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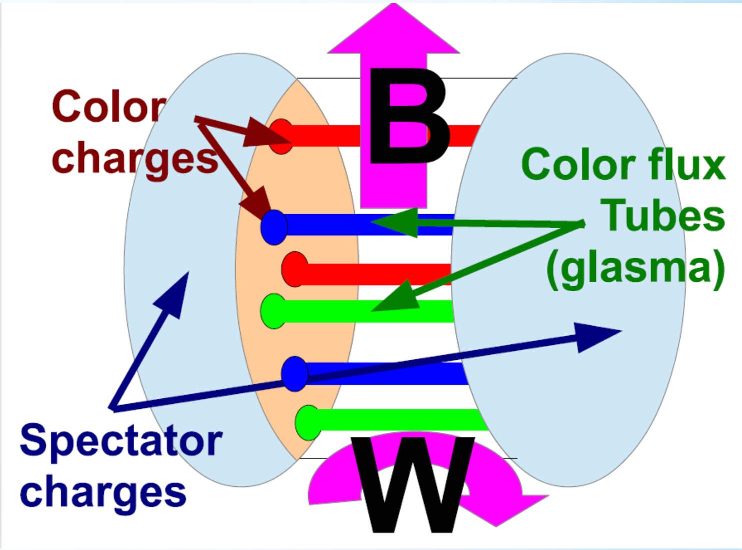
# Gaussian state approximation for real-time dynamics of gauge theories: Lyapunov exponents and entanglement entropy

# Motivation

Glasma state at early stages of HIC
Overpopulated gluon states
Almost "classical" gauge fields

Chaotic Classical Dynamics
[Saviddy, Susskind…]

- Positive Lyapunov exponents
- Gauge fields forget initial conditions



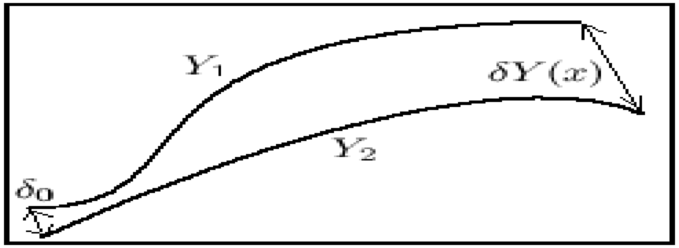

…but is it enough for <u>Thermalization?</u>

# Motivation

## Thermalization for quantum systems?

- Quantum extension of Lyapunov exponents - OTOCs $<[P(0),X(t)]^2>$
- Generation of entanglement between subsystems

$$\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$$




## Timescales: quantum vs classical?

☹ QFT tools extremely limited beyond strong-field classic regime…

☺ …Holography provides intuition

# Bounds on chaos

Reasonable physical assumptions
Analyticity of OTOCs ➡

$$\boxed{\lambda_L < 2\pi T}$$ (QGP ~0.1 fm/c)

[Maldacena Shenker Stanford'15]

- Holographic models with black holes saturate the bound(e.g. SYK)
- In contrast, for classical YM $\boxed{\lambda_L \sim T^{1/4}}$

What happens at low T ???

# Motivation

## N=1 Supersymmetric Yang-Mills in D=1+9:

### Reduce to a single point = BFSS matrix model [Banks, Fischler, Shenker, Susskind'1997]

$$L = \frac{1}{2g}\left[\mathrm{tr}\dot{X}^i\dot{X}^i + 2\theta^T\dot{\theta} - \frac{1}{2}\mathrm{tr}[X^i, X^j]^2 - 2\theta^T\gamma_i[\theta, X^i]\right]$$

N x N hermitian matrices

Majorana-Weyl fermions, N x N hermitian

System of *N* D0 branes joined by open strings [Witten'96]:

- $X^{ii}_{\mu}$ = D0 brane positions
- $X^{ij}_{\mu}$ = open string excitations

# Classical chaos and BH physics

## Stringy interpretation:

- Dynamics of gravitating D0 branes
- Thermalized state = black hole
- Classical chaos = info scrambling

**Expected to saturate the MSS bound at low temperatures!**

# In this talk:

**Numerical attempt to look at the real-time dynamics of BFSS and bosonic matrix models**

↓

**Of course, not an exact simulation, but should be good at early times**

↓

**Approximating all states by Gaussians**

# Gaussian state approximation

## Simple example: Double-well potential

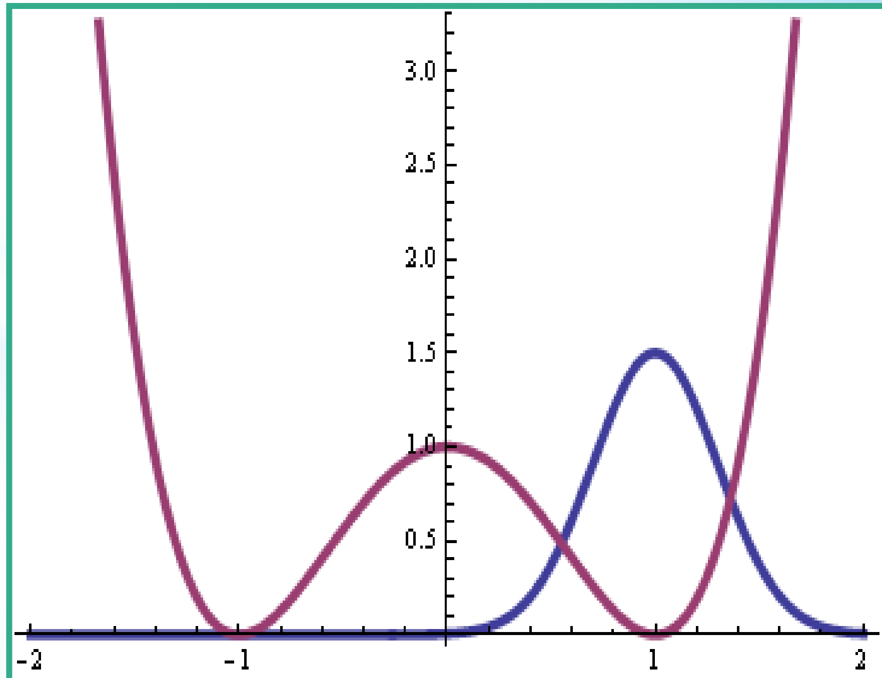
$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{a\hat{x}^2}{2} + \frac{b\hat{x}^3}{3} + \frac{c\hat{x}^4}{4}$$

## Heisenberg equations of motion

$$\partial_t \hat{x} = \hat{p},$$
$$\partial_t \hat{p} = -a\hat{x} - b\hat{x}^2 - c\hat{x}^3$$

## Also, for example

$$\partial_t \left(\hat{x}\hat{x}\right) = \hat{x}\hat{p} + \hat{p}\hat{x}$$



# Next step: Gaussian Wigner function

**Assume Gaussian wave function at any t**

**Simpler: Gaussian Wigner function**

$$\langle \hat{x}^2 \rangle = x^2 + \sigma_{xx},$$
$$\langle \hat{p}^2 \rangle = p^2 + \sigma_{pp},$$
$$\langle \frac{\hat{x}\hat{p}+\hat{p}\hat{x}}{2} \rangle = xp + \sigma_{xp}$$

**For other correlators: use Wick theorem!**

$$\langle \hat{x}^4 \rangle = x^4 + 6x^2\sigma_{xx} + 3\sigma_x x^2,$$
$$\langle \hat{x}^2\hat{p} \rangle = x^2 p + 2x\sigma_{xp} + p\sigma_{xx}$$

**Derive closed equations for**

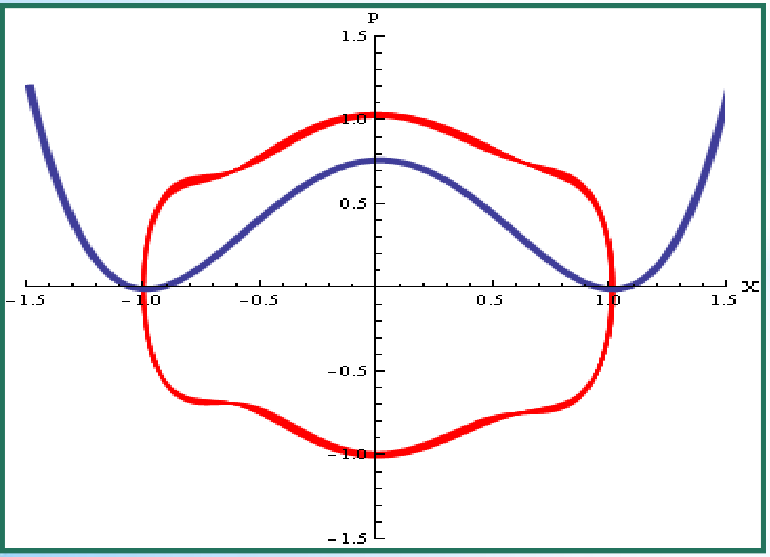
**$x$, $p$, $\sigma_{xx}$ , $\sigma_{xp}$ , $\sigma_{pp}$**

# Origin of tunnelling

$$\partial_t p = -ax - bx^2 - cx^3 - b\sigma_{xx} - 3cx\sigma_{xx},$$
$$\partial_t x = p$$

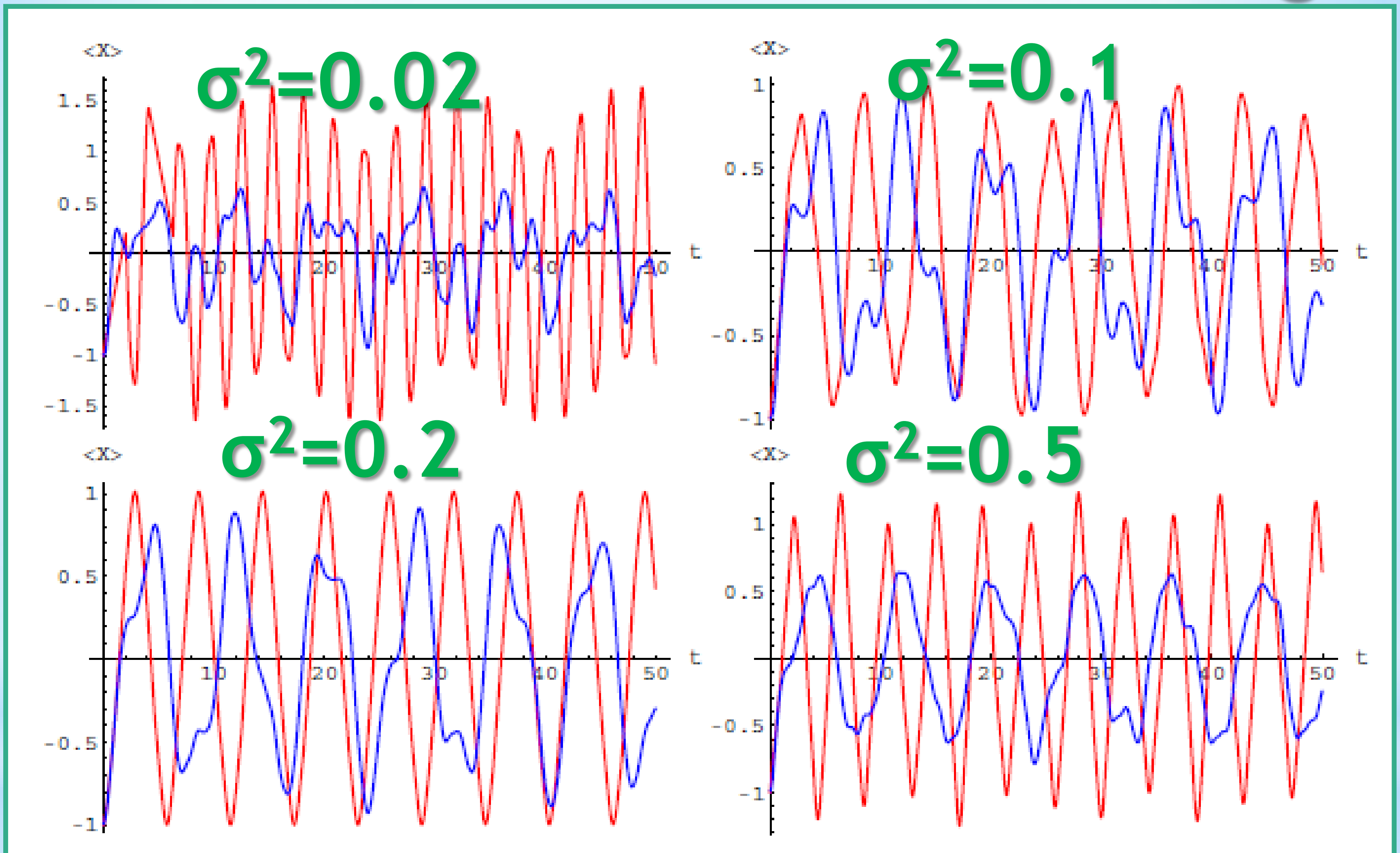
**Positive force even at *x=0* (classical minimum)**

$$\partial_t \sigma_{xx} = 2\sigma_{xp},$$
$$\partial_t \sigma_{xp} = \sigma_{pp} - a\sigma_{xx} - 2bx\sigma_{xx} - 3cx^2\sigma_{xx} - 3c\sigma_{xx}^2,$$
$$\partial_t \sigma_{pp} = -2\left(a\sigma_{xp} + 2bx\sigma_{xp} + 3cx^2\sigma_{xp} + 3c\sigma_{xx}\sigma_{xp}\right)$$




**Quantum force causes classical trajectory to leave classical minimum**

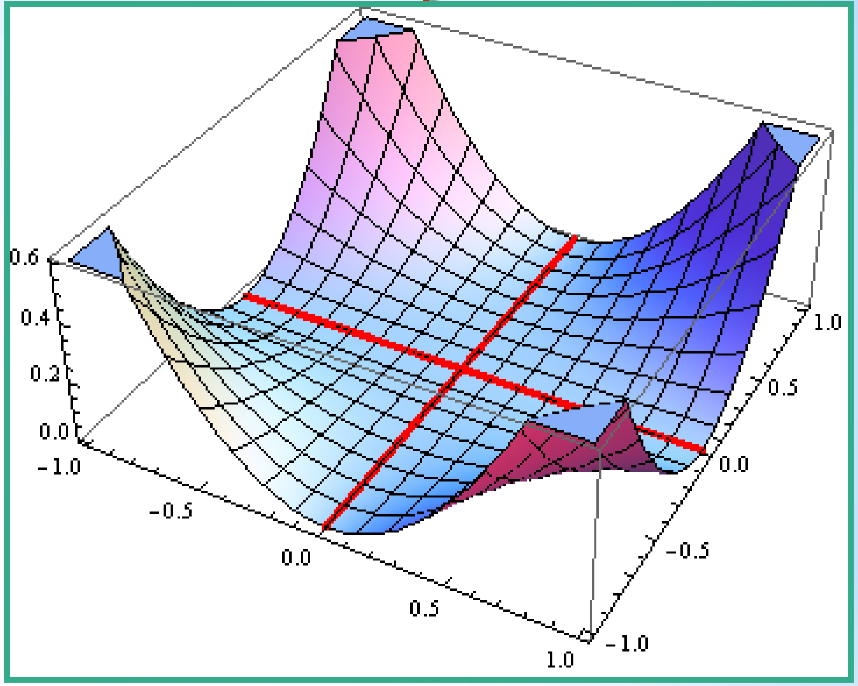
# Gaussian state vs exact Schrödinger




- Early-time evolution OK
- Tunnelling period qualitatively OK

# 2D potential with flat directions
## (closer to BFSS model)

$$\hat{H} = \frac{\hat{p}_x^2}{2} + \frac{\hat{p}_y^2}{2} + \frac{\kappa}{2}\hat{x}^2\hat{y}^2$$

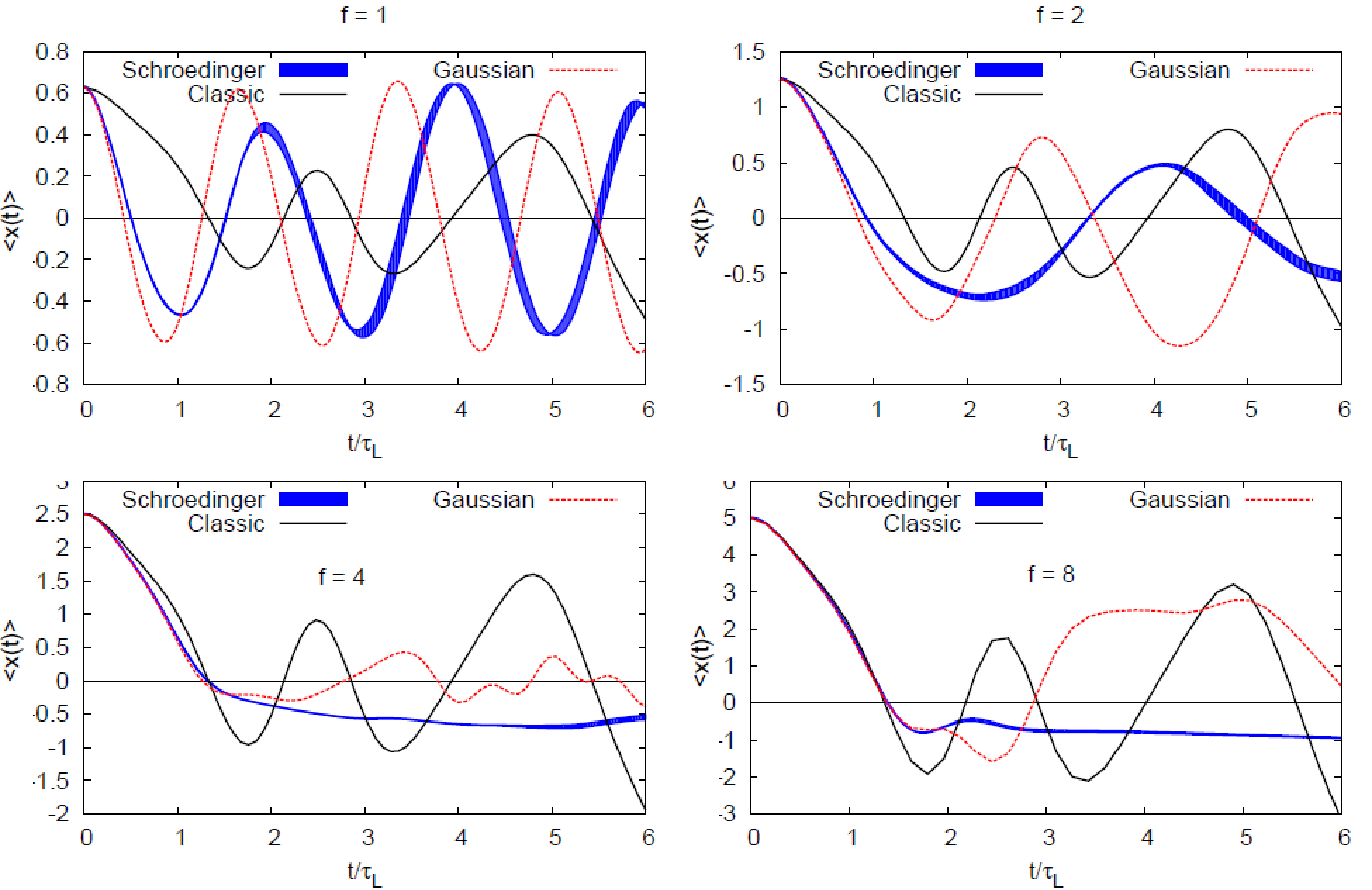



Classic runaway along *x=0 or y=0*

**Classically chaotic!**

We start with a Gaussian wave packet at distance *f* from the origin (away from flat directions)

# Gaussian state vs exact Schrödinger



# Gaussian state approximation

- ✓ Is good for at least two classical Lyapunov times
- ✓ Maps pure states to pure states
- ✓ Allows to study entanglement
- ✓ Closely related to semiclassics
- ✓ Is better for chaotic than for regular systems [nlin/0406054]
- ✓ Is likely safe in the large-N limit
- X Is not a unitary evolution

# BFSS matrix model: Hamiltonian formulation

$$\hat{H} = \frac{1}{2}\hat{P}_i^a \hat{P}_i^b + \frac{1}{4} C_{abc} C_{ade} \hat{X}_i^b \hat{X}_j^c \hat{X}_i^d \hat{X}_j^e + \frac{i}{2} C_{abc} \hat{\psi}_\alpha^a \left[\sigma_i\right]_{\alpha\beta} \hat{X}_i^b \hat{\psi}_\beta^c,$$

**a,b,c – su(N) Lie algebra indices**

**Heisenberg equations of motion**

$$\partial_t \hat{X}_i^a = \hat{P}_i^a$$

$$\partial_t \hat{P}_i^a = -C_{abc} C_{cde} \hat{X}_j^b \hat{X}_i^d \hat{X}_j^e - \frac{i}{2} C_{bac} \sigma^i_{\alpha\beta} \hat{\psi}_\alpha^b \hat{\psi}_\beta^c,$$

$$\partial_t \hat{\psi}_\alpha^a = C_{abc} X_i^b \sigma^i_{\alpha\beta} \hat{\psi}_\beta^c$$

# GS approximatio for BFSS model

$$\partial_t P_i^a = -C_{abc}C_{cde}X_j^b X_i^d X_j^e - \frac{i}{2}C_{bac}\sigma^i_{\alpha\beta}\langle \psi^b_\alpha \psi^c_\beta \rangle -$$
$$- C_{abc}C_{cde}X_j^b [XX]_{ij}^{de} - C_{abc}C_{cde}[XX]_{jj}^{be}X_i^d - C_{abc}C_{cde}[XX]_{ji}^{bd}X_j^e$$

$$\partial_t [XX]_{ij}^{ab} = [XP]_{ij}^{ab} + [XP]_{ji}^{ba},$$
$$\partial_t [XP]_{ik}^{af} = [PP]_{ik}^{af} - C_{abc}C_{cde}\left(X_i^d X_j^e + [XX]_{ij}^{de}\right)[XX]_{jk}^{bf} -$$
$$- C_{abc}C_{cde}\left(X_j^b X_j^e + [XX]_{jj}^{be}\right)[XX]_{ik}^{df} -$$
$$- C_{abc}C_{cde}\left(X_j^b X_i^d + [XX]_{ji}^{bd}\right)[XX]_{jk}^{ef},$$
$$\partial_t [PP]_{ik}^{af} = -C_{abc}C_{cde}\left(X_i^d X_j^e + [XX]_{ij}^{de}\right)[XP]_{jk}^{bf} -$$
$$- C_{abc}C_{cde}\left(X_j^b X_j^e + [XX]_{jj}^{be}\right)[XP]_{ik}^{df} -$$
$$- C_{abc}C_{cde}\left(X_j^b X_i^d + [XX]_{ji}^{bd}\right)[XP]_{jk}^{ef} + (\{a,i\} \leftrightarrow \{f,k\})$$

- **CPU time ~ *N^5* (double commutators)**
- **RAM memory ~ *N^4***
- ***SUSY broken, unfortunately …***

# Ungauging the BFSS model

- **Gauge constraints**

$$\hat{J}_a = C_{abc} \hat{X}_i^b \hat{P}_i^c - \frac{i}{2} C_{abc} \hat{\psi}_\alpha^b \hat{\psi}_\alpha^c \qquad \hat{J}_a |\psi\rangle = 0$$

- **For Gaussian states we can only have a weaker constraint** $$\langle\psi| \hat{J}_a |\psi\rangle = 0$$

- **We work with ungauged model [Maldacena,Milekhin'1802.00428]**
  **(e.g. LGT with unit Polyakov loops)**

- **Ungauging preserves most of the features of the original model [1802.02985]**

# Equation of state and temperature

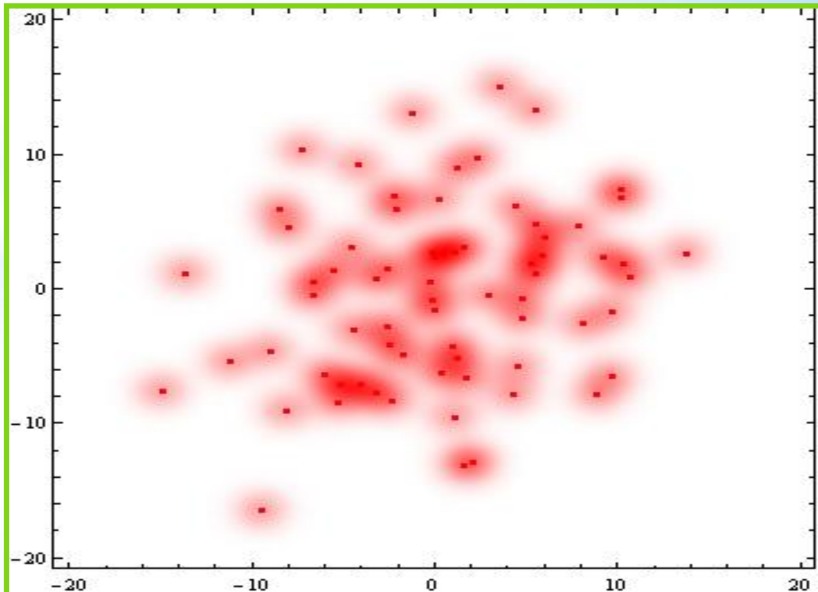
- Consider mixed Gaussian states with fixed energy $E = \langle H \rangle$
- *Maximize entropy w.r.t.* $\langle xx \rangle, \langle pp \rangle$
- Calculate temperature using

$$\boxed{T^{-1} = \frac{\partial S}{\partial E}}$$

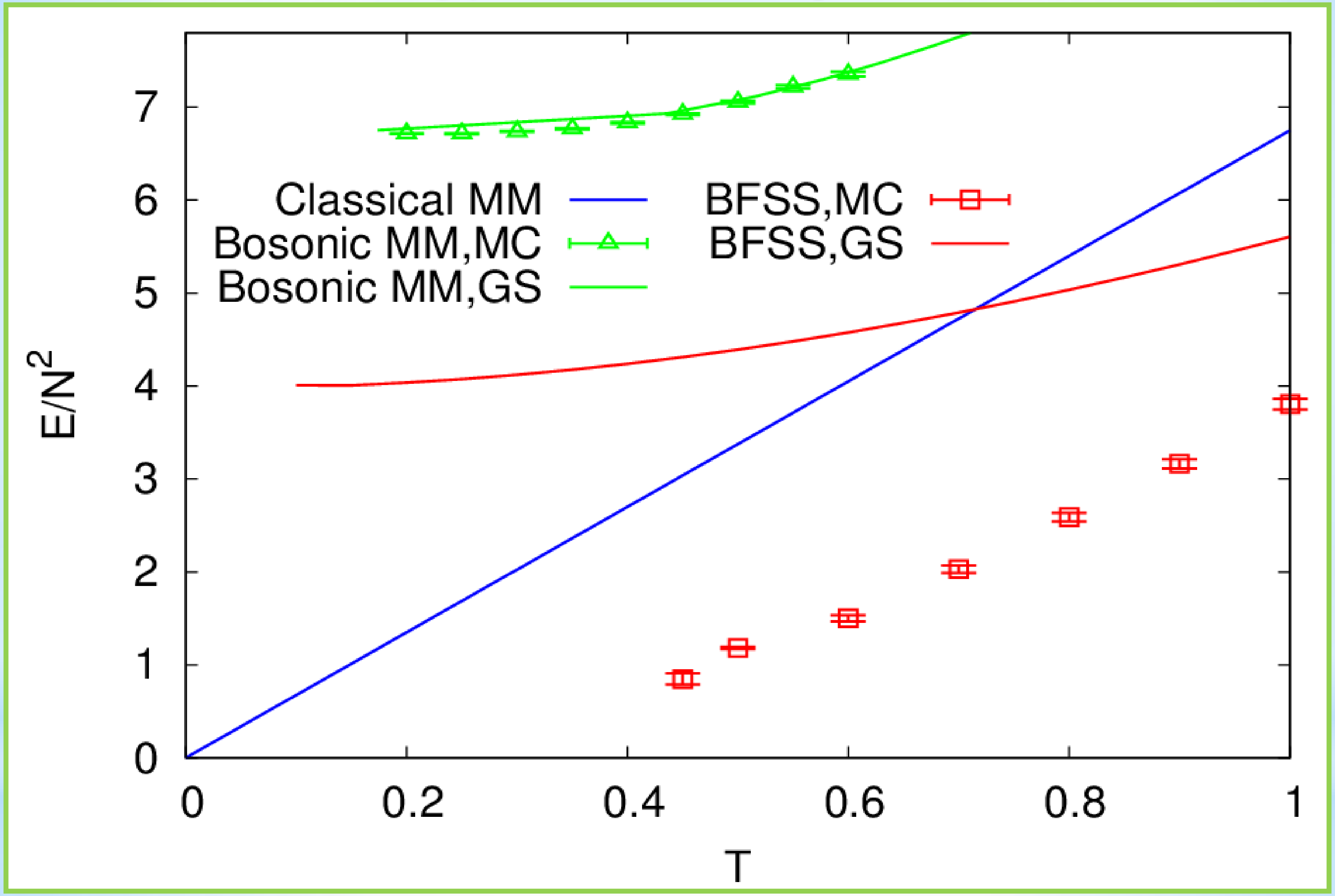
- Can be done analytically using rotational and SU(N) symmetries

# “Thermal” initial conditions

- At $T=0$ pure “ground” state with minimal $<pp>,<xx>$
- At $T>0$ mixed states, interpret as mixture of pure states, shifted by “classical” coordinates with dispersion $<xx>-<xx>_0$
- Makes difference for non-unitary evolution
- Fermions in ground state at fixed classical coordinates

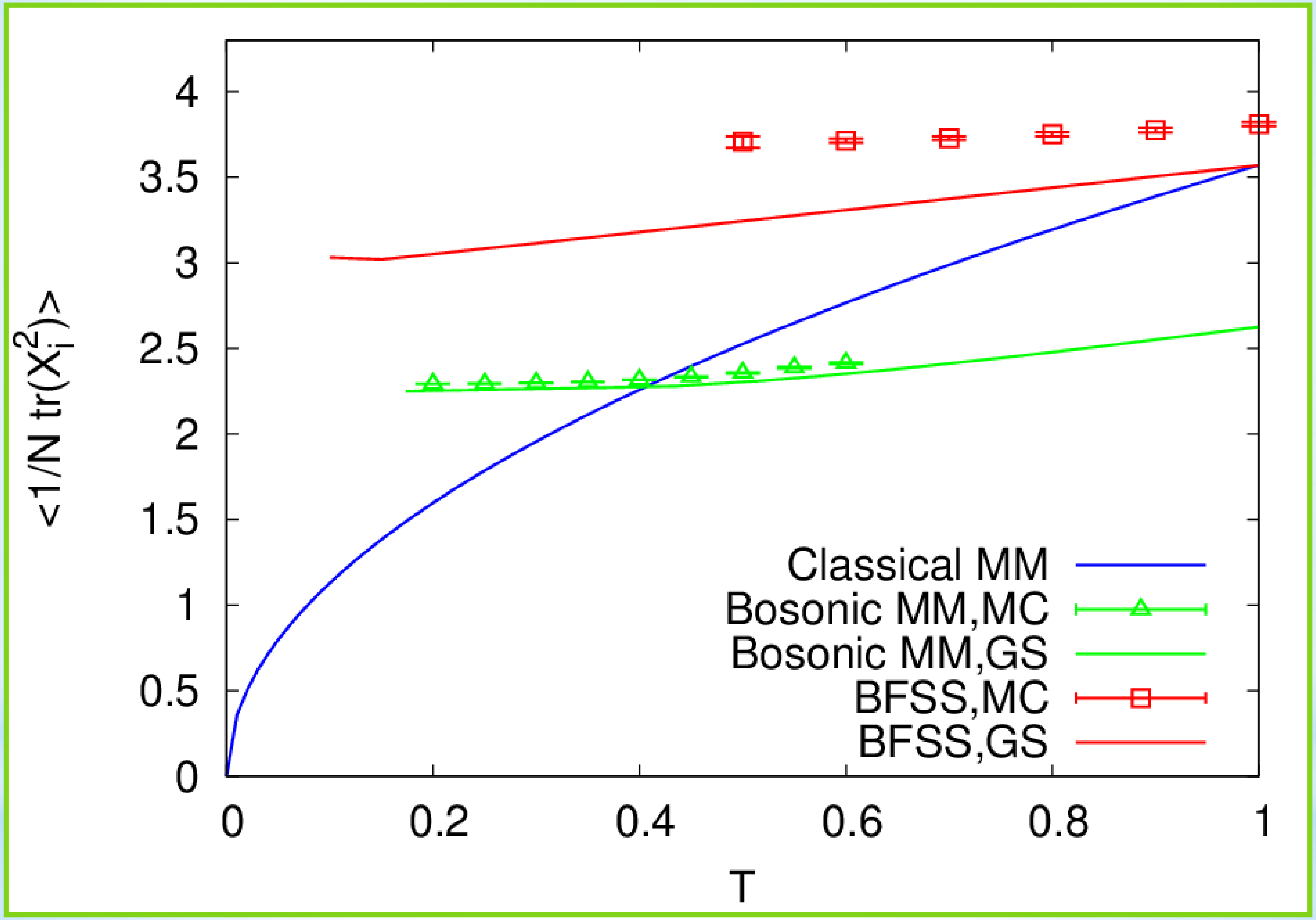


# Energy vs temperature



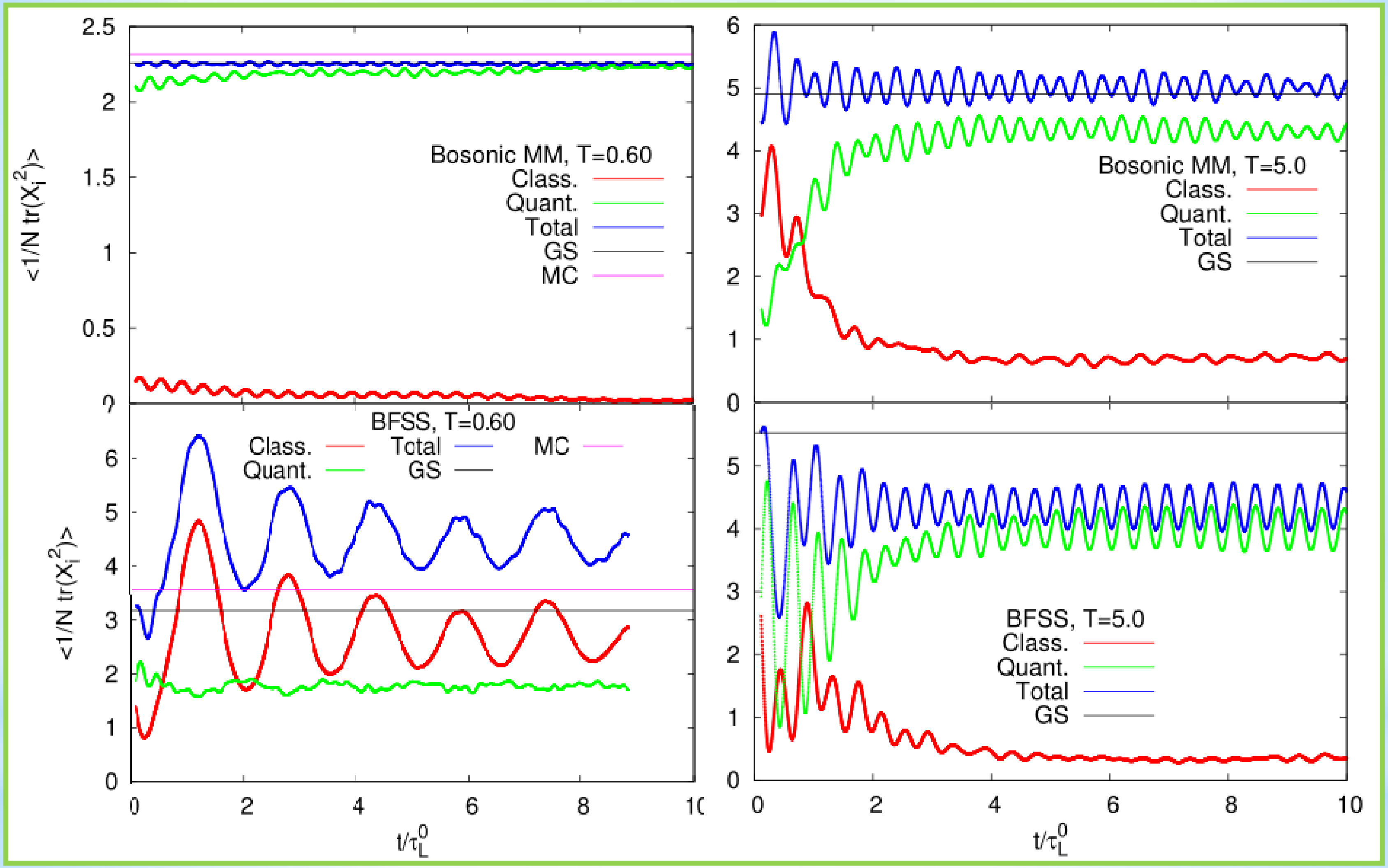

**MC data from [Berkowitz,Hanada, Rinaldi, Vranas, 1802.02985], we agree for pure gauge**

# $\langle 1/N\ Tr(X_i^2)\rangle$ vs temperature




MC data from [Berkowitz,Hanada, Rinaldi, Vranas, 1802.02985], we agree for pure gauge

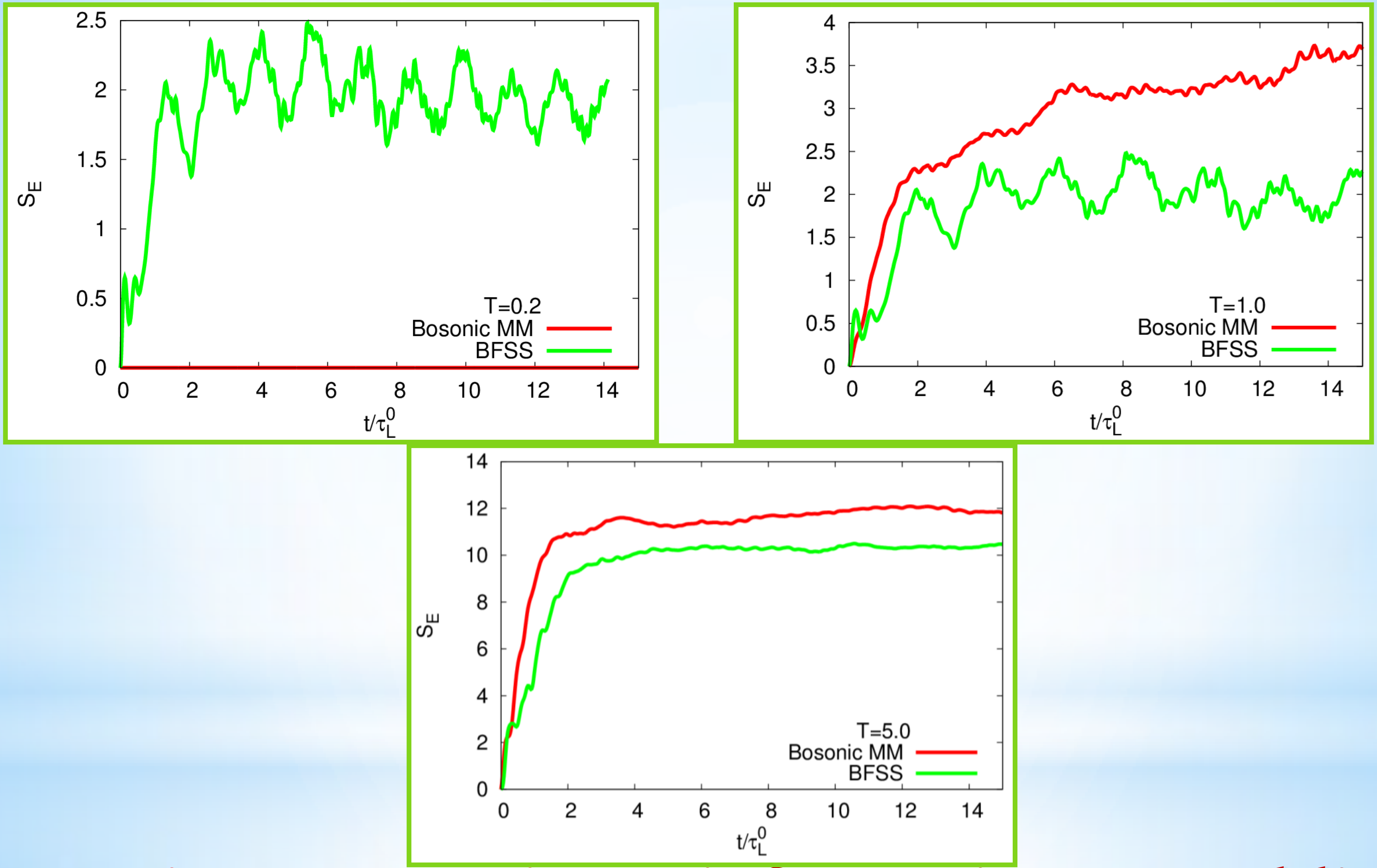
# Real-time evolution: $\langle 1/N\, Tr(X_i^2)\rangle$




**Wavepacket spread vs classical shrinking**

**For BFSS $\langle 1/N\, Tr(X_i^2)\rangle$ grows, instability?**

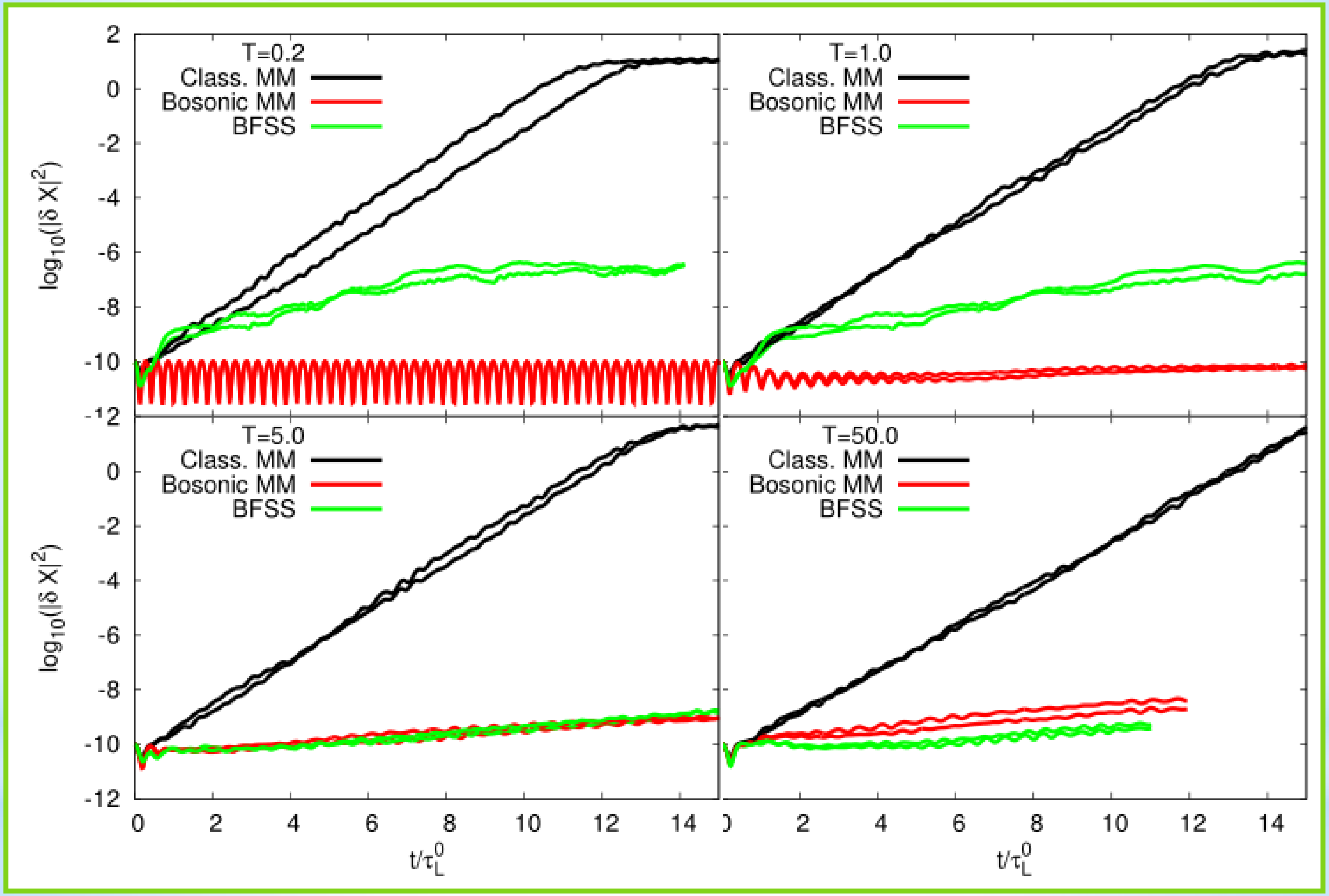
# Entanglement vs time




**Late-time saturation = information scrambling**

**Entanglement entropy ~ subsystem size**

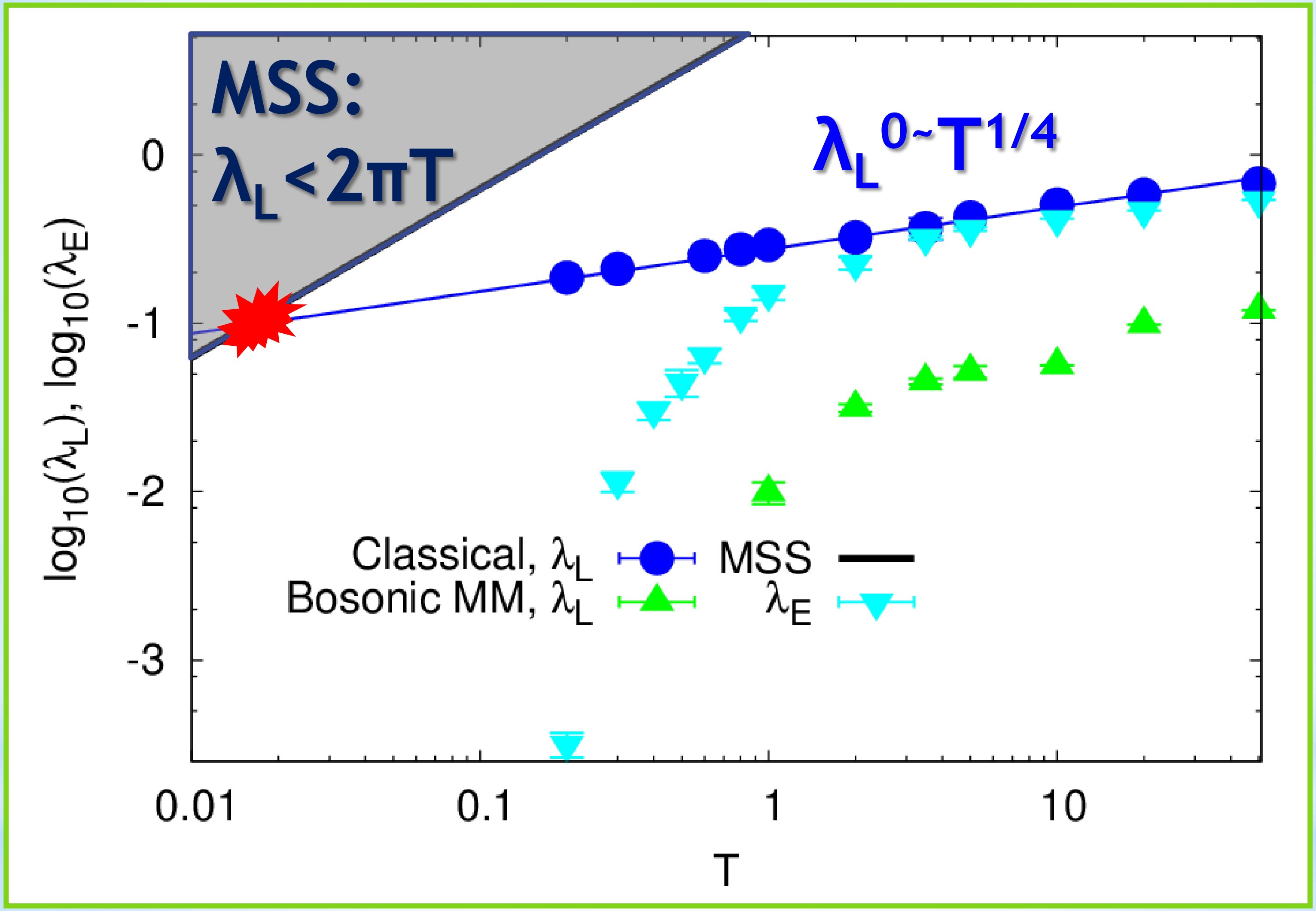
# Lyapunov distances vs time




**Early times: Very similar to classical dynamics**

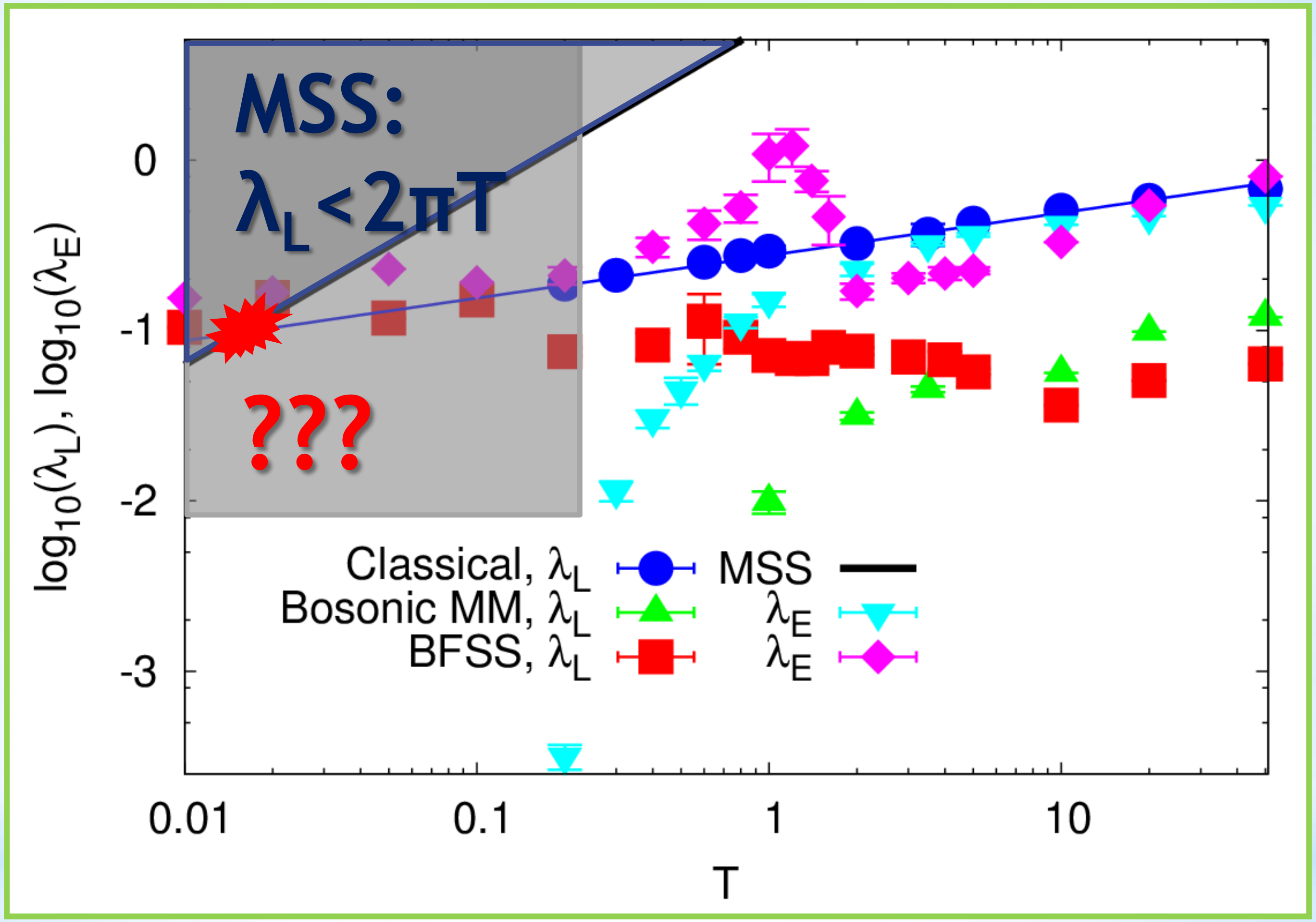
**Late times: significantly slower growth**

# Lyapunov vs entanglement: bosonic MM




**Entanglement saturates much faster than Lyapunov time, at high T - classical Lyapunov**

# Bosonic MM vs BFSS




- No strong statements at low T: loss of SUSY
- Non-chaotic confinement regime absent
- Shortest timescale still for entanglement

# Summary

- Longer quantum Lyapunov times vs. classical, important for MSS bound
- “Confining” regime non-chaotic
- Full BFSS model chaotic at all T
- “Scrambling” behavior for entanglement entropy
- Entanglement timescale is the shortest
- At high T governed by classic Lyapunov

# Summary

- Gaussian state approximation: $\sim V^2$ scaling of CPU time for QCD/ Yang-Mills
- Feasible on moderately large lattices
- Quantum effects on thermalization?
- Topological transitions in real time