

# The thermal photon rate from dynamical lattice QCD

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XQCD2018, Frankfurt, 23 May

# The thermal photon rate from the quark-gluon plasma

The differential *photon production rate* is, to  $O(\alpha_{\text{em}})$ ,

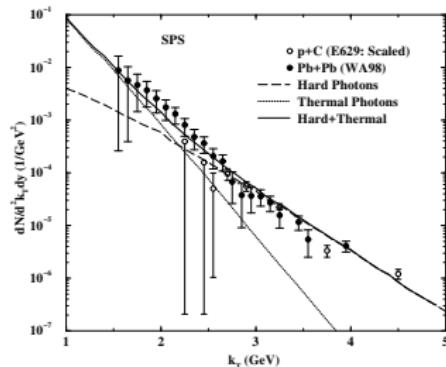
$$k \frac{d\Gamma_\gamma}{d^3 k} = \left( \sum_f Q_f^2 \right) \frac{\alpha_{\text{em}}}{4\pi^2} n_B(\omega = k) \rho^\mu{}_\mu(\omega = k, k = |k|),$$

McLerran & Toimela (1985)

where  $\rho^\mu{}_\mu(\omega, k)$ , the vector channel *spectral function*,  
and  $n_B(\omega)$ , the Bose-Einstein distribution, depend implicitly on the temperature,  $T$ .

In heavy-ion collisions, photons are produced *thermally* and in initial *hard* partonic reactions.

The temperature of the initial equilibrated thermal medium can be inferred from the thermal contribution to the photon spectra.



Alam et al. (2003)

By modelling the evolution of the medium, estimates for the photon spectra can be obtained from the photon rate.

# What is known about the photon rate?

The thermal photon rate has been estimated in weakly-coupled QCD and other theories at weak and strong coupling.

Arnold, Moore & Yaffe (2001); Huot et al. (2006)

Strongly-interacting media require a non-perturbative cross-check of weak-coupling approaches. Ghiglieri et al. (2016)

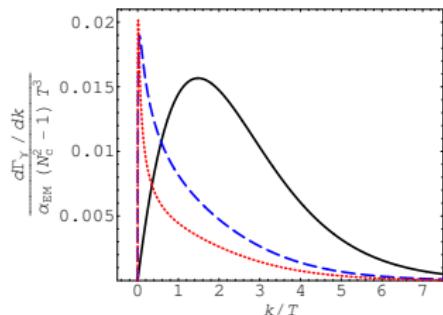


Figure: weakly-coupled (dashed) and strong-coupling  $\mathcal{N} = 4$  SYM (solid black)

Caron-Huot et al. (2006)

Due to the conservation of the electric charge, hydrodynamics predicts the  $k \rightarrow 0$  behaviour of  $\rho^{\mu}_{\mu}(\omega, k)$ ,

$$\frac{\rho^{\mu}_{\mu}(\omega, k)}{\omega} \xrightarrow{\omega^2 + (Dk^2)^2} \frac{4\chi_s Dk^2}{\omega^2 + (Dk^2)^2} \quad \text{as } \omega, k \rightarrow 0$$

# Real-time phenomena from Euclidean correlators

Photon emission is a *real-time* phenomenon challenging due to *analytic continuation*

Cuniberti et al. (2001)

Euclidean correlator has a spectral representation

$$\tilde{G}_{\mu\nu}(\omega_n, k) \equiv \int_0^\beta d\tau e^{i\omega_n \tau} \int d^3x e^{-ikx} \text{Tr} \left\{ e^{-\beta \hat{H}} \hat{V}_\mu(\tau, x) \hat{V}_\nu(0) \right\} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\rho_{\mu\nu}(\omega, k)}{\omega - i\omega_n}$$

or, for the coordinate-space correlation function

$$G_{\mu\nu}(\tau, k) = \int_0^\infty \frac{d\omega}{2\pi} \underbrace{\frac{\cosh(\omega(\beta/2 - \tau))}{\sinh(\beta\omega/2)}}_{K(\omega, \tau)} \rho_{\mu\nu}(\omega, k)$$

which is *underdetermined* for  $\rho_{\mu\nu}(\omega)$  when  $G_{\mu\nu}(\tau)$  is known only at  $\tau = a, \dots, \beta$

$$\begin{bmatrix} G_{\mu\nu} \\ \vdots \\ G_{\mu\nu} \end{bmatrix} = \begin{bmatrix} K & & \\ \cdots & \ddots & \\ \cdots & \cdots & \ddots \end{bmatrix} \begin{bmatrix} \rho_{\mu\nu} \\ \vdots \\ \rho_{\mu\nu} \end{bmatrix}$$

especially if the spectral function is *steep*, e.g. a diverging background

# A UV-finite spectral density for the photon rate

Take a special linear combination of  $\rho^{\mu\nu}(\omega, k)$

$$\rho_\lambda(\omega, k) \equiv \left( \delta^{ij} - \frac{k^i k^j}{k^2} \right) \rho^{ij}(\omega, k) - \frac{\lambda}{k^2} (k^2 \rho^{00}(\omega, k) - k^i k^j \rho^{ij}(\omega, k))$$

which is equal to  $\rho^\mu{}_\mu(\omega, k)$  for  $\lambda = 1$  and furthermore, due to vector Ward identity

$$\omega^2 \rho^{00}(\omega, k) - k^i k^j \rho^{ij}(\omega, k) = 0,$$

is independent of  $\lambda$  on the light-cone,  $\omega = k$ .

$$\begin{aligned}\rho^\mu{}_\mu(\omega, k) &\xrightarrow{\omega \rightarrow \infty} \omega^2 \\ \rho_{\lambda=-2}(\omega, k) &\xrightarrow{\omega \rightarrow \infty} 0\end{aligned}$$

↔ only need *current conservation* and *Lorentz-invariance* to show this!

The photon rate *vanishes* at tree-level.

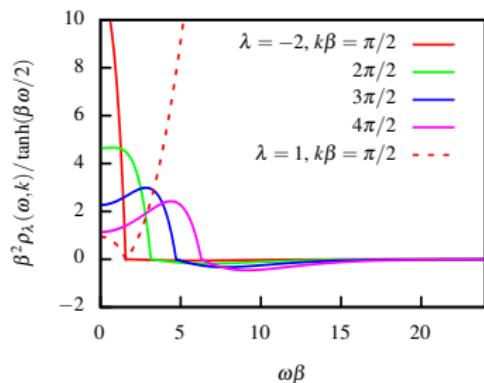


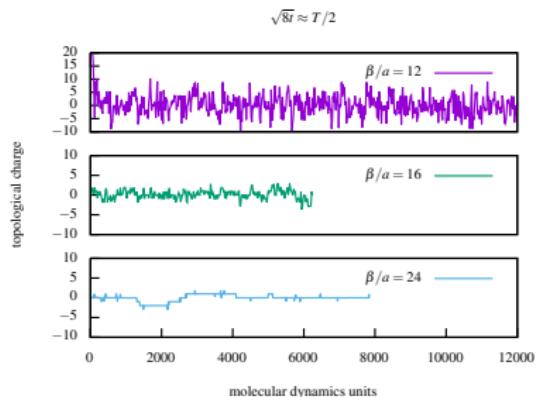
Figure: Free massless spectral function

Laine (2013)

# Lattice set-up with $N_f = 2$ O( $a$ )-improved Wilson fermions

$T$ (MeV)	$T/T_c$	$\beta_{\text{LAT}}$	$\beta/a$	$L/a$	$m_{\overline{\text{MS}}(2 \text{ GeV})}$ (MeV)	$N_{\text{meas}}$
250	1.2	5.3	12	48	12	8256
"	"	5.5	16	64	"	4880
"	"	5.83	24	96	"	9600
500	2.4	6.04	16	64	"	8064

- enables continuum limit at  $T = 250$  MeV



- no dependence of observable on topological charge is seen
- autocorrelations are under control

## Continuum limit 1/3

Four independent discretizations of the  $\lambda = -2$  isovector vector correlator

$$G^{\lambda=-2}(\tau, k) = \left( \delta^{ij} - \frac{3k^i k^j}{k^2} \right) G^{ij}(\tau, k) + 2G^{00}(\tau, k)$$

where  $G^{\mu\nu}(\tau, k) = \int d^3x e^{-ikx} V^\mu(\tau, x) V^\nu(0)$  using both the

- local vector current
- exactly-conserved vector current

In the local-conserved case, there are two discretizations possible by defining the local current on the link, or the conserved current on the site

$$G^{ij}(\tau + a/2, k) = \frac{1}{2} (G^{ij}(\tau, k) + G^{ij}(\tau + a, k))$$

$$G^{00}(\tau, k) = \frac{1}{2} (G^{00}(\tau - a/2, k) + G^{00}(\tau + a/2, k))$$

Project to all spatial momenta, on and off-axis, with  $k\beta \leq 2\pi$

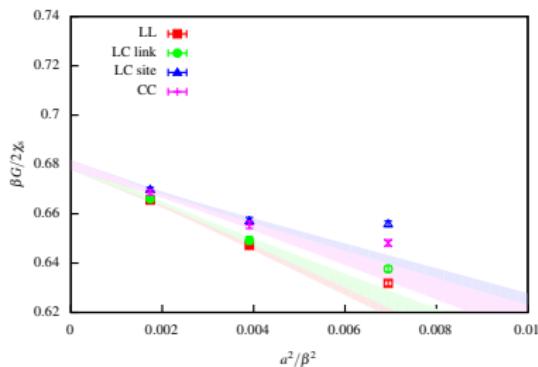
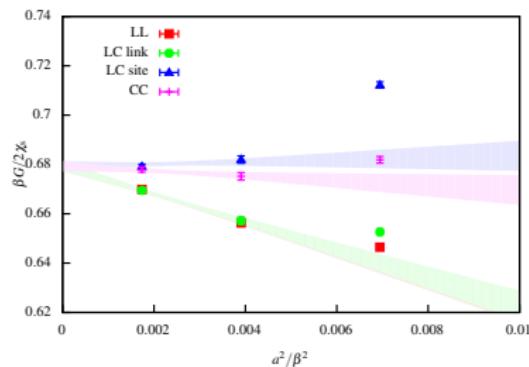
## Continuum limit 2/3

In the chirally-symmetric phase, the matrix-elements of the  $O(a)$ -improvement counterterms are suppressed, so we perform a continuum limit in  $a^2/\beta^2$

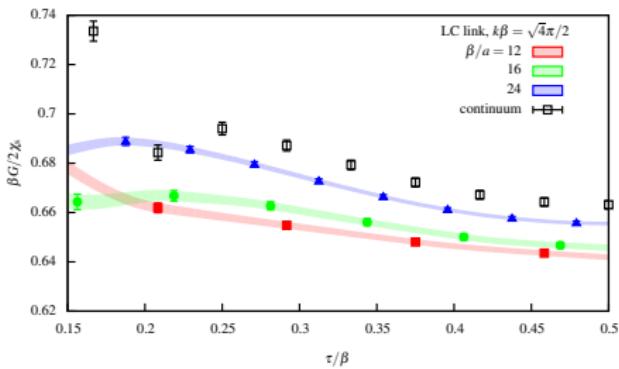
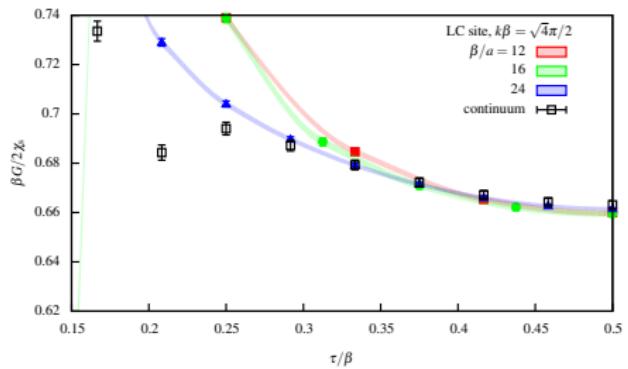
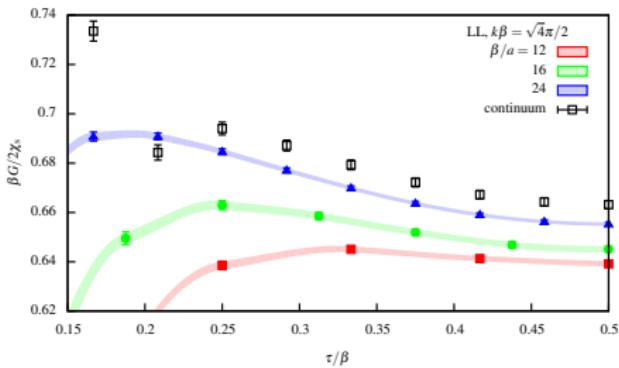
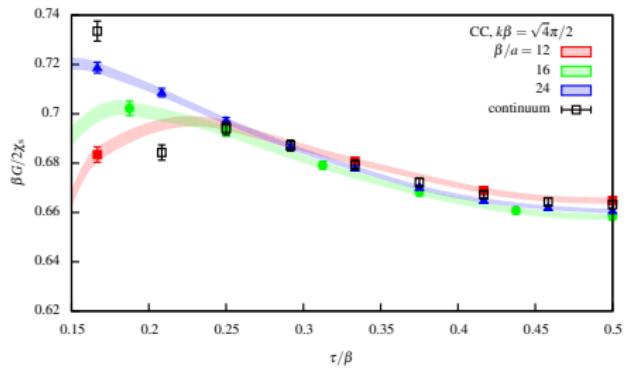
Instead we perform tree-level improvement, defined via

$$G^{\lambda=-2}(\tau, k) \rightarrow \frac{G_{\text{cont.t.l.}}^{\lambda=-2}(\tau, k)}{G_{\text{lat.t.l.}}^{\lambda=-2}(\tau, k)} G^{\lambda=-2}(\tau, k)$$

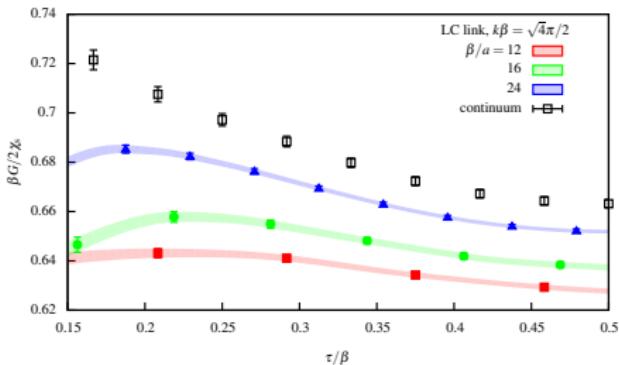
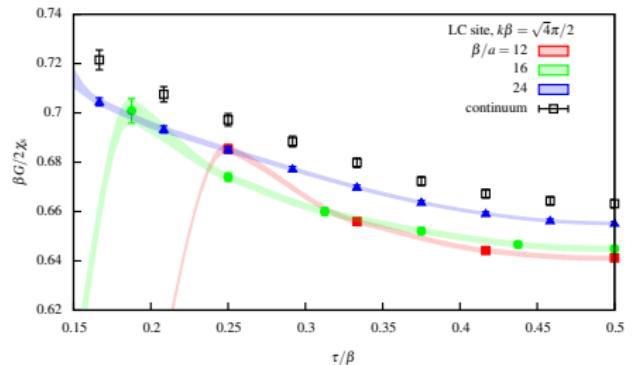
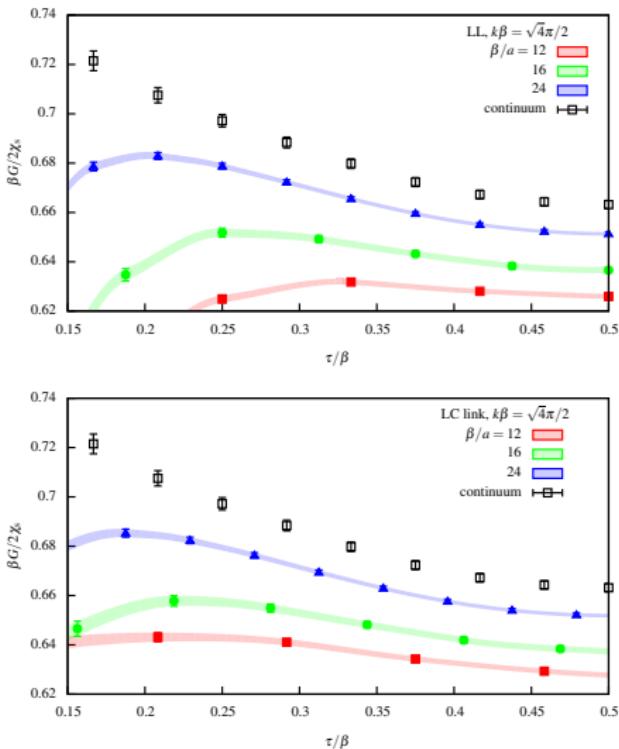
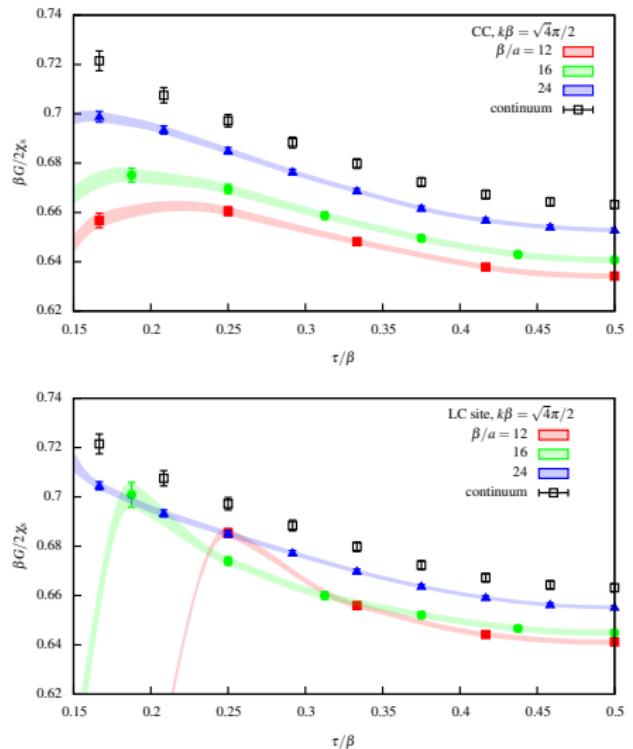
A piecewise spline interpolation is used before taking the combined continuum limit of the four discretizations of  $\beta G^{\lambda=-2}(\tau, k)/\chi_s$



# Continuum limit 3/3 - no improvement



# Continuum limit 3/3 - tree-level improved



## Backus-Gilbert method 1/3

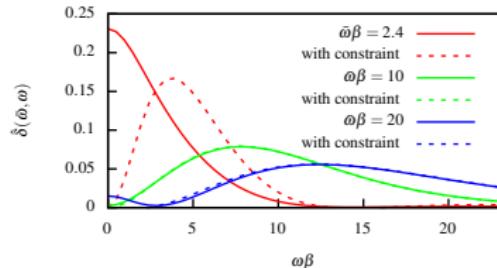
$$G_E(\tau) = \int_0^\infty \frac{d\omega}{2\pi} \underbrace{\frac{\cosh(\omega(\beta/2 - \tau))}{\sinh(\beta\omega/2)}}_{K(\omega, \tau)} \rho(\omega)$$

goal: find set of inverse kernels  $q_i(\bar{\omega})$  and construct an estimator  $\rho_{BG}$  such that

$$\begin{aligned}\rho_{BG} &= \int_0^\infty \frac{d\omega}{2\pi} \hat{\delta}(\bar{\omega}, \omega) \rho(\omega), \quad \hat{\delta}(\bar{\omega}, \omega) = \sum_{i=1}^{N_f} q_i(\bar{\omega}) K(\omega, \tau_i) \\ \rho_{BG} &= \sum_{i=1}^{N_\tau} q_i(\bar{\omega}) G_E(\tau_i), \quad \text{Var} [\rho_{BG}] = q_i(\bar{\omega}) \mathcal{C}_{ij} q_j(\bar{\omega})\end{aligned}$$

- $q_i(\bar{\omega})$  determined by the covariance matrix of the *Euclidian data* and *known kernel functions*
- linear method aiming at *maximum stability* of the estimator and *maximum agreement* with the “true” solution
- works best for smooth and/or not rapidly varying spectral functions!!
- no maximum likelihood interpretation of  $\rho_{BG}(\bar{\omega})$

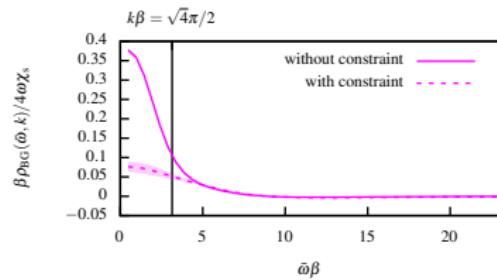
## Backus-Gilbert method 2/3



← resolution function  $\hat{\delta}(\bar{\omega}, \omega)$

acts like a smearing kernel

a linear constraint  $\hat{\delta}(\bar{\omega} = 0, \omega) = 0$  removes contributions from  $\rho(\omega = 0, k)$



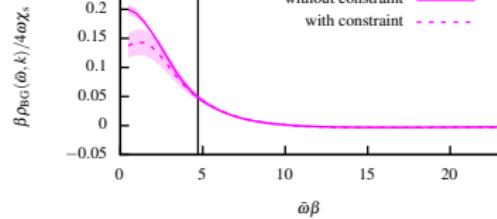
← spectral function  $\rho_{\text{BG}}(\bar{\omega}, k)$

at  $k\beta \gtrsim \pi$ , the photon rate is consistent with or without the constraint

estimate the effective diffusion constant

$$D_{\text{eff}}(k, \beta) = \frac{\rho_{\text{BG}}(k, k)}{4k\chi_s}$$

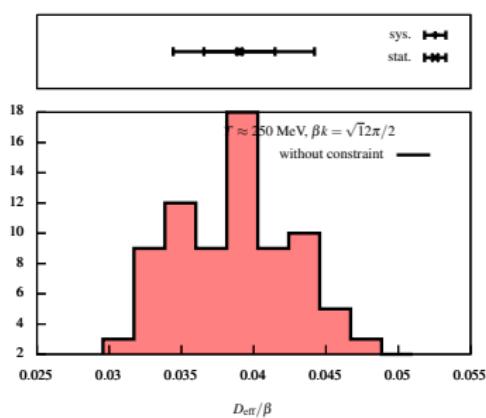
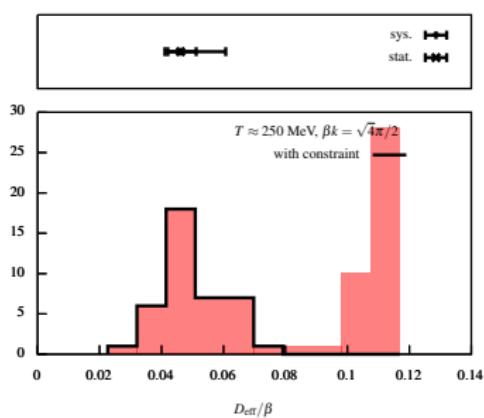
Ghiglieri et al. (2016)



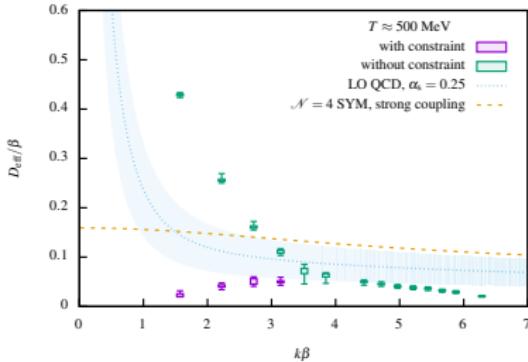
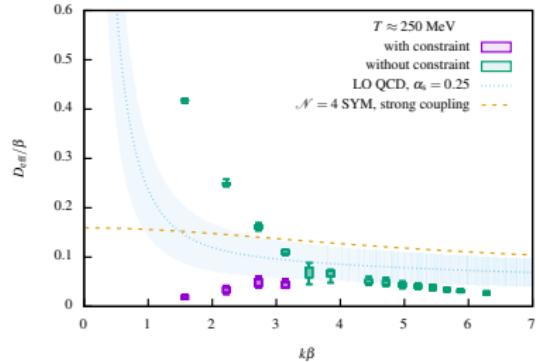
## Backus-Gilbert method 3/3

Estimate a systematic uncertainty by repeating with many variations.

variation	values
$\tau_{\min}/\beta$	$\{0.1, \dots, 0.25\}$
extra constraint	{yes, no}
$\alpha$	$\{10^{-2}, \dots, 10^{-4}\}$
tree-level improved	{yes, no}
discretization (@ $T = 500$ MeV)	{LL, LC site, LC link, CC}



# Preliminary results from the BG method



Results display virtually no visible temperature effects

Inverse problem appears to be controlled when  $k\beta > \pi$

Improved momentum resolution using on- and off-axis momenta

## Padé fit to the spectral function

- so far model independent approach (BG),

now explore a maximum likelihood fit ansatz:

- tanh-regulated spectral function can be modeled as

$$\frac{\rho(\omega, k)}{\tanh(\omega\beta/2)} = \frac{A(1 + B\omega^2)}{[\omega^2 + a^2] [(\omega + \omega_0)^2 + b^2] [(\omega - \omega_0)^2 + b^2]}$$

two linear parameters  $A$  and  $B$ ,

three nonlinear parameters  $(a, \omega_0, b)$

- inspired by a superconvergent sum rule derived from Lorentz invariance, charge conservation and the operator product expansion (OPE)

## Padé fit to the spectral function

$$\frac{\rho(\omega, k)}{\tanh(\omega\beta/2)} = \frac{A}{[\omega^2 + a^2]}$$

- inspired by the diffusion pole as it arises in hydrodynamics prediction in the infrared limit
- with  $a \leftrightarrow Dk^2$  for small  $k$

$$\frac{\rho(\omega, k)}{\omega} \approx \frac{4\chi_s D k^2}{\omega^2 + (Dk^2)^2}, \quad \omega, k \ll D^{-1} \quad \text{Caron - Huot et al. (2006)}$$

## Padé fit to the spectral function

$$\frac{\rho(\omega, k)}{\tanh(\omega\beta/2)} = \frac{A(1 + B\omega^2)}{[\omega^2 + a^2] [(\omega + \omega_0)^2 + b^2] [(\omega - \omega_0)^2 + b^2]}$$

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- satisfies the large- $\omega$  behaviour known from an OPE:

$$\rho(\omega, k) \propto k^2/\omega^4, \quad \omega \gg \pi T, k$$

- satisfies the superconvergent sum rule

$$\int_0^\infty d\omega \omega \rho(\omega, k) = 0$$

⇒ second linear parameter  $B$  becomes a function of  $(a, \omega_0, b)$

## Padé fit to the spectral function

in total:

$$\frac{\rho(\omega, k)}{\tanh(\omega\beta/2)} = \frac{A(1 + B\omega^2)}{[\omega^2 + a^2] [(\omega + \omega_0)^2 + b^2] [(\omega - \omega_0)^2 + b^2]}$$

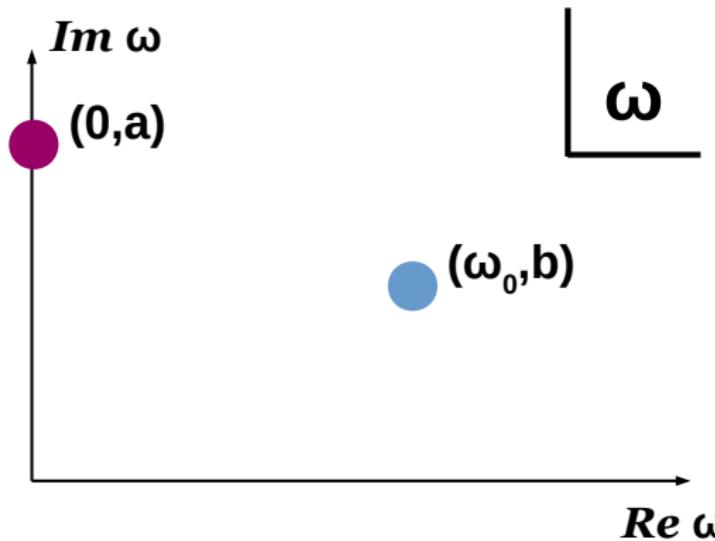
- $a \simeq Dk^2$  for small  $k$
- second pole of thermal size ( $\sim T$ )
- satisfies UV behaviour and sum rule
- we know there is spectral positivity below the light cone:

$$\rho(\omega) \geq 0, \quad \omega \leq k$$

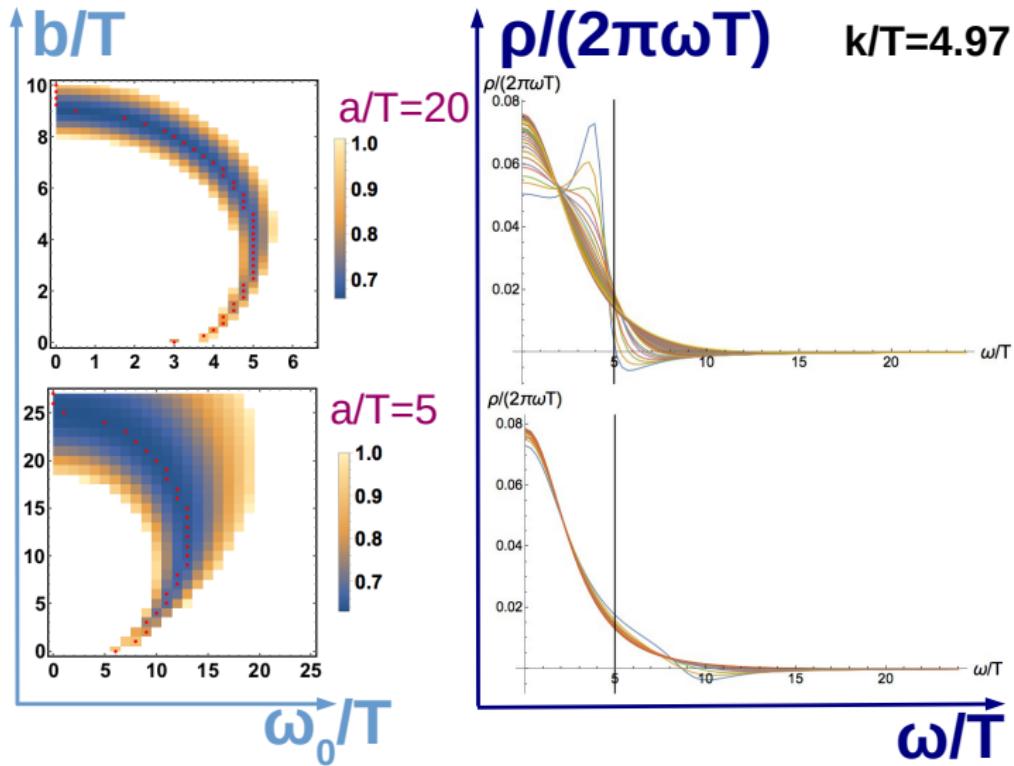
this is checked *a posteriori*.

## Padé fit to the spectral function - pole structure

$$\frac{\rho(\omega, k)}{\tanh(\omega\beta/2)} = \frac{A(1 + B\omega^2)}{[\omega^2 + a^2] [(\omega + \omega_0)^2 + b^2] [(\omega - \omega_0)^2 + b^2]}$$



# Padé fit to the spectral function - uncorrelated $\chi^2(\omega_0, b)$ -landscape



## Padé fit to the spectral function - uncorrelated $\chi^2$

- 4 fit parameters, 3 degrees of freedom  
for continuum: 7 data points from  $t_{\min}/\beta = 0.25$  up to  $t_{\max}/\beta = 0.5$
- nonlinear fits are very difficult
- rather than minimizing uncorrelated  $\chi^2$ :  
in order to bound the photon rate: take the *min* and *max* values of all photon rates with  $\chi^2(A, a, \omega_0, b) < 1$
- exclude photon rates which are incompatible with the data ( $\chi^2 > 1$ )
- another exclusion criterion for  $(a, b)$ :

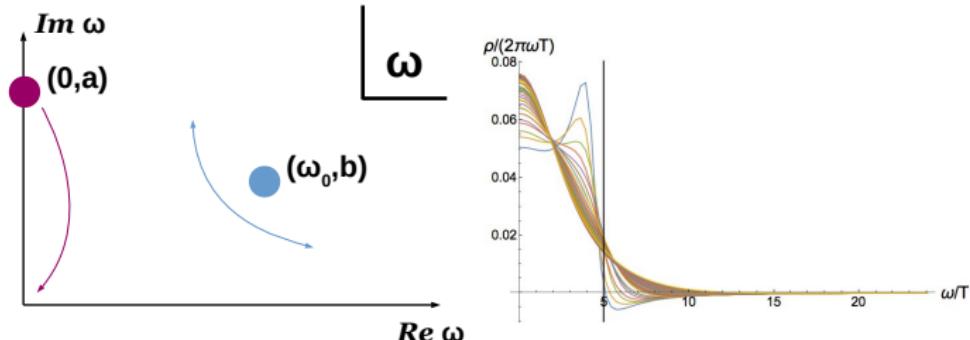
$$\min(a, b) > \min(D_{\text{AdS/CFT}} \cdot k^2, D_{\text{PT}}^{-1})$$

where

$$\begin{aligned} D_{\text{AdS/CFT}} &= 1/(2\pi T) \\ D_{\text{PT}}^{-1} &= \mathcal{O}(\alpha_s^2) \cdot T, \quad \alpha_s = 0.25 \end{aligned}$$

Caron-Huot et al. (2006)  
Arnold et al. (2003)

## Padé fit to the spectral function - constraining $a, b$

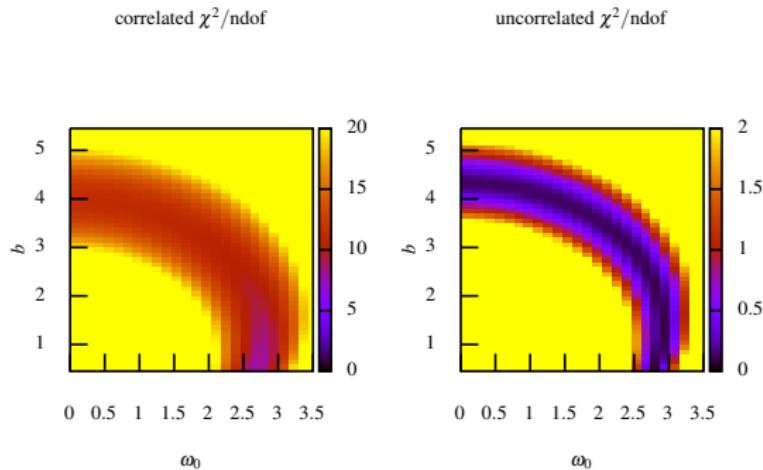


$$\min(a, b) > \min(D_{\text{AdS/CFT}} \cdot k^2, D_{\text{PT}}^{-1})$$

- corresponds to constraining any additional excitation to be shorter-lived than the largest possible relaxation times in the system  
(this amounts to the most conservative constraint based on physics):
- $D_{\text{AdS/CFT}} \cdot k^2 \sim \text{diffusion of electric charge}$  ( $D_{\text{AdS/CFT}} = 1/(2\pi T)$ )
- $D_{\text{PT}}^{-1} \sim \text{damping of static current}$  ( $D_{\text{PT}}^{-1} = \mathcal{O}(\alpha_s^2) \cdot T, \quad \alpha_s = 0.25$ )

Caron-Huot et al. (2006)  
Arnold et al. (2003)

## Correlated vs uncorrelated $\chi^2$ -landscape

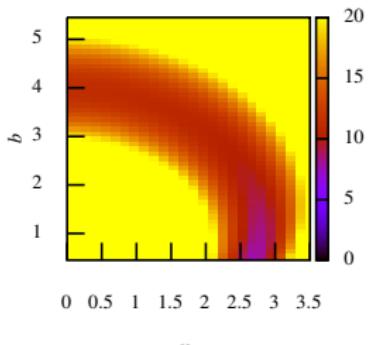


we find the two extremes:

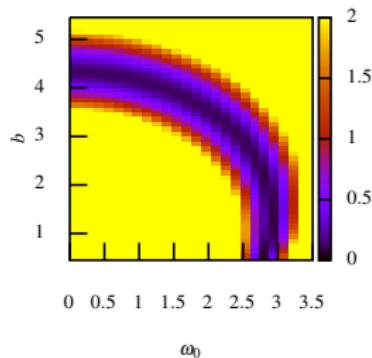
- a plethora of acceptable solutions (uncorrelated case)
- no solution at all for certain momenta (correlated case)  
⇒ truth lies somewhere in between?

# Regulating the correlations

correlated  $\chi^2/\text{ndof}$



uncorrelated  $\chi^2/\text{ndof}$



regulate the covariance matrix  $C$

$$C \longrightarrow x \cdot C + (1 - x) \cdot \text{diag}(C), \quad x \in (0, 1)$$

## Using PT to further constrain the results

with the *perturbative results* of M. Laine (JHEP11(2013)120),

one constructs  $\rho_{\lambda=-2}^{\text{LO}}$   
and defines the *curvature*

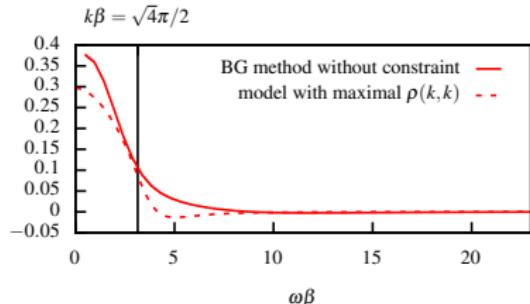
$$\begin{aligned}\text{Curv}^{(\text{LO}, \text{Ans})}(k, \tau) &\equiv \partial_\tau^2 G_{\lambda=-2}^{(\text{LO}, \text{Ans})}(k, \tau) \\ &= \int_0^\infty \frac{d\omega}{2\pi} \omega^2 \rho_{\lambda=-2}^{(\text{LO}, \text{Ans})}(\omega, k) \frac{\cosh(\omega(\beta/2 - \tau))}{\sinh(\beta\omega/2)},\end{aligned}$$

then after computing the family of  $\rho_{\lambda=-2}^{(\text{Ans})}(A, \omega_0, a, b)$  for each  $k$   
one accepts only those solutions that satisfy

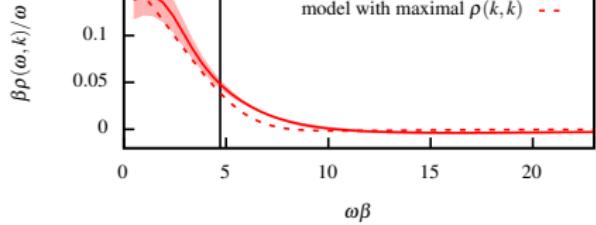
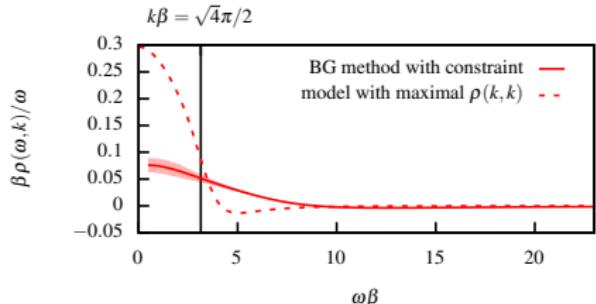
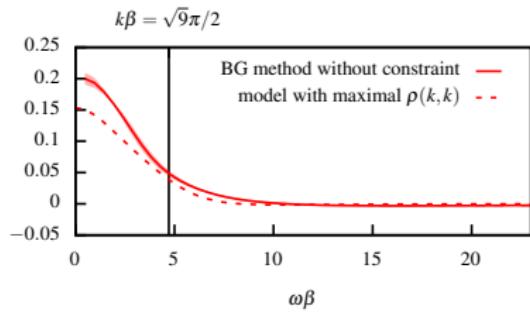
$$\left| \frac{\text{Curv}^{(\text{Ans})}(k, 0)}{\text{Curv}^{(\text{LO})}(k, 0)} - 1 \right| < 0.3$$

# Comparison of BG and model

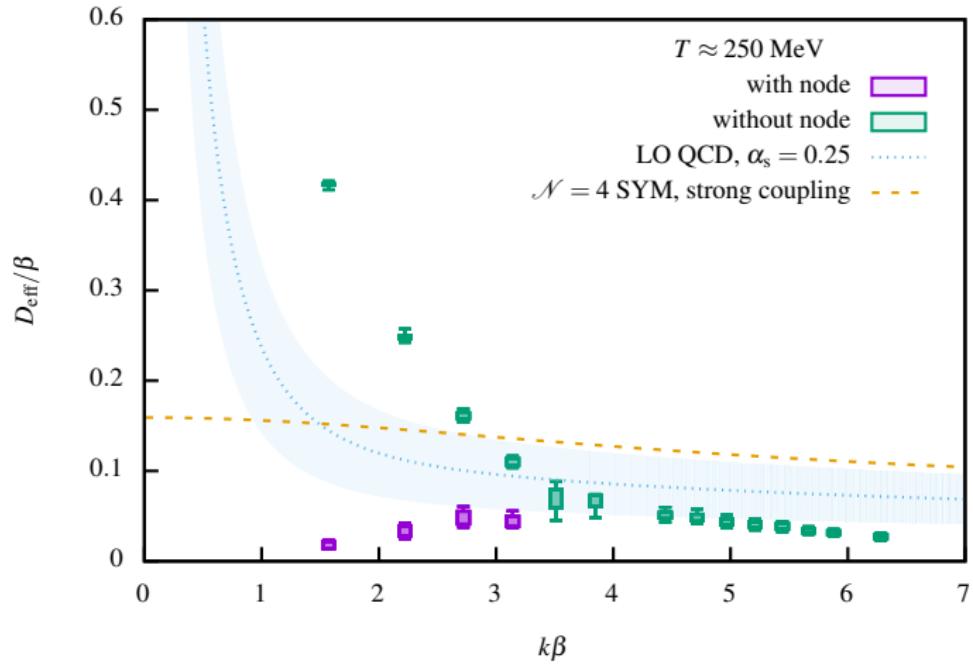
$\beta \rho(\omega, k) / \omega$



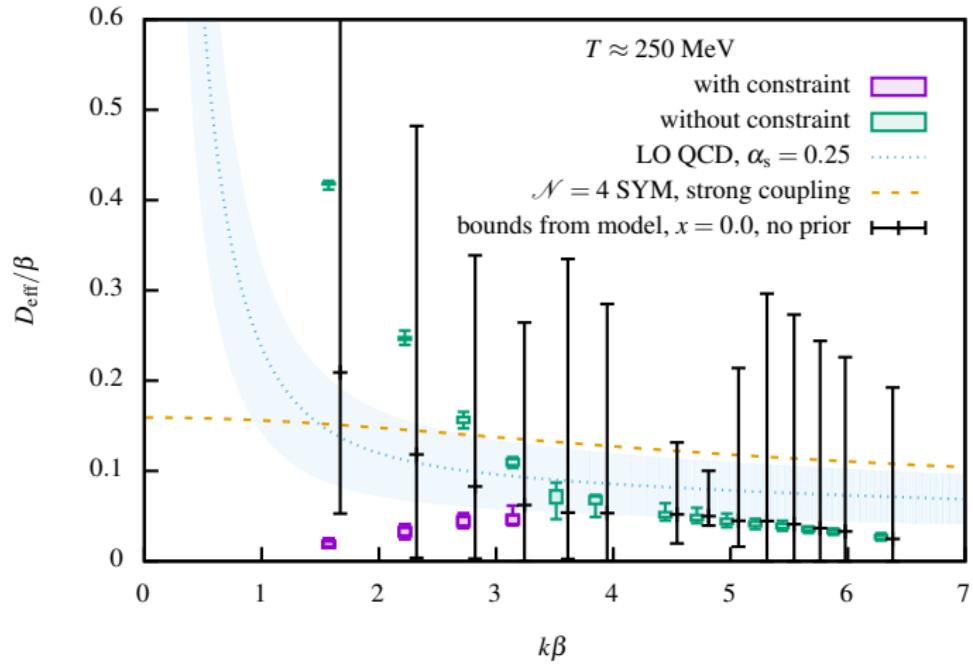
$\beta \rho(\omega, k) / \omega$



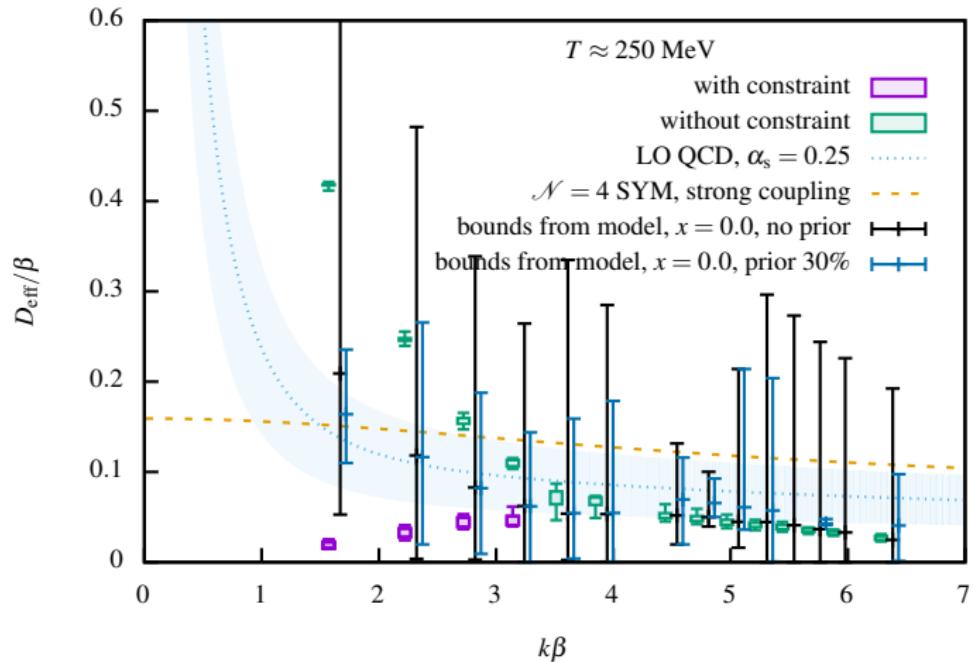
# Plot of the effective diffusion constant $D_{\text{eff}}/\beta = \rho(k, k)/(4\chi_s k)$



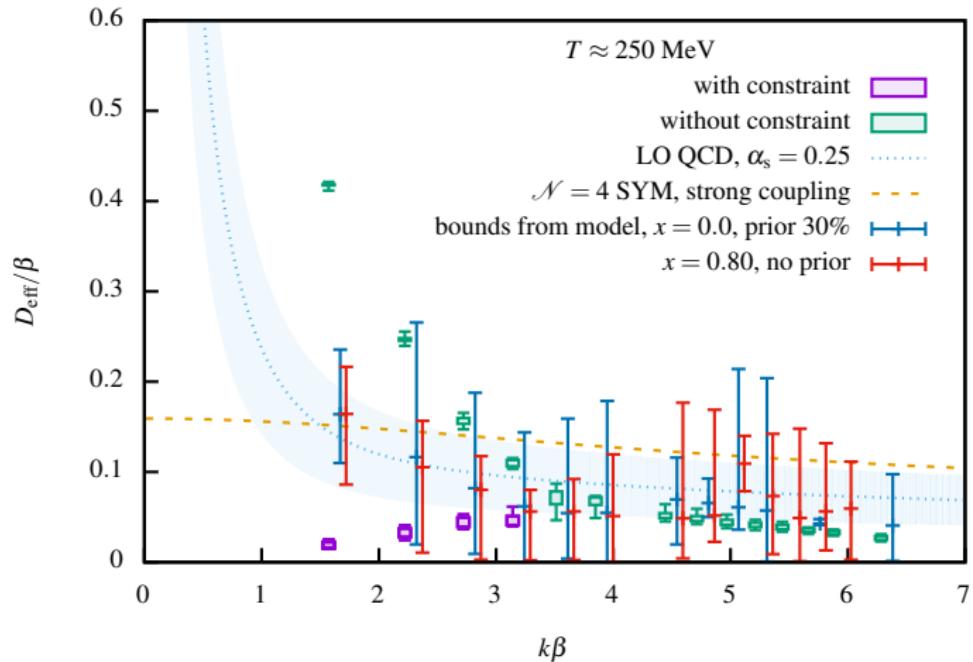
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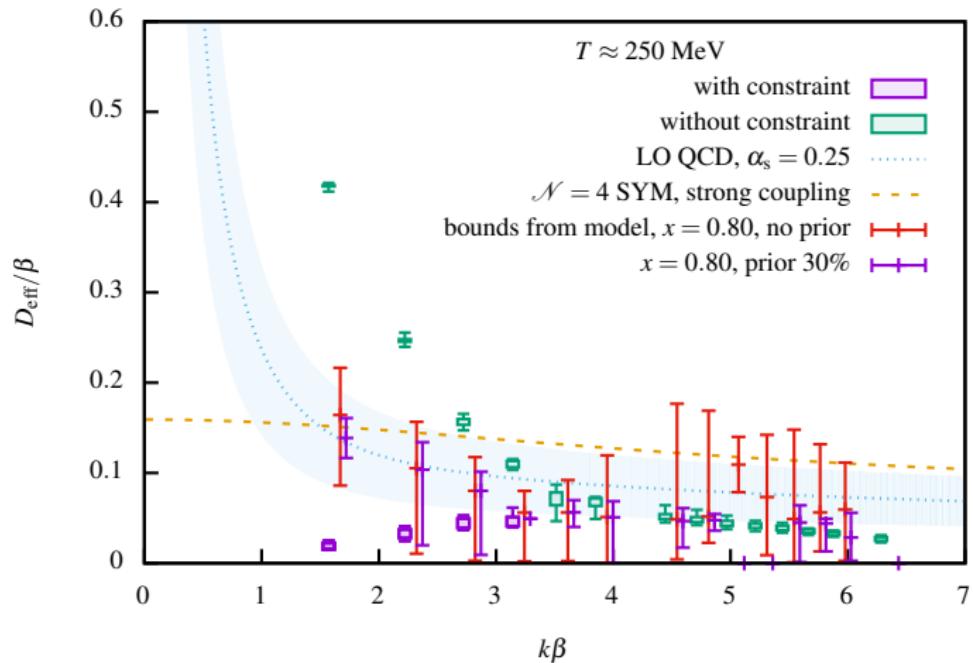
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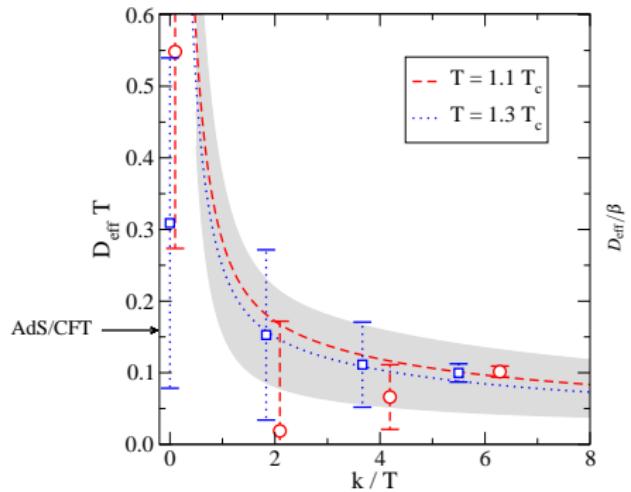
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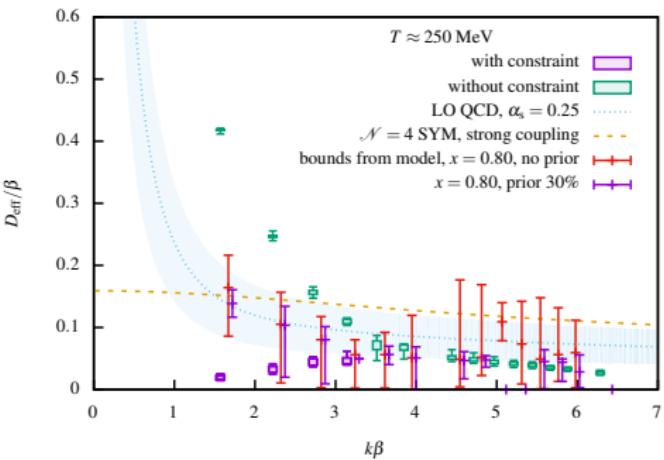
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Ghiglieri et al. (2016)



## Summary and Outlook

### CONCLUDING REMARKS:

- first continuum estimate of the photon rate from dynamical QCD
- employing an alternative linear combination of the vector-vector correlator to eliminate the UV contamination
- exploiting UV behavior known from OPE and imposing a superconvergent sum rule
- applying a model independent method to reconstruct the spectral function
- applying a Padé fit inspired by hydro and physics constraints
- combining with perturbative results about the curvature at the origin seems to help

### AGENDA:

- with higher statistics and finer lattices one could aim at computing the curvature from the lattice to constrain fit results
- define another effective  $D_{\text{eff}}(\xi, k) = \frac{\xi \rho(\xi k, k, -2)}{4\chi_s k} \rightarrow D$  for  $k \rightarrow 0$  at fixed  $\xi \in [0, 1]$ )  
⇒ more control in small  $k$  region
- further examination of systematics including the continuum limit and regularization
- extension to higher temperature

## Derivation of a sum rule for $\rho \equiv \rho_{\lambda=-2}$ short version

- i. Lorentz invariance and transversity  $\Rightarrow \tilde{G}_E(\omega_n, k) = 0$  in vacuum and UV finite at  $T > 0$
- ii. UV finite correlation admits an OPE  $\tilde{G}_E(\omega_n, k) \sim \frac{\mathcal{O}_4}{\omega_n^2}$

Furthermore, charge conservation demands  $\tilde{G}_E(\omega_n, k) \rightarrow 0$  as  $k \rightarrow 0$  and  $\omega > 0$ , so

$$\tilde{G}_E(\omega_n, k) \sim \frac{k^2 \mathcal{O}_4}{\omega_n^4}$$

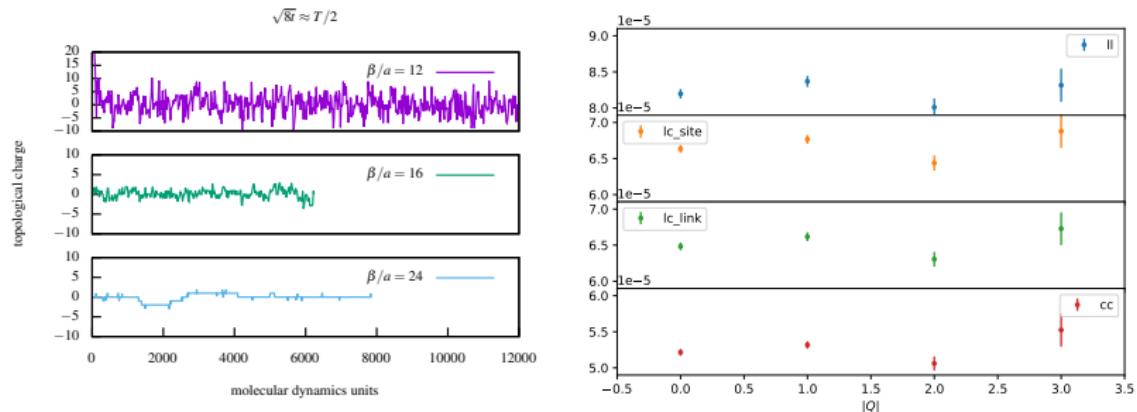
- iii. Matching the large  $\omega_n$ -behaviour

$$\tilde{G}_E(\omega_n, k) = \int_0^\infty \frac{d\omega}{\pi} \omega \frac{\rho(\omega, k)}{\omega^2 + \omega_n^2}$$

$$\stackrel{\omega_n \rightarrow \infty}{\longrightarrow} \frac{1}{\pi \omega_n^2} \int_0^\infty d\omega \omega \rho(\omega, k) \sim \frac{k^2 \mathcal{O}_4}{\omega_n^4}$$

results in the sum rule  $\int_0^\infty d\omega \omega \rho(\omega, k) = 0$

# Lattice set-up with $N_f = 2$ O( $a$ )-improved Wilson fermions



- no dependence of observable on topological charge is seen
- autocorrelations are under control, e.g.  $G_{00}(\tau = \beta/2)$

	$ll$	$lc_{\text{site}}$	$lc_{\text{link}}$	$cc$
naive error/ $10^{-5}$	8.229(47)	6.653(39)	6.505(38)	5.231(34)
error incl. autocorrs./ $10^{-5}$	8.229(70)	6.653(55)	6.505(57)	5.231(45)

## Backus-Gilbert method

$$G_E(\tau) = \int_0^\infty \frac{d\omega}{2\pi} \underbrace{\frac{\cosh(\omega(\beta/2 - \tau))}{\sinh(\beta\omega/2)}}_{K(\omega, \tau)} \rho(\omega)$$

goal: find set of inverse kernels  $q_i(\bar{\omega})$  and construct an estimator  $\rho_{BG}$  such that

$$\begin{aligned}\rho_{BG} &= \int_0^\infty \frac{d\omega}{2\pi} \hat{\delta}(\bar{\omega}, \omega) \rho(\omega), \quad \hat{\delta}(\bar{\omega}, \omega) = \sum_{i=1}^{N_f} q_i(\bar{\omega}) K(\omega, \tau_i) \\ \rho_{BG} &= \sum_{i=1}^{N_\tau} q_i(\bar{\omega}) G_E(\tau_i), \quad \text{Var} [\rho_{BG}] = q_i(\bar{\omega}) \mathcal{C}_{ij} q_j(\bar{\omega})\end{aligned}$$

challenges of the inversion problem:

- maximize stability of estimator  $\rho_{BG} \Rightarrow \text{minimize } \text{Var} [\rho_{BG}]$
- maximize agreement of estimator  $\rho_{BG}$  and “true” solution  $\rho$  subject to some constraint  $\Rightarrow$  minimize the functional (second moment of  $\hat{\delta}^2$ )

$$\begin{aligned}F [q_i(\bar{\omega})] &= q_i(\bar{\omega}) \cdot \int d\omega K(\omega, \tau_i) (\omega - \bar{\omega})^2 K(\omega, \tau_j) \cdot q_j(\bar{\omega}) \\ &\quad - \lambda \left( \int d\omega \hat{\delta}(\bar{\omega}, \omega) - 1 \right)\end{aligned}$$

## Backus-Gilbert method

$$G_E(\tau) = \int_0^\infty \frac{d\omega}{2\pi} \underbrace{\frac{\cosh(\beta/2 - \tau)}{\sinh(\beta\omega/2)}}_{K(\omega, \tau)} \rho(\omega)$$

$$\begin{aligned} F[q_i(\bar{\omega})] &= q_i(\bar{\omega}) \cdot \underbrace{\int d\omega K(\omega, \tau_i)(\omega - \bar{\omega})^2 K(\omega, \tau_j) \cdot q_j(\bar{\omega})}_{\mathcal{W}_{ij}(\bar{\omega})} \\ &\quad - \lambda \left( \int d\omega \hat{\delta}(\bar{\omega}, \omega) - 1 \right) \end{aligned}$$

in total minimize

$$\alpha \cdot F[\mathbf{q}(\bar{\omega})] + (1 - \alpha) \cdot \text{Var} [\rho_{BG}], \quad \alpha \in (0, 1)$$

and obtain

$$\mathbf{q}(\bar{\omega}) = \frac{[\alpha \cdot \mathcal{W}(\bar{\omega}) + (1 - \alpha) \cdot \mathcal{C}]^{-1} \cdot \mathbf{R}}{\mathbf{R} \cdot [\alpha \cdot \mathcal{W}(\bar{\omega}) + (1 - \alpha) \cdot \mathcal{C}]^{-1} \cdot \mathbf{R}}, \quad R_i = \int d\omega K(\omega, \tau_i)$$

$$\rho_{BG(\bar{\omega})} = \sum_i q_i(\bar{\omega}) G_E(\tau_i)$$