

Extension of Busch's Theorem to Particle Beams

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Outline

- Hamiltonian, conjugate variables (short & basic)
- Busch theorem on single particle
- Applications:
 - electron cooling
 - magnetic bottle
- Projected rms-emittances, eigen emittances
- Extension of Busch theorem to beams
- Applications:
 - increase of FEL gain & collider luminosity
 - increase of injecton efficiency into ion rings



Hamiltonian, conjugate variables

Hamiltonian is total energy:

$$H \coloneqq E_{kin} + E_{pot}$$
$$H(\vec{p}, \vec{r}) \coloneqq \frac{\vec{P}^2}{2m} + V(\vec{r})$$
$$\vec{r} \coloneqq \text{position}$$
$$\vec{P} \coloneqq \text{mechanical momentum} = m\vec{v} \coloneqq m\dot{\vec{r}}$$



equations of motion through derivatives of H:

$$\dot{\vec{P}} := -\frac{\partial H}{d\vec{r}} = -\frac{\partial V}{d\vec{r}} = \vec{F}$$
$$\dot{\vec{r}} := \frac{\partial H}{\partial \vec{P}} = \frac{\vec{P}}{m}$$

 \vec{r} and \vec{P} are conjugate variables



Force from magnetic field

- force on particle by constant magnetic field $\vec{B} \coloneqq \begin{bmatrix} 0\\0\\B_c \end{bmatrix} = \vec{\nabla} \times \vec{A}$
- $\vec{F} = eq(\vec{v} \times \vec{B})$
- $F_{x} = \dot{P}_{x} = eqv_{y}B_{s}$ $F_{y} = \dot{P}_{y} = -eqv_{x}B_{s}$

to obtain equations of motion from Hamiltonian mechanics → use "generalized" momentum

 $\vec{p} := \vec{P} + eq\vec{A}$ in the Hamiltonian:

 $H(\vec{r},\vec{P}) = \tilde{H}(\vec{r},\vec{p}) = \frac{(\vec{p}-eq\vec{A})^2}{2m}$ and apply previous formalism :

$$\dot{\vec{p}} = \frac{\partial \tilde{H}}{\partial \vec{r}}$$
 together with $\vec{A} = \frac{1}{2} \begin{bmatrix} -yB_s \\ xB_s \\ 0 \end{bmatrix}$ and $\dot{\vec{p}} = \dot{\vec{P}} + eq\dot{\vec{A}}$



Generalized angular momentum

 analogue to generalized momentum, the generalized angular momentum is defined

 $\vec{L} \coloneqq \vec{r} \times \vec{p}$

- if there is just magnetic field, the generalized angular momentum is preserved $\vec{L} = \vec{r} \times (\vec{P} + eq\vec{A}) = const$
- Busch theorem is special case for :
 - cylindrical symmetric magnetic field $\vec{B} = \vec{B}(s)$
 - s-component of \vec{L} : $L_s = const$



Busch Theorem

cylindrically symmetric magnetic field with $\vec{\nabla} \vec{B} = 0$

$$\vec{B} = \frac{1}{2} \begin{bmatrix} -xB_{s}' \\ -yB_{s}' \\ 2B_{s} \end{bmatrix} = \vec{\nabla} \times \vec{A} = \frac{1}{2} \vec{\nabla} \times \begin{bmatrix} -yB_{s} \\ xB_{s} \\ 0 \end{bmatrix}$$

$$L_s = \left[\vec{r} \times (\vec{P} + eq\vec{A})\right] \cdot \vec{e}_s = const$$

- particle entering into region with \vec{B} (hence \vec{A}) acquires orbital angular momentum
- $\vec{r} \times \vec{A}$ has angular momentum as well
- sum of both acquired angular momenta is zero



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Busch Theorem

Busch theorem from 1926 states

$$L_s = \left[\vec{r} \times (\vec{P} + eq\vec{A})\right] \cdot \vec{e}_s = const$$

using cylindrical coordinates :

 $x = r \cos \theta, \qquad y = r \sin \theta$ $\vec{P} = m \begin{bmatrix} \dot{x} \\ \dot{y} \\ \beta c \end{bmatrix}$

$$L_{s} = mr^{2}\dot{\theta} + \frac{1}{2}eqB_{s}r^{2} = const$$
$$L_{s} = mr^{2}\dot{\theta} + \frac{eq}{2\pi}\Psi = const$$



H. Busch, Z. Phys. **81** (5) 924 (1926)

 L_s = orbital angular momentum + flux through area of cyclotron motion = const



Busch Theorem

$$mr_0^2 \dot{\theta}_0 + \frac{eq}{2}B_{s0}r_0^2 = mr^2 \dot{\theta} + \frac{eq}{2}B_sr^2 = const$$

preservation of magn. flux:

 $B_{s0}r_0^2 = B_s r^2 = const$





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Applications: Electron beam size shaping



$$r = r_0 \sqrt{\frac{B_{s0}}{B_s}}$$

- cyclotron radius much smaller than beam size → beam size shaped by magnetic field
- applied at low energy electron beams





Applications: Electron beam temperature reduction



mean transv. velocity is measure for transv. beam temperature

$$T_{\perp} \sim < v_{\perp}^2 >$$

using
$$v_{\perp} = \frac{reqB_s}{m}$$
 and $r = r_0 \sqrt{\frac{B_{s0}}{B_s}}$

results into
$$v_{\perp} = v_{\perp 0} \frac{B_s}{B_{s0}}$$

electron beam expansion by magn. field expansion $B_s < B_{s0}$:

- increases beam radius
- lowers beam temperature
- lowers electron density

technique applied at many e-coolers: 2*IMP, LEIR, TSR, CRYRING, SIS-18

Applications: Magnetic bottle



from Busch theorem:

 $B_{0s}r_0^2 = B_s r^2, \qquad \dot{\theta} = \frac{eqB_s}{m} = \frac{v_\perp}{r}$

preservation of total kin. energy :

$$v_{||0}^{2} + v_{\perp 0}^{2} = v_{||}^{2} + v_{\perp}^{2}$$

$$\rightarrow v_{||}^{2}(B_{s}) = v_{||0}^{2} + v_{\perp 0}^{2} - \left[\frac{eqr_{0}}{m}\right]^{2} B_{s0}B_{s}$$



- \rightarrow beam confinement by strong B_s in magn. bottles :
 - traps
 - ECR sources





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Generalized Busch Theorem

Busch theorem can be further generalized to :



$$C_i := circles with constant r_i \rightarrow m\gamma r^2 \dot{\theta} + \frac{eq}{2\pi} \Psi = const$$



Projected rms emittance

rms emittances defined through beam's second moments :

- a_i, b_i : two coordinates of particle i
- <ab>: mean of product a_ib_i
- C is moment matrix (symmetric)

$$\varepsilon_x^2 = \langle xx \rangle \langle x'x' \rangle - \langle xx' \rangle^2$$

$$C_{x} = \begin{bmatrix} \langle xx \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'x' \rangle \end{bmatrix}, \quad \varepsilon_{\chi}^{2} = \det C_{\chi}$$
$$C_{y} = \begin{bmatrix} \langle yy \rangle & \langle yy' \rangle \\ \langle y'y \rangle & \langle y'y' \rangle \end{bmatrix}, \quad \varepsilon_{y}^{2} = \det C_{y}$$

(x,y,x',y') are laboratory coordinates which can be measured



Transport of moments

linear transport from point_1 \rightarrow point_2 through matrices :

$$\begin{bmatrix} x \\ x' \end{bmatrix}_2 = M_x \begin{bmatrix} x \\ x' \end{bmatrix}_1 \qquad \qquad M_x = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \ det \ M_x = 1$$

beam moments transport by matrix equation :

$$C_{x2} = M_x C_{x1} M_x^T$$

analogue in y



4d linear beam dynamics

$$\varepsilon_{4d}^{2} = det \begin{bmatrix} \langle xx \rangle & \langle xx' \rangle & \langle xy \rangle & \langle xy' \rangle \\ \langle x'x \rangle & \langle x'x' \rangle & \langle x'y \rangle & \langle x'y' \rangle \\ \langle yx \rangle & \langle yx' \rangle & \langle yy \rangle & \langle yy' \rangle \\ \langle y'x \rangle & \langle y'x' \rangle & \langle y'y \rangle & \langle y'y' \rangle \end{bmatrix}$$

transport of moments from $1 \rightarrow 2$ as usual :

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_{2} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_{1}, det M = 1$$

 $C_2 = M C_1 M^T$

if x & y planes are not coupled

$$\varepsilon_{4d}^{2} = det \begin{bmatrix} \langle xx \rangle \langle xx' \rangle & 0 & 0 \\ \langle x'x \rangle \langle x'x' \rangle & 0 & 0 \\ 0 & 0 & \langle yy \rangle \langle yy' \rangle \\ 0 & 0 & \langle y'y \rangle \langle y'y' \rangle \end{bmatrix} = (\varepsilon_{x} \cdot \varepsilon_{y})^{2}$$

transport of moments from $1 \rightarrow 2$ as usual :

$$M = \begin{bmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{bmatrix}, \quad \det M = \det M_x \cdot \det M_y = 1 \cdot 1 = 1$$

$$C_2 = M C_1 M^4$$



Eigen-emittances

• linear (4d), Hamiltonian beam line elements preserve :

• rms emittance
$$\varepsilon_{4d}^2 = det \begin{bmatrix} \langle xx \rangle \langle xx' \rangle \langle xy \rangle \langle xy' \rangle \\ \langle x'x \rangle \langle x'x' \rangle \langle x'y \rangle \langle x'y' \rangle \\ \langle yx \rangle \langle yx' \rangle \langle yy \rangle \langle yy' \rangle \\ \langle y'x \rangle \langle y'x' \rangle \langle y'y \rangle \langle y'y' \rangle \end{bmatrix}$$

A.J. Dragt, Phys. Rev. A 45 4 (1992)

$$\varepsilon_1 = \frac{1}{2}\sqrt{-tr[(CJ)^2] + \sqrt{tr^2[(CJ)^2] - 16det(C)}}$$

• the two eigen-emittances

 $\varepsilon_{4d} = \varepsilon_1 \cdot \varepsilon_2$

$$\varepsilon_2 = \frac{1}{2}\sqrt{-tr[(CJ)^2] - \sqrt{tr^2[(CJ)^2] - 16det(C)}}$$

$$C = \begin{bmatrix} \langle xx \rangle & \langle xx' \rangle & \langle xy \rangle & \langle xy' \rangle \\ \langle x'x \rangle & \langle x'x' \rangle & \langle x'y \rangle & \langle x'y' \rangle \\ \langle yx \rangle & \langle yx' \rangle & \langle yy \rangle & \langle yy' \rangle \\ \langle y'x \rangle & \langle y'x' \rangle & \langle y'y \rangle & \langle y'y' \rangle \end{bmatrix} \qquad J := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

• the formulas are pretty ugly and their application consumes much time & paper



eigen-emittances

• if, and only if there is no
$$x \leftrightarrow y$$
 coupling, i.e. $C = \begin{bmatrix} \langle xx \rangle \langle x'x \rangle & 0 & 0 \\ \langle x'x \rangle \langle x'x' \rangle & 0 & 0 \\ 0 & 0 & \langle yy \rangle \langle yy' \rangle \\ 0 & 0 & \langle y'y \rangle \langle y'y' \rangle \end{bmatrix}$

- rms emittances = eigen-emittances
- if there is any coupling
 - rms emittances ≠ eigen-emittances

• coupling parameter
$$t = \frac{\varepsilon_x \varepsilon_y}{\varepsilon_1 \varepsilon_2} - 1 \ge 0$$

term "eigen-emittance" is quite unknown, since generally coupling is just ignored

rms vs eigen-emittances: Example





coutesy P. Spädtke

4d distribution behind ECR source

- $\varepsilon_x = 123 \text{ mm mrad}$ $\varepsilon_y = 125 \text{ mm mrad}$
- $\epsilon_1 = 17 \text{ mm mrad}$ $\epsilon_2 = 231 \text{ mm mrad}$
- $\begin{aligned} \epsilon_{4d} &= \epsilon_1 \cdot \epsilon_2 = 3927 \text{ (mm mrad)}^2 \\ \epsilon_x \cdot \epsilon_y &= 15375 \text{ (mm mrad)}^2 \\ \epsilon_x \cdot \epsilon_y &= 3.9 \epsilon_{4d} \end{aligned}$

Coupling linear elements: Skew quadrupole





Coupling linear elements: FAIR **ES I** Solenoid $\kappa := \frac{B}{2(B\rho)}$ $\alpha(L) = -2\kappa L$ axial field entrance fringe exit fringe $M_{fi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \kappa & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \dots & 0 & 0 & 1 \end{bmatrix} \qquad M_{||} = \begin{bmatrix} 1 & -\frac{1}{2\kappa}sin(\alpha) & 0 & \frac{1-cos(\alpha)}{2\kappa} \\ 0 & cos(\alpha) & 0 & -sin(\alpha) \\ 0 & \frac{cos(\alpha)-1}{2\kappa} & 1 & -\frac{sin\alpha}{2\kappa} \\ 0 & sin(\alpha) & 0 & cos(\alpha) \end{bmatrix} \qquad M_{fo} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\kappa & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

complete solenoid matrix $M_{sol} = M_{fo} \cdot M_{||} \cdot M_{fi}$



How elements change emittances

applying ugly formulas \rightarrow

element	rms _{x,y}	4d rms	eigen _{1,2}
drift	no	no	no
quadrupole	no	no	no
tilted quadrupole	yes	no	no
dipole	no	no	no
tilted dipole	yes	no	no
solenoid	yes	no	no
solenoid fringe	yes	no	yes
solenoid axial field	yes	no	yes



How elements change emittances

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drift	no	no	no
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solenoid	yes	no	no
solenoid fringe	yes	no	yes
solenoid axial field	yes	no	yes



- eigen-emittances seem to change by magnetic flux through beam surface (note: in front of and behind solenoid flux is zero !)
- in the following this will be proven ...

Preservation of eigen emittances in conjugate coordinates



• reminder: generalized momentum $\vec{p} := \vec{P} + eq\vec{A}$, i.e.,

$$p_x := x' + \frac{\mathcal{A}_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)}$$
$$p_y := y' + \frac{\mathcal{A}_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}$$

- original ansatz of Busch for single particle :
 - angular momentum including the contribution from $\vec{r} \times eq \vec{A}$ is preserved
 - "generalized angular momentum" is preserved

- ansatz for extension to beams :
 - eigen-emittances including contribution from $\dot{\vec{r}} \coloneqq \vec{A}/(B\rho)$ are preserved
 - "generalized" eigen-emittances are preserved

Preservation of eigen emittances in conjugate coordinates



• calculation of "generalized" eigen-emittance through replacing (x',y') by (p_x,p_y)

$$C = \begin{bmatrix} \langle xx \rangle \langle xy \rangle \langle xy \rangle \langle xy \rangle \langle xp_y \rangle \\ \langle x'x \rangle \langle x'x' \rangle \langle x'y \rangle \langle x'y' \rangle \\ \langle yx \rangle \langle yx \rangle \langle yy \rangle \langle yy \rangle \langle yy \rangle \rangle \\ \langle y'x \rangle \langle y'x \rangle \langle y'y \rangle \langle y'y \rangle \langle y'y' \rangle \end{bmatrix} \longrightarrow \tilde{C} = \begin{bmatrix} \langle x^2 \rangle \langle xp_x \rangle \langle xp_x \rangle \langle xy \rangle \langle xy \rangle \langle xp_y \rangle \\ \langle xp_x \rangle \langle p_x^2 \rangle \langle yp_x \rangle \langle p_xp_y \rangle \\ \langle xp_y \rangle \langle yp_x \rangle \langle yp_y \rangle \langle yp_y \rangle \\ \langle xp_y \rangle \langle p_xp_y \rangle \langle yp_y \rangle \langle yp_y \rangle \langle p_y^2 \rangle \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad \tilde{\varepsilon}_{1/2} = \frac{1}{2}\sqrt{-tr[(\tilde{C}J)^2] \pm \sqrt{tr^2[(\tilde{C}J)^2] - 16\det(\tilde{C})}}$$

- if $\tilde{\varepsilon}_{1/2}$ are preserved, also $\tilde{\varepsilon}_1^2 + \tilde{\varepsilon}_2^2$ must be preserved
- state $\tilde{\varepsilon}_1^2 + \tilde{\varepsilon}_2^2 = const$ by substituting

$$p_x := x' + \frac{\mathcal{A}_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)}$$
$$p_y := y' + \frac{\mathcal{A}_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}$$

to obtain expression for useful "laboratory" eigen-emittances

Preservation of eigen emittances in conjugate coordinates



• state
$$\tilde{\varepsilon_1}^2 + \tilde{\varepsilon_2}^2 = const$$
 by substituting
 $p_x := x' + \frac{A_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)}$
 $p_y := y' + \frac{A_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}$
with $A := \sqrt{\langle x^2 \rangle \langle y^2 \rangle - \langle xy \rangle^2}$

• this delivers:

$$(\varepsilon_1 - \varepsilon_2)^2 + \left[\frac{AB_s}{(B\rho)}\right]^2 + 2\frac{B_s}{(B\rho)}\left[\langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yx' \rangle + \langle xy \rangle (\langle xx' \rangle - \langle yy' \rangle)\right] = const$$

- confirmed: change of laboratory eigen-emittances just through long. magn. field B_s
- quadrupoles and dipoles (even) skewed: $B_s = 0 \rightarrow \varepsilon_{1/2} = const$



Sum of quantities is invariant

$$(\varepsilon_1 - \varepsilon_2)^2 + \left[\frac{AB_s}{(B\rho)}\right]^2 + 2\frac{B_s}{(B\rho)}\left[\langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yx' \rangle + \langle xy \rangle (\langle xx' \rangle - \langle yy' \rangle)\right] = const$$

- sum of three quantities forms an invariant
- difference of eigen-emittances, flux through beam area,
- what is the third ?
 - has dimension m³
 - scales with beam rms area as for y = ax it vanishes
 - vanishes for uncorrelated beams
 - invariant under rotation around beam axis
 - investigate term for some examples ...

Understanding third term: Example objects



$$\mathcal{W}_A := \langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yx' \rangle + \langle xy \rangle (\langle xx' \rangle - \langle yy' \rangle)$$





object under shear :

 $W_A = -aA^2 \neq 2AL$

solenoid fringe field performs rigid beam rotation



Understanding *W_A*: Transformations



$$\mathcal{W}_A := \langle y^2 \rangle \langle xy' \rangle - \langle x^2 \rangle \langle yx' \rangle + \langle xy \rangle (\langle xx' \rangle - \langle yy' \rangle)$$

check, how W_A is changed under transformation through :

• thin reg. quadrupole: $x' \to x' - qx$ and $y' \to y' + qy$

. . .

- thin skew quadrupole: ...
- short solenoid:
- \rightarrow non of them changes W_A



Understanding W_A : Pick idea from gen. Busch theorem







C_i enclose possible single particle trajectories

sort of mean rotation around beam area

try ansatz :

$$\mathcal{W}_A = 2A \oint_C \vec{r'}(x, y, s) \cdot d\vec{C}$$

mean angle integrated along curve enclosing beam area ... multiplied with beam area

S

 $\vec{\bar{r'}}(x,y,s) := [\bar{x'}(x,y,s), \bar{y'}(x,y,s), 1]$

mean: average (xʻ,yʻ) at given (x,y)



Calculating W_A

$$\mathcal{W}_A = 2A \oint_{\mathcal{C}} \vec{r'}(x, y, s) \cdot d\vec{C}$$



as W_A is invariant under rotation \rightarrow calculated for ellipse being turned upright $\rightarrow \langle xy \rangle = 0 !$



Calculating W_A







Busch theorem for particle beams

using expression for W_A finally delivers :

$$(\varepsilon_1 - \varepsilon_2)^2 + \left[\frac{AB_s}{(B\rho)}\right]^2 + \frac{4AB_s}{(B\rho)} \oint_{\mathcal{C}} \vec{r'} d\vec{C} = const$$

acceleration can be included by initially multiplying

$$p_x := x' + \frac{\mathcal{A}_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)}$$
$$p_y := y' + \frac{\mathcal{A}_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}$$

with $m\gamma\beta c$ at both sides

... resulting in Busch's theorem extended to accelerated particle beams :

$$(\varepsilon_{n1} - \varepsilon_{n2})^2 + \left[\frac{eq\psi}{mc\pi}\right]^2 + \frac{4eq\psi\beta\gamma}{mc\pi} \oint_{\mathcal{C}} \vec{r'} \cdot d\vec{C} = const$$



Busch theorem for particle beams

original Busch theorem for single particle :

$$\frac{eq}{m\gamma}\psi + \oint_{\mathcal{C}} \vec{v} \cdot d\vec{C} = const$$

theorem extended to accelerated particle beams :

$$(\varepsilon_{n1} - \varepsilon_{n2})^2 + \left[\frac{eq\psi}{mc\pi}\right]^2 + \frac{4eq\psi\beta\gamma}{mc\pi} \oint_{\mathcal{C}} \vec{r'} \cdot d\vec{C} = const$$

- both expressions include flux and <u>"vorticity</u>" $\vec{v} \cdot d\vec{C} \simeq (\vec{\nabla} \times \vec{v}) d\vec{A}$ (Stoke's law)
- the extended theorem additionally includes eigen-emittances
- theorem allows very fast modelling of setups for emittance gymnastics



Varification through simulations

- beam tracking through three solenoids
- extended fringe fields from \vec{B} -maps
- invariance confirmed



flat electron beams, i.e., $\varepsilon_x << \varepsilon_y$ useful for :

- increase of luminosity in e⁻ / e⁺ colliders
- production of X-ray pulses with femto seconds in duration

test accelerator at FERMILAB demonstrated $\varepsilon_v / \varepsilon_x = 100$:

- create beam at photo cathode being immersed into $B_s = B_0$
- reduce B_s to zero, accelerate, and decouple x/y planes

Northern Illinois Center for Accelerator and Detector Development (NICADD)

P. Piot et al., Phys. Rev. ST Accel. Beams 9 031001 (2006)

prior to formulation of extended Busch theorem, the deviaton of final beam emittances took several pages ...

$$\varepsilon_{nfx/y} = \pm L\beta\gamma + \sqrt{(L\beta\gamma)^2 + \varepsilon_{4d}^2}$$

with $\varepsilon_{4d} = \varepsilon_{ni1} \cdot \varepsilon_{ni2}$

and $L \coloneqq (eB_0A_0)/(2m\gamma\beta c)$

Kwang-Je Kim, Phys. Rev. ST Accel. Beams 6 104002 (2006)

PRST-AB 6 ROUND-TO-PLAT TRANSP.	RMATION OF ANGULAR 104002 (2003)	PRST-AB 6	KWANG-JE
vanishing initial emittance. Finally, Sec. VII contains concluding remarks. II, 4D SYMPLECTIC TRANSFORMATION AND	Here, for example, $\langle X \hat{Y} \rangle$ is the 2 × 2 matrix: $\langle X \hat{Y} \rangle = \begin{bmatrix} \langle xy \rangle & \langle xp_j \rangle \\ \langle p_x \rangle & \langle p_x p_y \rangle \end{bmatrix}$. (9)	where $T_d = D(d)T_0L$	Š(d), (20)
TWO INVARIANTS We consider particle distribution in 4D transverse	The transformation, Eq. (3), induces the transforma- tion of the beam matrix:	$D(d) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	i (21)
phase space. The coordinates in this space are specified by a four-component vector. We find it convenient to represent the four-component vector in terms of two	$\Sigma \rightarrow M \Sigma \tilde{M}$. (10)	$T_0 = \begin{bmatrix} \rho & 0\\ 0 & 1/ \end{bmatrix}$ Here the quantity e_{st} and β ca	β. (22) B in be interpreted as the
two-component vectors: $\mathbf{y} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix}$ (1)	Because of the simplecticity of M, the transformation (10) leaves the following two quantities invariant:	emitance and the Courant-Say respectively [8]. Collecting these results, we ca	der envelope function, an write a cylindrically in
~ [p,] · [p,] ···	$\kappa_{4D} = det(\Sigma),$ (11)	symmetric beam in the following	form:
respectively, and	$I_2(\Sigma) = -\frac{1}{2}T_R(J_4\Sigma J_4\Sigma).$ (12)	$\begin{bmatrix} z \\ -LJ \\ -LJ \end{bmatrix} = \mathbb{Z}$	$\begin{bmatrix} LJ \\ atT_d \end{bmatrix}$. (23) T
$(p_s, p_y) = \frac{p_s}{mc}(x', y') = \frac{p_s}{mc}\left(\frac{d}{ds}x \frac{d}{ds}y\right).$ (2)	In the above, det denotes the determinant and T_R the trace. The quantity $\kappa_{\rm 4D}$ is well known and can be inter-	With cylindrical symmetry, the greatly simplified, requiring on B, L, and d.	beam matrix is thus ly the parameters eer, is
where p_i is the momentum in the axial direction, w is the particle mass, c is the speed of light, and s is the distance along the axial direction. As the beam is transported	ant Eq. (12) was pointed out by Rangarajan et al. in the context of beam physics [11].	The quantity <i>L</i> is one-half of since	the angular momentum
along an accelerator, the phase-space coordinates are transformed as	IIL PROPERTIES OF ROUND BEAMS	$(xp_y - yp_x) =$ Heaveyer, it is not the kinetic and	ular momentum since it
$\begin{bmatrix} X \\ Y \end{bmatrix} \rightarrow M \begin{bmatrix} X \\ Y \end{bmatrix}$ (3)	The beam matrix Σ is symmetric and thus in general contains ten independent elements. In many cases of interest, however, beams are generated, accelerated, and	is measured in the rotating frame the canonical angular momentum [10]. The canonical angular more	e defined in Sec. V. It is in the laboratory frame mentum is conserved in
We will limit our discussion in this paper to the case where M is linear, and the motion is Hamiltonian. The M matrix is then symplectic:	transported in a cylindrically symmetric environment. The beam matrix must then be cylindrically symmetric:	the presence of a cylindrically sy- field. Experimental measurement momentum was discussed recent	nmetric axial magnetic d t of the beam angular d hv [13].
$\vec{M}J_LM = J_L$ (4)	$\Sigma = M_R(\theta)\Sigma M_R^{-1}(\theta). (13)$	Equation (23) can also be write	ien as n
Here – denotes the transpose operation, and J_4 is the	Here $M_R(\theta)$ is the matrix representing a rotation around the beam axis:	$\Sigma = M_d \Sigma_0 \dot{b}$ where	ί _d , (25) s
$J_4 = \begin{bmatrix} J & 0\\ 0 & J \end{bmatrix}$ (5)	$M_R(\theta) = \begin{bmatrix} I \cos\theta & I \sin\theta \\ -I \sin\theta & I \cos\theta \end{bmatrix}$ (14)	$M_d = \begin{bmatrix} D(d) \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ D(d) \end{bmatrix}$ (26)
where we have introduced the 2×2 unit symplectic matrix	where I is the 2 × 2 unity matrix. By demanding that Eq. (13) be satisfied for an arbitrary θ , we obtain the following conditions:	$\Sigma_0 = \begin{bmatrix} \sigma_{eB}T_0 \\ -LJ \end{bmatrix} = \sigma_0 \Sigma$	$\begin{bmatrix} LJ \\ c_{eff}T_0 \end{bmatrix}$ (27)
$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$ (6)	$\langle X \bar{X} \rangle = \langle Y \bar{Y} \rangle$ (15)	In obtaining Eq. (25), we have un	ed the relation
An extensive discussion of linear and nonlinear symplec- tic transformations can be found in [12], including the fact that if <i>M</i> is symplectic then so is <i>M</i> .	$\langle X \vec{X} \rangle = - \langle V \vec{X} \rangle.$ (16) From Eq. (16), it follows:	D _d /D _d = 2 which follows from the symplecti Equation (25) represents the trans	 (28) city of D_d in x subspace. slation to the location of
The global properties of a beam are described by beam moments. Assuming that the beam is centered properly.	$\langle \hat{X}\hat{Y} \rangle = -\langle \hat{x}\hat{X} \rangle = -\langle \hat{x}\hat{Y} \rangle.$ (17)	the beam waist. In the following beam is at the waist since the tr	we will assume that the F anslation to other loca-
the first-order moments vanish: (Y) = (Y) = 0 (7)	Therefore the 2 \times 2 matrix (XF) is antisymmetric and can be written as	A cylindrically symmetric be round in this paper	sam is also said to be p
$\sqrt{\alpha_f} - \sqrt{\beta} = 0.$ (7) Here the angular brackets imply taking the average. The	(XF) = LJ. (18)	IV TRANSFORMATION O	F ROUND BEAMS
second-order beam moments can be organized into the 4 × 4 beam matrix:	where J is the 2×2 unit symplectic matrix, Eq. (6). The symmetric 2×2 matrix $(X\hat{X})$ can be written as follows:	To compute the two invarian corresponding to the cylindrical	ts, Eqs. (11) and (12), o ly symmetric Σ matrix, o
$\Sigma = \begin{pmatrix} \chi \chi \end{pmatrix} \langle \chi Y \rangle$ (8)			

 $-(e^2 - L^2)$

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at cathode beam is symmetric:

- eigen = rms emittances: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- eigen (rms) emittances are equal: $\varepsilon_{1/x} = \varepsilon_{2/y}$
- immersed into B-flux
- no x/y coupling $\rightarrow \mathcal{W}_{A} = 0$

final beam:

- eigen = rms emittances: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- eigen (rms) emittances differ: $\varepsilon_{1/x} \neq \varepsilon_{2/y}$
- no B-flux
- x/y coupling removed $\rightarrow W_A = 0$

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replacing $\varepsilon_{nfy} = \varepsilon_{4d}/\varepsilon_{nfx}$

and $eB_0A_0 = 2m\gamma\beta c \cdot L$

results into $\varepsilon_{nfx} = L\beta\gamma \pm \sqrt{(L\beta\gamma)^2 + \varepsilon_{4d}^2}$

using upper sign gives $\varepsilon_{nfx/y} = \pm L\beta\gamma + \sqrt{(L\beta\gamma)^2 + \varepsilon_{4d}^2}$

Application to ion beams: EmTEx at GSI (Emitt. Transf. Exp.)

- beams from linacs: $\varepsilon_x \approx \varepsilon_y$
- hor. multi-turn injection into rings profits from $\varepsilon_x < \varepsilon_y$
- EmTEx @ transfer channel :
 - place charge state stripper inside short solenoid
 - x/y-decoupling afterwards

Application to ion beams: EmTEx at GSI (Emitt. Transf. Exp.)

prior to formulation of extended Busch theorem, the deviaton of final beam emittances took several pages ...

$$(\varepsilon_{x,7+} - \varepsilon_{y,7+})^2 = (\varepsilon_{x,3+} - \varepsilon_{y,3+})^2 + (A_f B_0)^2 \left[\frac{1}{(B\rho)_{7+}} - \frac{1}{(B\rho)_{3+}}\right]^2$$

C. Xiao et. al, Phys. Rev. ST Accel. Beams 16 044201 (2013)

L. Groening, arXiv 1403.6962 (2014)

Application to ion beams: EmTEx at GSI (Emitt. Transf. Exp.)

change of q is a non-Hamiltonian action → "splitting" into two Hamiltonian actions

extended theorem before stripping (q=3+) :

$$(\varepsilon_{1f} - \varepsilon_{2f})^2 + \left[\frac{A_f B_0}{(B\rho)_{3+}}\right]^2 + \frac{2B_0}{(B\rho)_{3+}} \mathcal{W}_{Af} = const$$

extended theorem after stripping (q=7+) :

$$(\varepsilon_{1f} - \varepsilon_{2f})^2 + \left[\frac{A_f B_0}{(B\rho)_{7+}}\right]^2 + \frac{2B_0}{(B\rho)_{7+}} \mathcal{W}_{Af} = \overline{const}$$

Application to ion beams: EmTEx up to stripping foil

beam line entrance :

- eigen = rms emittances: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- measured: $\varepsilon_{x,3+}$ and $\varepsilon_{y,3+}$
- no B-flux
- no x/y coupling $\rightarrow \mathcal{W}_{\mathcal{A}} = 0$

inside solenoid & just before foil :

- eigen \neq rms emittances: $\varepsilon_{1/2} \neq \varepsilon_{x/y}$
- eigen emittances differ: $\varepsilon_1 \neq \varepsilon_2$
- B-flux
- x/y coupling $\rightarrow W_{Af} = -(B_0 A_f^2)/(B\rho)_{3+}$

Application to ion beams: EmTEx up to stripping foil

plugging in
$$W_{Af} = -(B_0 A_f^2)/(B\rho)_{3+}$$
:
 $(\varepsilon_{x,3+} - \varepsilon_{y,3+})^2 + 0 + 0 = (\varepsilon_{1f} - \varepsilon_{2f})^2 - \left[\frac{A_f B_0}{(B\rho)_{3+}}\right]^2$

using experiment's parameters :

$$(\varepsilon_{1f} - \varepsilon_{2f})^2 = 2.755 \ (mm \ mrad)^2$$

Application to ion beams: EmTEx behind the stripping foil

- foil changes just q, i.e., $(B\rho)$ ٠ • $\varepsilon_{1,2,x,y}$, B-flux, and W_{Af} do not change $(\varepsilon_{1f} - \varepsilon_{2f})^2 + \left[\frac{A_f B_0}{(B\rho)_{7\perp}}\right]$ $+ \frac{2B_0}{(B\rho)_{7+}} \mathcal{W}_{Af} = \left[(\varepsilon_{x,7+} - \varepsilon_{y,7+})^2 + 0 + 0 \right]$ inside solenoid & just after foil :
- eigen \neq rms emittances: $\varepsilon_{1/2} \neq \varepsilon_{x/y}$
- eigen emittances differ: $\varepsilon_1 \neq \varepsilon_2$ ٠
- **B**-flux ٠
- x/y coupling $\rightarrow W_{Af} = -(B_0 A_f^2)/(B\rho)_{3+}$ •

- beam line exit :
- eigen = rms emittances: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- to be calculated/measured: $\varepsilon_{\chi,7+}$ and $\varepsilon_{\gamma,7+}$ •
- no B-flux •
- no x/y coupling $\rightarrow W_{A} = 0$

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Application to ion beams: EmTEx behind the stripping foil

plugging in
$$W_{Af} = -(B_0 A_f^2)/(B\rho)_{3+}$$
 and $(\varepsilon_{1f} - \varepsilon_{2f})^2 = 2.755 \ (mm \ mrad)^2$

finally delivers $|\varepsilon_{x,7+} - \varepsilon_{y,7+}| = 2.21 \ mm \ mrad$

the measured values are :

- $\varepsilon_{x,7+} = 2.76(14) mm mrad$
- $\varepsilon_{y,7+} = 0.72(4) mm mrad$
- $|\varepsilon_{x,7+} \varepsilon_{y,7+}| = 2.04(14) \, mm \, mrad$
- L. Groening et. al, Phys. Rev. Lett. 113 044201 (2014)
- \rightarrow very good agreement to extended Busch theorem

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EmTEx increases injection efficiency into SIS18

Summary

- Busch's original theorem for single particle was extended to particle beams
- Original and extended theorem look very similar

$$\frac{eq}{m\gamma}\psi + \oint_{\mathcal{C}} \vec{v} \cdot d\vec{C} = const$$
$$(\varepsilon_{n1} - \varepsilon_{n2})^2 + \left[\frac{eq\psi}{mc\pi}\right]^2 + \frac{4eq\psi\beta\gamma}{mc\pi} \oint_{\mathcal{C}} \vec{r'} \cdot d\vec{C} = const$$

- Extended theorem for very fast modelling of emittance gymnastic exp. as
 - flat electron beams at FERMILAB
 - flat ion beams at GSI
- Its power is through provision of an invariant
- Using invariants is much more convenient than solving equs. of motion