}_2^2 = \text{const}$ by substituting

  $$p_x := x' + \frac{A_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)}$$

  $$p_y := y' + \frac{A_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}$$

  with $A := \sqrt{\langle x^2 \rangle \langle y^2 \rangle - \langle xy \rangle^2}$

- this delivers:

  $$\langle \varepsilon_1 - \varepsilon_2 \rangle^2 + \left( \frac{AB_s}{(B\rho)} \right)^2 + \frac{2B_s}{(B\rho)} \left[ \langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yx' \rangle + \langle xy \rangle \left( \langle xx' \rangle - \langle yy' \rangle \right) \right] = \text{const}$$

- confirmed: change of laboratory eigen-emittances just through long. magn. field $B_s$

- quadrupoles and dipoles (even) skewed: $B_s = 0 \Rightarrow \varepsilon_{1/2} = \text{const}$
Sum of quantities is invariant

\[
(\varepsilon_1 - \varepsilon_2)^2 + \frac{AB_s}{(B\rho)}^2 + 2\frac{B_s}{(B\rho)} \left[ \langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yy' \rangle + \langle xy \rangle (\langle xx' \rangle - \langle yy' \rangle) \right] = \text{const}
\]

- sum of three quantities forms an invariant
- difference of eigen-emittances, flux through beam area, ......
- what is the third?
  - has dimension m^3
  - scales with beam rms area as for \( y = ax \) it vanishes
  - vanishes for uncorrelated beams
  - invariant under rotation around beam axis
  - investigate term for some examples ...
Understanding third term: Example objects

\[ W_A := \langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yx' \rangle + \langle xy \rangle (\langle xx' \rangle - \langle yy' \rangle) \]

**rigid object rotating with \( \omega \):**

\[ W_A = 2\omega A^2 = 2AL \]

**object under shear:**

\[ W_A = -aA^2 \neq 2AL \]

solenoid fringe field performs rigid beam rotation

thin skew quad performs shear
Understanding $W_A$: Transformations

\[ W_A := \langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yx' \rangle + \langle xy \rangle (\langle xx' \rangle - \langle yy' \rangle) \]

check, how $W_A$ is changed under transformation through:

- thin reg. quadrupole: $x' \rightarrow x' - qx$ and $y' \rightarrow y' + qy$
- thin skew quadrupole: ...
- short solenoid: ...

- $\rightarrow$ non of them changes $W_A$

scheme of mutual cancellation of constituents of $W_A$
Understanding $W_A$:

Pick idea from gen. Busch theorem

$$\oint_C \vec{v} \cdot d\vec{C} + \frac{eq}{m\gamma} \psi = \text{const}$$

sort of mean rotation around beam area

Ci enclose possible single particle trajectories

Try ansatz:

$$W_A = 2A \oint_C \vec{r}'(x, y, s) \cdot d\vec{C}'$$

mean angle integrated along curve enclosing beam area ... multiplied with beam area

$$\vec{r}'(x, y, s) := [x'(x, y, s), y'(x, y, s), 1]$$

mean: average $(x', y')$ at given $(x, y)$
Calculating $W_A$

\[ W_A = 2A \int_C \vec{r}(x, y, s) \cdot d\vec{c} \]

as $W_A$ is invariant under rotation
→ calculated for ellipse being turned upright
→ $<xy> = 0$!
Calculating $W_A$

\[
W_A = 2A \oint_C \mathbf{r}'(x, y, s) \cdot d\mathbf{C}
\]

Finally confirms:

\[
x = \sqrt{\langle x^2 \rangle} \cos \theta, \\
y = \sqrt{\langle y^2 \rangle} \sin \theta, \\
d\mathbf{C} = \begin{bmatrix} -x \\ y \end{bmatrix} d\theta
\]

Tylor expansion of $\mathbf{r}'$ to first order:

\[
\bar{x}'(x, y) := \bar{x}'(0, 0) + \frac{\partial \bar{x}'}{\partial x} x + \frac{\partial \bar{x}'}{\partial y} y, \\
\bar{y}'(x, y) := \bar{y}'(0, 0) + \frac{\partial \bar{y}'}{\partial x} x + \frac{\partial \bar{y}'}{\partial y} y.
\]

Finally confirms:

\[
W_A := \langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yx' \rangle + \langle x \rangle \langle \langle xy' \rangle - \langle yy' \rangle \rangle
\]
Busch theorem for particle beams

using expression for $W_A$ finally delivers:

$$(\varepsilon_1 - \varepsilon_2)^2 + \left[ \frac{AB_s}{(B\rho)} \right]^2 + \frac{4AB_s}{(B\rho)} \oint \vec{r}'d\vec{C} = \text{const}$$

acceleration can be included by initially multiplying with $m\gamma\beta c$ at both sides

$$p_x := x' + \frac{A_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)}$$

$$p_y := y' + \frac{A_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}$$

... resulting in Busch's theorem extended to accelerated particle beams:

$$(\varepsilon_{n1} - \varepsilon_{n2})^2 + \left[ \frac{eq\psi}{mc\pi} \right]^2 + \frac{4eq\psi\beta\gamma}{mc\pi} \oint \vec{r}' \cdot d\vec{C} = \text{const}$$
Busch theorem for particle beams

original Busch theorem for single particle:
\[ \frac{eq}{m\gamma} \psi + \oint_C \vec{v} \cdot d\vec{C} = \text{const} \]

theorem extended to accelerated particle beams:
\[ (\varepsilon_{n1} - \varepsilon_{n2})^2 + \left[ \frac{eq\psi}{mc\pi} \right]^2 + \frac{4eq\psi\beta\gamma}{mc\pi} \oint_C \vec{r}' \cdot d\vec{C} = \text{const} \]

- both expressions include flux and "vorticity" \( \vec{v} \cdot d\vec{C} \approx (\vec{V} \times \vec{v}) \, d\vec{A} \) (Stoke's law)
- the extended theorem additionally includes eigen-emittances
- theorem allows very fast modelling of setups for emittance gymnastics
Varification through simulations

- beam tracking through three solenoids
- extended fringe fields from $\vec{B}$-maps
- invariance confirmed
Flat electron beams, i.e., $\epsilon_x << \epsilon_y$ useful for:

- increase of luminosity in $e^-$ / $e^+$ colliders
- production of X-ray pulses with femto seconds in duration

Test accelerator at FERMILAB demonstrated $\epsilon_y / \epsilon_x = 100$:

- create beam at photo cathode being immersed into $B_s = B_0$
- reduce $B_s$ to zero, accelerate, and decouple x/y-planes
prior to formulation of extended Busch theorem, the deviaton of final beam emittances took several pages ...

\[ \varepsilon_{nf x/y} = \pm L \beta \gamma + \sqrt{(L \beta \gamma)^2 + \varepsilon_{4d}^2} \]

with \( \varepsilon_{4d} = \varepsilon_{ni1} \cdot \varepsilon_{ni2} \)

and \( L := (eB_0 A_0)/(2m \gamma \beta c) \)

Application to electron beams:
Flat beam creation at FERMILAB

applying theorem:
\[(\varepsilon_{n1} - \varepsilon_{n2})^2 + \left[ \frac{e\psi}{mc} \right]^2 + \frac{4e\psi\beta\gamma}{mc} \int \vec{r}'d\vec{C} = \text{const} \]

at cathode beam is symmetric:
- eigen = rms emittances: \(\varepsilon_{1/2} = \varepsilon_{x/y}\)
- eigen (rms) emittances are equal: \(\varepsilon_{1/x} = \varepsilon_{2/y}\)
- immersed into B-flux
- no x/y coupling \(\Rightarrow W_A = 0\)

final beam:
- eigen = rms emittances: \(\varepsilon_{1/2} = \varepsilon_{x/y}\)
- eigen (rms) emittances differ: \(\varepsilon_{1/x} \neq \varepsilon_{2/y}\)
- no B-flux
- x/y coupling removed \(\Rightarrow W_A = 0\)
Application to electron beams: Flat beam creation at FERMILAB

Replacing \( \varepsilon_{nf_y} = \varepsilon_{4d} / \varepsilon_{nf_x} \)

And \( eB_0A_0 = 2m\gamma\beta c \cdot L \)

Results into \( \varepsilon_{nf_x} = L\beta\gamma \pm \sqrt{(L\beta\gamma)^2 + \varepsilon_{4d}^2} \)

Using upper sign gives \( \varepsilon_{nf_x/y} = \pm L\beta\gamma + \sqrt{(L\beta\gamma)^2 + \varepsilon_{4d}^2} \)
Application to ion beams: EmTEx at GSI (Emitt. Transf. Exp.)

- beams from linacs: $\varepsilon_x \approx \varepsilon_y$
- hor. multi-turn injection into rings profits from $\varepsilon_x < \varepsilon_y$
- EmTEx @ transfer channel:
  - place charge state stripper inside short solenoid
  - $x/y$-decoupling afterwards
prior to formulation of extended Busch theorem, the deviation of final beam emittances took several pages ...

\[
(\varepsilon_{x,7} - \varepsilon_{y,7})^2 = (\varepsilon_{x,3} - \varepsilon_{y,3})^2 + (A_f B_0)^2 \left[ \frac{1}{(B\rho)_7} - \frac{1}{(B\rho)_3} \right]^2
\]

L. Groening, arXiv 1403.6962 (2014)
Application to ion beams: EmTEx at GSI (Emitt. Transf. Exp.)

change of q is a non-Hamiltonian action → „splitting“ into two Hamiltonian actions

extended theorem before stripping (q=3+) :

\[(\varepsilon_{1f} - \varepsilon_{2f})^2 + \left( \frac{A_f B_0}{(B\rho)_{3+}} \right)^2 + \frac{2B_0}{(B\rho)_{3+}} \mathcal{W}_{A_f} = \text{const} \]

extended theorem after stripping (q=7+) :

\[(\varepsilon_{1f} - \varepsilon_{2f})^2 + \left( \frac{A_f B_0}{(B\rho)_{7+}} \right)^2 + \frac{2B_0}{(B\rho)_{7+}} \mathcal{W}_{A_f} = \text{const} \]
Application to ion beams: EmTEx up to stripping foil

beam line entrance:

- eigen = rms emittances: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- measured: $\varepsilon_{x,3+}$ and $\varepsilon_{y,3+}$
- no B-flux
- no x/y coupling $\Rightarrow W_A = 0$

inside solenoid & just before foil:

- eigen $\neq$ rms emittances: $\varepsilon_{1/2} \neq \varepsilon_{x/y}$
- eigen emittances differ: $\varepsilon_1 \neq \varepsilon_2$
- B-flux
- x/y coupling $\Rightarrow W_{A_f} = -(B_0A_f^2)/(B\rho)_{3+}$

$A_f :=$ beam area at foil, from measurements, $\approx$ const along short solenoid

\[
(\varepsilon_{x,3+} - \varepsilon_{y,3+})^2 + 0 + 0 = (\varepsilon_{1f} - \varepsilon_{2f})^2 + \left[\frac{A_fB_0}{(B\rho)_{3+}}\right]^2 + \frac{2B_3}{(B\rho)_{3+}}W_{A_f}
\]
Application to ion beams: EmTEx up to stripping foil

\[(\varepsilon_{x,3+} - \varepsilon_{y,3+})^2 + 0 + 0 = (\varepsilon_{1f} - \varepsilon_{2f})^2 + \left[\frac{A_f B_0}{(B \rho)_{3+}}\right]^2 + \frac{2B_0}{(B \rho)_{3+}} W_{Af}\]

plugging in \(W_{Af} = -(B_0 A_f^2)/(B \rho)_{3+}\):  

\[(\varepsilon_{x,3+} - \varepsilon_{y,3+})^2 + 0 + 0 = (\varepsilon_{1f} - \varepsilon_{2f})^2 - \left[\frac{A_f B_0}{(B \rho)_{3+}}\right]^2\]

using experiment's parameters:

\[(\varepsilon_{1f} - \varepsilon_{2f})^2 = 2.755 \text{ (mm mrad)}^2\]
Application to ion beams: 
EmTEx behind the stripping foil

• foil changes just \( q \), i.e., \((B\rho)\)

• \( \varepsilon_{1,2,x,y} \), B-flux, and \( W_{Af} \) do not change

inside solenoid & just after foil:

• eigen ≠ rms emittances: \( \varepsilon_{1/2} \neq \varepsilon_{x/y} \)

• eigen emittances differ: \( \varepsilon_1 \neq \varepsilon_2 \)

• B-flux

• \( x/y \) coupling \( \rightarrow W_{Af} = -(B_0 A_f^2)/(B\rho)_{3+} \)

beam line exit:

• eigen = rms emittances: \( \varepsilon_{1/2} = \varepsilon_{x/y} \)

• to be calculated/measured: \( \varepsilon_{x,7+} \) and \( \varepsilon_{y,7+} \)

• no B-flux

• no \( x/y \) coupling \( \rightarrow W_A = 0 \)
Application to ion beams: EmTEx behind the stripping foil

\[
(\varepsilon_1 f - \varepsilon_2 f)^2 + \left[ \frac{A_f B_0}{(B \rho)_{7+}} \right]^2 + \frac{2B_0}{(B \rho)_{7+}} W_{Af} = (\varepsilon_{x,7+} - \varepsilon_{y,7+})^2 + 0 + 0
\]

plugging in \( W_{Af} = -(B_0 A_f^2)/(B \rho)_{3+} \) and \( (\varepsilon_1 f - \varepsilon_2 f)^2 = 2.755 \text{ (mm mrad)}^2 \)

finally delivers \( |\varepsilon_{x,7+} - \varepsilon_{y,7+}| = 2.21 \text{ mm mrad} \)

the measured values are:

- \( \varepsilon_{x,7+} = 2.76(14) \text{ mm mrad} \)
- \( \varepsilon_{y,7+} = 0.72(4) \text{ mm mrad} \)
- \( |\varepsilon_{x,7+} - \varepsilon_{y,7+}| = 2.04(14) \text{ mm mrad} \) → very good agreement to extended Busch theorem
EmTEx increases injection efficiency into SIS18

\[(\varepsilon_{x,y}^+ - \varepsilon_{x,y}^+) \times (\varepsilon_{x,y}^+ - \varepsilon_{x,y}^+) + (A_f B_0)^2 \left[ \frac{1}{(B \rho)^{x+}} - \frac{1}{(B \rho)^{y+}} \right]^2 \]

emittance shaping

L. Groening / Extension of Busch's Theorem to Particle Beams

B_{sol} = 0.9 T
B_{sol} = 0.4 T
B_{sol} = 0.0 T
Summary

- Busch’s original theorem for single particle was extended to particle beams

- Original and extended theorem look very similar

\[
\frac{e}{m\gamma} \psi + \int_C \vec{v} \cdot d\vec{C} = \text{const}
\]

\[
(e_{n1} - e_{n2})^2 + \left( \frac{e\psi}{mc\pi} \right)^2 + \frac{4e\psi\beta\gamma}{mc\pi} \int_C \vec{r}' \cdot d\vec{C} = \text{const}
\]

- Extended theorem for very fast modelling of emittance gymnastic exp. as
  - flat electron beams at FERMILAB
  - flat ion beams at GSI

- Its power is through provision of an invariant

- Using invariants is much more convenient than solving equs. of motion