Diffusive dynamics of net-baryon fluctuations near the QCD critical point

Marcus Bluhm Marlene Nahrgang Thomas Schäfer Steffen Bass

Constraining the QCD Phase Boundary with Data from Heavy Ion Collisions – GSI Darmstadt, Germany – February 13, 2018

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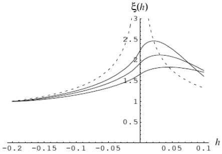
Dynamical effects are very important...

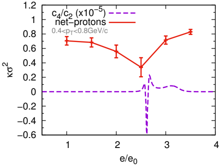
- **At the critical point:** $\xi \to \infty \Rightarrow$ fluctuations of the critical mode diverge!
- Higher moments more sensitive to ξ:

$$\langle \Delta\sigma^2 \rangle \propto \xi^2 \,, \quad \langle \Delta\sigma^3 \rangle \propto \xi^{9/2} \,, \quad \langle \Delta\sigma^4 \rangle_c \propto \xi^7 \,.$$

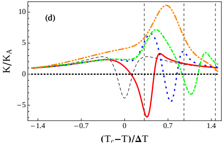
- Relaxation time $\tau_{\rm rel} \propto \xi^z$ diverges $\Rightarrow$ critical slowing down!



B. Berdnikov, K. Rajagopal PRD61 (2000)



C. Herold, MN, Y. Yan and C. Kobdaj PRC93 (2016) no.2



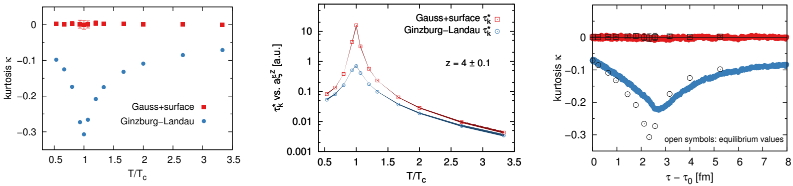
S. Mukherjee, R. Venugopalan, Y. Yin PRC92 (2015)

$\Rightarrow$ Study fluctuations within diffusive dynamics of net-baryon number!

Goal of this talk

Present a numerical implementation of the diffusive dynamics of net-baryon density fluctuations near the QCD critical point:

- thoroughly tested against equilibrium expectations
- discuss effects of resolution, system-size and net-baryon charge conservation



⇒ generation of non-Gaussian fluctuations from Gaussian white noise!

⇒ dynamical critical scaling of model B! Hohenberg, Halperin, Rev.Mod.Phys.49 (1977)

⇒ nonequilibrium and memory effects in the dynamics!

M. Nahrgang, MB, T. Schäfer, S. Bass, in preparation

Diffusive dynamics of the net-baryon density

$$\partial_\mu N_B^\mu = 0 \quad \text{net-baryon charge conservation}$$

The diffusive dynamics follows the minimized free energy $\mathcal{F}$:

$$\partial_t n_B(t,x) = \kappa \nabla^2 \left(\frac{\delta \mathcal{F}[n_B]}{\delta n_B} \right)$$

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To study intrinsic fluctuations include a stochastic current:

$$J(t,x) = \sqrt{2T\kappa}\,\zeta(t,x)$$

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$\rightarrow$ $\zeta(t,x)$ is Gaussian and uncorrelated (white noise):

$$\langle \zeta(x,t) \zeta(0,0) \rangle = \delta(x)\delta(t)$$

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$\Rightarrow$ respects the fluctuation-dissipation theorem:

$$P_{\mathrm{eq}}[n_B] = \tfrac{1}{\mathcal{Z}} \exp\left(-\mathcal{F}[n_B]/T\right)$$

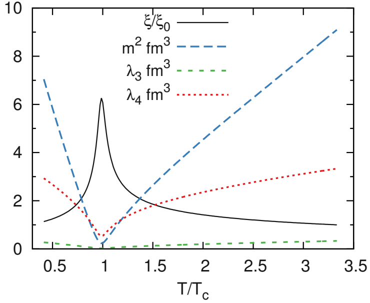
Couplings motivated by 3-dimensional Ising model

$$\mathcal{F}[n_B] = T\int \mathrm{d}^3 r \left(\frac{m^2}{2n_c^2}\Delta n_B^2 + \frac{K}{2n_c^2}(\nabla \Delta n_B)^2 + \frac{\lambda_3}{3n_c^3}\Delta n_B^3 + \frac{\lambda_4}{4n_c^4}\Delta n_B^4 + \frac{\lambda_6}{6n_c^6}\Delta n_B^6 \right)$$

The couplings depend on temperature via the correlation length $\xi(T)$:

$$m^2 = 1/(\xi_0 \xi^2)$$
$$K = \tilde{K}/\xi_0$$
$$\lambda_3 = n_c\, \tilde{\lambda}_3\, (\xi/\xi_0)^{-3/2}$$
$$\lambda_4 = n_c\, \tilde{\lambda}_4\, (\xi/\xi_0)^{-1}$$
$$\lambda_6 = n_c\, \tilde{\lambda}_6$$

M. Tsypin PRL73 (1994); PRB55 (1997)



in this Fig.: $\tilde{\lambda}_3 = 1$, $\tilde{\lambda}_4 = 10$

parameter choice: $\Delta n_B = n_B - n_c$

$\xi_0 \sim 0.5$ fm, $T_c = 0.15$ GeV, $n_c = 1/3$ fm^{-3}

$K = 1/\xi_0$ (surface tension)

$\tilde{\lambda}_3$, $\tilde{\lambda}_4$, $\tilde{\lambda}_6$ (universal, but mapping to QCD)

Different physical scenarios

The diffusion equation:

$$\partial_t n_B = \frac{D}{n_c}\left(m^2 - K\nabla^2\right)\nabla^2 n_B + D\nabla^2\left(\frac{\lambda_3}{n_c^2}\Delta n_B^2 + \frac{\lambda_4}{n_c^3}\Delta n_B^3 + \frac{\lambda_6}{n_c^5}\Delta n_B^5\right) + \sqrt{2Dn_c}\nabla\zeta$$

Different physical scenarios

The diffusion equation:

Gauss+surface **Ginzburg-Landau** **noise**

$$\partial_t n_B = \frac{D}{n_c}\left(m^2 - K\nabla^2\right)\nabla^2 n_B + D\nabla^2\left(\frac{\lambda_3}{n_c^2}\Delta n_B^2 + \frac{\lambda_4}{n_c^3}\Delta n_B^3 + \frac{\lambda_6}{n_c^5}\Delta n_B^5\right) + \sqrt{2Dn_c}\nabla\zeta$$

- consider $3+1$d system with propagation in 1 spatial dimension
- diffusion coefficient $D = \kappa T / n_c$
  - for equilibrium calculations $D = 1$ fm
  - for dynamical calculations T-dependent
- equilibrium: let system equilibrate for long times
- dynamics: equilibrate first at high T, then evolve
- $\langle N_B \rangle$ perfectly conserved in the numerics!

Different physical scenarios

The diffusion equation:

Gauss+surface **advection**

$$\partial_t n_B = \frac{D}{n_c}\left(m^2 - K\nabla^2\right)\nabla^2 n_B \quad -\vec{v}\cdot\nabla n_B$$

Ginzburg-Landau **noise**

$$+D\nabla^2\left(\frac{\lambda_3}{n_c^2}\Delta n_B^2 + \frac{\lambda_4}{n_c^3}\Delta n_B^3 + \frac{\lambda_6}{n_c^5}\Delta n_B^5\right) + \sqrt{2Dn_c}\nabla\zeta$$

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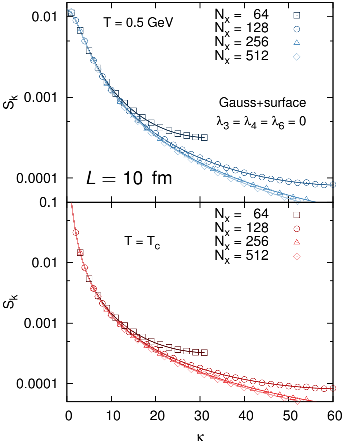
- equilibrium: let system equilibrate for long times
- dynamics: equilibrate first at high T, then evolve
- $\langle N_B \rangle$ perfectly conserved in the numerics!
- (including advection does not change results qualitatively)

Equilibrium fluctuations

Gaussian model in equilibrium - structure factor



static structure factor in continuum ($\xi^2 = K/m^2$):

$$S(k) = \frac{n_c^2}{m^2} \frac{1}{1 + \xi^2 k^2}$$

discretized form ($k = 2\pi\kappa/L$):

(semi-implicit predictor-corrector scheme)

$$S_k = \frac{n_c^2}{m^2} \frac{1}{1 + \frac{2K}{m^2 \Delta x^2} \left(1 - \cos(k\Delta x)\right)}$$

$\Rightarrow$ perfectly reproduced!

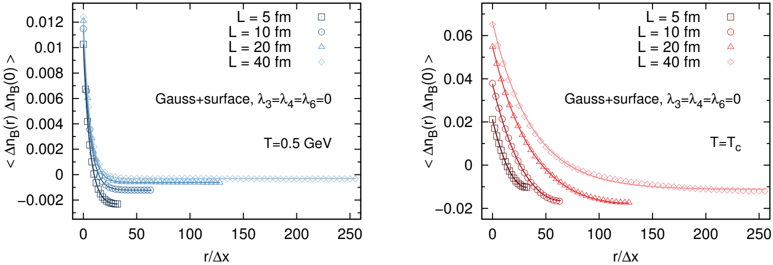
$\rightarrow S(k)$ for $\Delta x = L/N_x \rightarrow 0$!

non-zero surface tension suppresses short-wavelength fluctuations!

(in contrast to purely Gaussian model)

Gaussian model in equilibrium - correlation function

- for $K = 0$: fluctuations are delta-correlated!
- for $K \neq 0$: numerical correlation length $\xi > \Delta x$, where spatial correlations broaden near T_c!



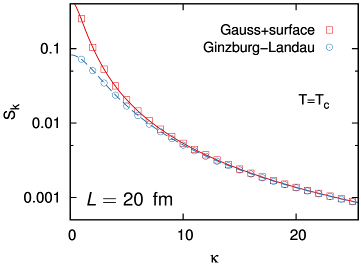
impact of exact $\langle N_B \rangle$-conservation: local n_B-fluctuations need to be balanced within L

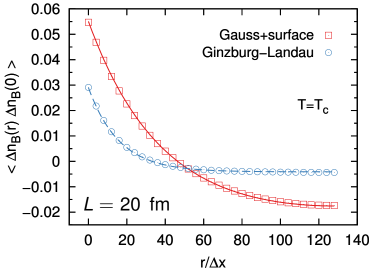
$\int_L \mathrm{d}r \, \langle \Delta n_B(r) \Delta n_B(0) \rangle = 0 \;\rightarrow\;$ negative correction term expected

$\Rightarrow$ perfectly reproduced by the numerics!

Note: with increasing L larger equilibration times needed!

Ginzburg-Landau model in equilibrium



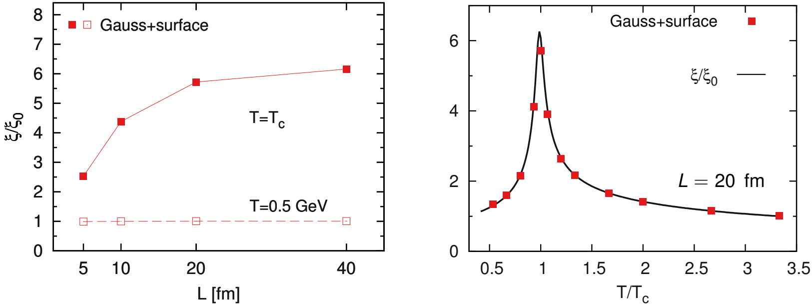


- nonlinear interactions reduce S_k for long-wavelength fluctuations!
- spatial correlations are significantly smaller!

$\Rightarrow$ at the level of 2-point correlations, results of Ginzburg-Landau model can be described by a **renormalized** Gauss+surface model (with m^2 modified, K essentially unaffected)!

Correlation length and finite-size effects

For finite size L, the numerical correlation length is strongly limited due to $\langle N_B \rangle$-conservation.



- very good realization of the input $\xi(T)$! (only achievable with $K \neq 0$!)

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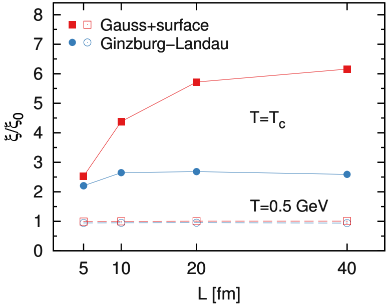
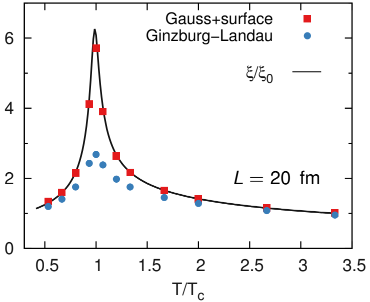
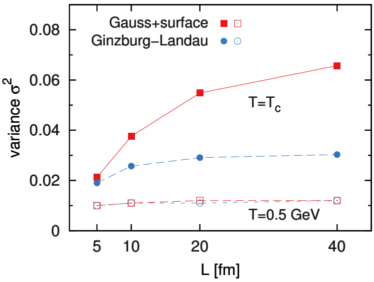


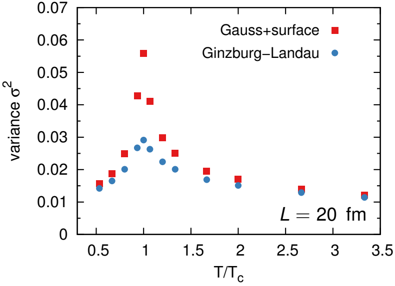


- very good realization of the input $\xi(T)$! (only achievable with $K \neq 0$!)
- reduction of the numerically realized correlation length for Ginzburg-Landau model in accordance with the renormalized m^2!

T-dependence of Gaussian fluctuations

In finite systems: variance is reduced compared to TD expectations.



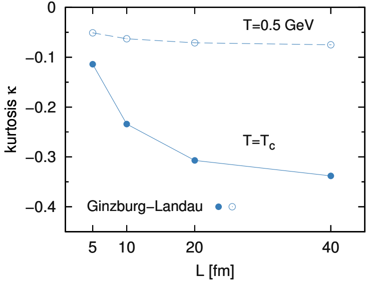


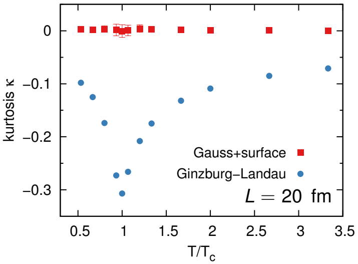
- nonlinear couplings in the Ginzburg-Landau model reduce the variance!

$\Rightarrow$ Could explain why no sign of criticality is observed in second-order moments at NA49 and RHIC BES phase I.

T-dependence of non-Gaussian fluctuations

Kurtosis vanishes in the absence of nonlinear interactions!





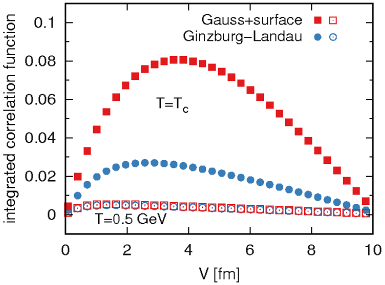
- negative kurtosis observed for Ginzburg-Landau model with a pronounced signal near T_c!

Note: we choose $L = 20$ fm ($N_x = 256$) for the following studies!

Integrated fluctuation measures

- so far local fluctuations considered, i.e. taken over $V = \Delta x$
  $\Rightarrow \sigma^2 \propto \xi$!
- integrated variance: $\sigma_V^2 = \frac{1}{V^2} \int dx \int dy \, \langle \Delta n_B(x) \Delta n_B(y) \rangle \propto \xi^2$

M. Stephanov et al. PRL81 (1998); PRD60 (1999); PRL102 (2009)



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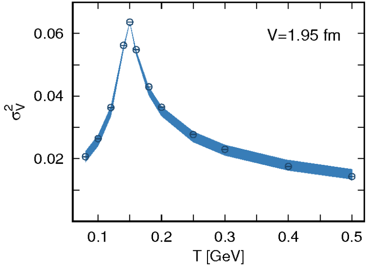
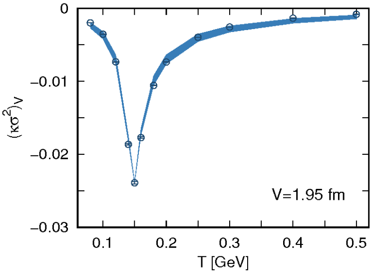


- including finite-size and $\langle N_B \rangle$-conservation we find for $V \simeq 2$ fm:

$$\sigma_V^2 \propto \xi^n \text{ with } n \simeq 1.4 \pm 0.1\,!$$

Integrated fluctuation measures

- integrated kurtosis:



- including finite-size and $\langle N_B \rangle$-conservation we find for $V \simeq 2$ fm:

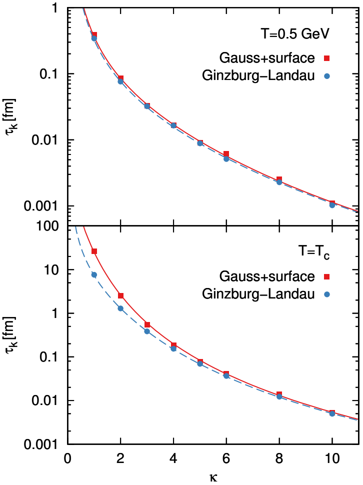
$$(\kappa\sigma^2)_V \propto \xi^n \text{ with } n \simeq 2.9 \pm 0.2\,!$$

Dynamics of fluctuations

Dynamics: relaxation time

dynamic structure factor for Gaussian model in continuum:

$$S(k,t) = S(k)\exp\left(-t/\tau_k\right) \quad \text{with} \quad \tau_k^{-1} = \frac{Dm^2}{n_c}\left(1 + \frac{K}{m^2}k^2\right)k^2$$

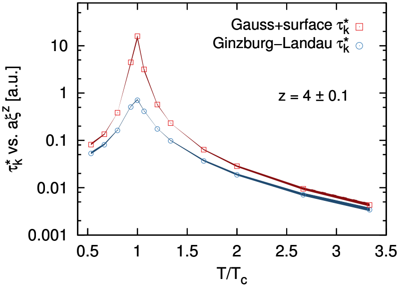


- analytic results reproduced by numerics!
- long-wavelength modes relax slowly!
- τ_k significantly enhanced near T_c!
- nonlinear interactions reduce τ_k for modes with small k!

  (described by renormalized Gauss+surface model)

Dynamics: dynamical critical scaling

analyze ξ-dependence of relaxation time for modes with $k^* = 1/\xi$:



- for both models: $\tau_k^* = a\,\xi^z$
- best fit gives:

$$z = 4 \pm 0.1$$

$$a = \frac{n_c \xi_0}{D(1+\tilde{K})}$$

$\Rightarrow$ Simulations reproduce scaling of model B!

Hohenberg, Halperin, Rev.Mod.Phys.49 (1977)

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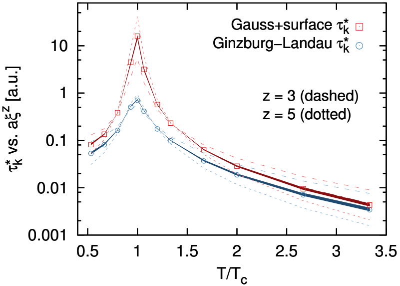

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- $z = 3, 5$ ruled out! (nice accuracy!)

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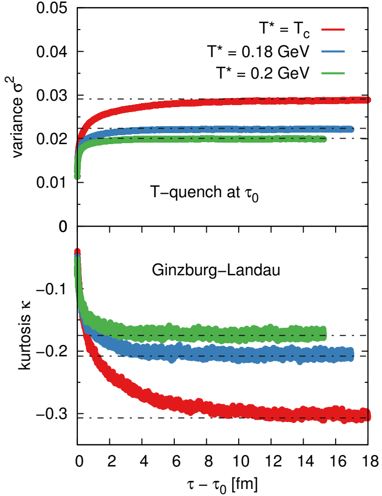
Hohenberg, Halperin, Rev.Mod.Phys.49 (1977)

Dynamics: temperature quench and equilibration



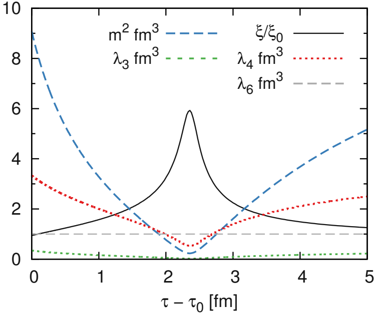
- temperature quench:
  at τ_0 temperature drops from $T_0 = 0.5$ GeV to T^*
- fast initial relaxation
- variance approaches equilibrium value faster than kurtosis
- long relaxation times near T_c

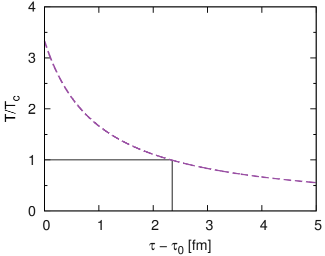
B. Berdnikov, K. Rajagopal PRD61 (2000)

Dynamics: time-dependent couplings

time-dependent temperature:

$$T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)^{dc_s^2}$$

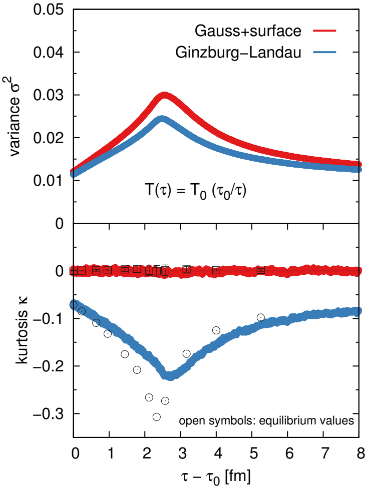




- choose $c_s^2 = 1/3$

  (should be $c_s^2 = c_s^2(T)$)
- initialize system at $T_0 = 0.5$ GeV, $D(T_0) = 1$ fm
- T_c reached at $\tau - \tau_0 = 2.33$ fm

Dynamics: critical (non-)Gaussian fluctuations



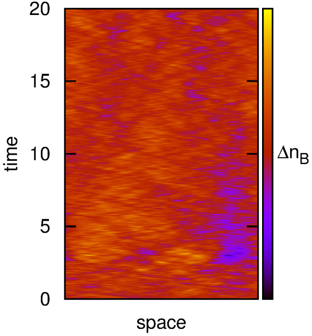
- shift of extrema for variance and kurtosis (memory effects)

  B. Berdnikov, K. Rajagopal PRD61 (2000), S. Mukherjee et al., PRC92(3) (2015)

- |extremal values| in dynamical simulations below equilibrium values (nonequilibrium effects)
- no dynamical effects on non-Gaussianity in Gauss+surface model
- expected behavior with varying D and c_s^2 (expansion rate)

Conclusions

- successful test of numerical implementation vs. analytic expectations!
- significant effect of net-baryon charge conservation!
- in presence of nonlinear interactions:
  - variance is reduced (consistent with experimental data)!
  - generation of non-Gaussian fluctuations (from purely Gaussian white noise)!
- dynamics: critical scaling of model B, nonequilibrium and memory effects!

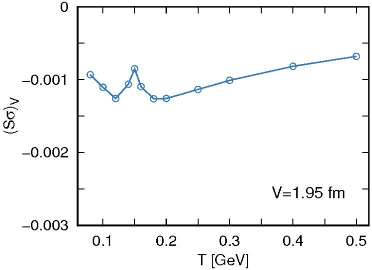


Thanks to:

National Science Centre Poland

BEST COLLABORATION

What about the integrated skewness?



Fluid dynamical fluctuations in heavy-ion collisions

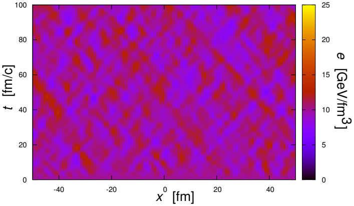
The average particle spectra are successfully described using conventional fluid dynamics. $\rightarrow$ What about fluctuations?

$$\partial_\mu T^{\mu\nu} = \partial_\mu \left(T^{\mu\nu}_{\text{eq}} + \Delta T^{\mu\nu}_{\text{visc}} + \Xi^{\mu\nu} \right) = 0$$

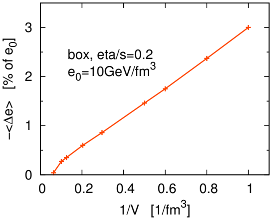
$$\partial_\mu N^{\mu} = \partial_\mu \left(N^{\mu}_{\text{eq}} + \Delta N^{\mu}_{\text{visc}} + I^{\mu} \right) = 0$$

- important consequence: nonlinearities cause cutoff dependent corrections to EoS, η ... (e.g. $\langle \Delta e \rangle \sim -\Lambda^3$) P. Kovtun et al., JHEP1407 (2014)

box, $\Delta x = 1$ fm, $e_0 = 10$ GeV/fm^3, $\eta/s = 0.2$



based on vHLLE code



$\Rightarrow$ Implementing fluid dynamical fluctuations is important, but requires a sustained and systematic effort!

M. Nahrgang et al., CPOD2016

I. Karpenko et al., Comput.Phys.Commun.185 (2014)