Onset of mesonic condensation phenomena from higher order susceptibilities

Vladimir Skokov (RBRC BNL)

in collaboration with Swagato Mukherjee

- **Introduction** : Signatures of phase transitions
  talks on Monday and Tuesday

- Two questions:

  Phenomenological question: Can we extract more from experimental data?

  Theoretical question: Can we extract the location of a phase transition from analysis of cumulants/Taylor coefficients?

Outline I

- **Introduction** : Signatures of phase transitions
  talks on Monday and Tuesday

- Two questions:

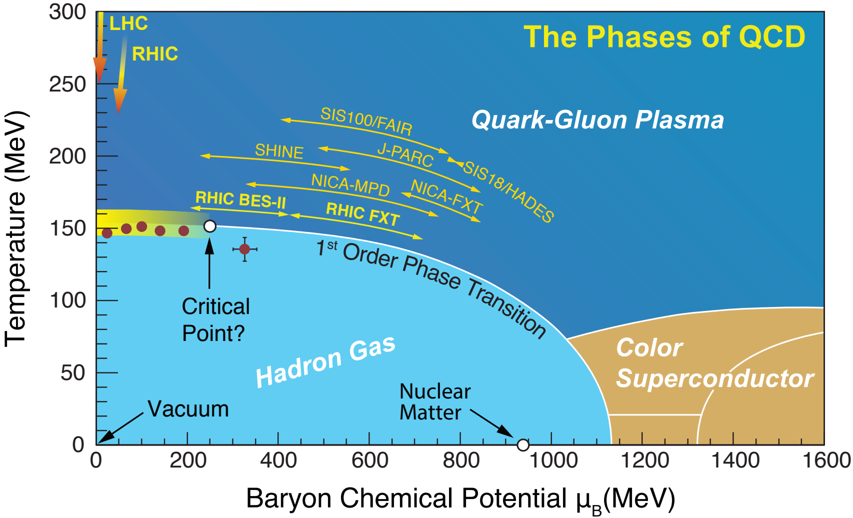
  Phenomenological question: Can we extract more from experimental data?

  Theoretical question: Can we extract the location of a phase transition from analysis of cumulants/Taylor coefficients?

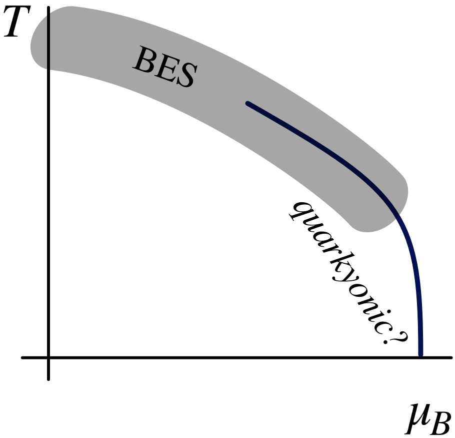
  Real question: Is it feasible?

Outline II

- **Fluctuations**: electric/isospin charge to study Bose-Einstein condensation
- **Demonstration I**: Free pions and hadron resonance gas
- **Demonstration II**: Non-perturbative Chiral Matrix Model
- **Outlook and Conclusions**



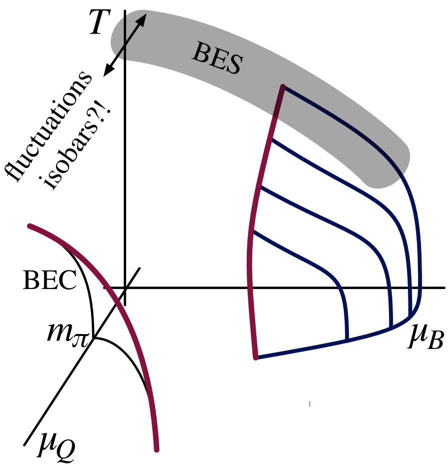
Courtesy of Thomas Ullrich



HIC provide unique opportunity to study $T - \mu_B$ phase diagram

It is possible that phase diagram is very different:
Inhomogeneous phase: M. Buballa . . .
R. Pisrski, V.S., A. Tsvelik, arXiv:1801.08156

Extending to 3d dimension



- What happens at non-zero μ_Q, μ_I?
- We may probe those regions with fluctuations. Is it feasible?
- $\mu_I \neq 0$ can be studied on lattice owing to absence of sign problem

Radius of convergence I

- Lets consider an even function and its Taylor expansion at $z_0 = 0$

$$f(z) = \sum_k c_{2k} z^{2k} \quad \text{with} \quad c_{2k} = \frac{1}{(2k)!} \frac{\partial^{2k} f}{\partial z^{2k}}$$

- **Negative statement**: Taylor expansion converges only for certain values of z. Series converges in the interval $(-R, R)$ where r is the radius of convergence.
- The **ratio** and root tests:

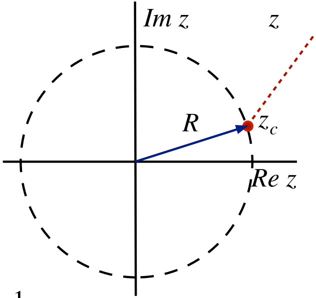
$$R = \lim_{k\to\infty} \inf R_{2k}, \quad R_{2k} = \left|\frac{c_{2k-2}}{c_{2k}}\right|^{1/2} \quad \text{or} \quad R_k = \left|\frac{1}{c_{2k}}\right|^{1/(2k)}$$

- Examples:

$$\frac{1}{1-z^2} = \sum_n z^{2n}, \quad R = 1; \qquad \mathrm{Li}_2(z^2) = \sum_n \frac{1}{n^2} z^{2n}; \quad R = 1$$

Radius of convergence II

- **Positive statement**: Closest singularity (z_c) to expansion point ($z_0 = 0$) defines radius of convergence R of series



- Examples pole singularity and branch point singularity:

$$\frac{1}{1-z^2} = \sum_n z^{2n}, \quad R = 1; \qquad \mathrm{Li}_2(z^2) = \sum_n \frac{1}{n^2} z^{2n}; \quad R = 1$$

- According to Darboux theorem (1878):
  "late terms in the development of a function as a Taylor series are determined by the behaviour of the function in the vicinity of singular points on the circle of convergence"

- One can determine not only location but also type of singularity.

Why is it interesting?

- Singularities of partition function $\rightsquigarrow$ phase transitions.

- Type of singularity $\rightsquigarrow$ critical exponent.

Radius of convergence: relation to cumulants

- Pressure can be expanded to

$$p/T^4 = \sum_k \frac{\chi_k}{k!} \cdot \left(\frac{\mu - \mu_0}{T}\right)^k$$

where χ_k is k-th order cumulant computed at μ_0/T.

- Cumulants are related to fluctuations of corresponding charge, i.e.

$$\chi_4 = \frac{\langle(\delta N)^4\rangle - 3\langle(\delta N)^2\rangle^2}{VT^3}$$

Pressure ≡ cumulant generating function, $p \propto \ln \sum_N \mathrm{tr}\, e^{-\beta(H-\mu N)}$

- Ratio test: explicit factors of VT^3 cancel

SINGULARITIES ON THE COMPLEX PLANE ARE RELATED TO

- critical point of second-order phase transition. Singularity lies on real μ axis.

- crossover transition. Singularity is at complex μ.

See P. C. Hemmer and E. H. Hauge, Phys. Rev. **133***, A1010 (1964); C. Itzykson, R. B. Pearson and J. B. Zuber, Nucl. Phys. B* **220***, 415 (1983); M. A. Stephanov, Phys. Rev. D* **73***, 094508 (2006).*

- spinodal lines for first-order phase transition.
  Singularities are either at real (mean-field) or complex values of μ.

Analyticity conjecture by Fonseca and Zamolodchikov arXiv:hep-th/0112167: complexified spinodal point is the nearest singularity

- "thermal singularities" associated with zeros of inverse Fermi-Dirac function.

See F. Karbstein and M. Thies, Phys. Rev. D **75***, 025003 (2007); V.S., K. Morita, B. Friman Phys.Rev. D83 (2011) 071502*

Thermal singularities: an illustration

- Fermi-Dirac function

$$f(\mu) = \frac{1}{e^{-(\omega-\mu)/T} + 1}$$

- Consider $\mu = \pm i\pi T + \mathrm{Re}\,\mu$

$$f(\mu) = \frac{1}{e^{\pm i\pi} e^{-(\omega-\mathrm{Re}\mu)/T} + 1} = \frac{1}{-e^{-(\omega-\mathrm{Re}\mu)/T} + 1} = -\frac{1}{e^{-(\omega-\mathrm{Re}\mu)/T} - 1} \to -f_{\mathrm{Bose}}(\mathrm{Re}\,\mu)$$

"Bose-Einstein condensation" at $\mathrm{Re}\,\mu = m$.

- In QCD, owing to Z(3) the singularities are located at

$$\mathrm{Im}\,\mu_q = \pm(\pi T + 2\pi nkT)/3$$

Roberge-Weiss symmetry: a shift in imaginary chemical potential can be compensated by a gauge transformation (Z(3) transformation)

Free pions

Pressure:

$$\frac{p_{\pi^+} + p_{\pi^-}}{T^4} = \frac{m_\pi^2}{\pi^2 T^2} \sum_{l=1}^{\infty} K_2(l m_\pi / T) \cosh(l \mu_Q / T)$$

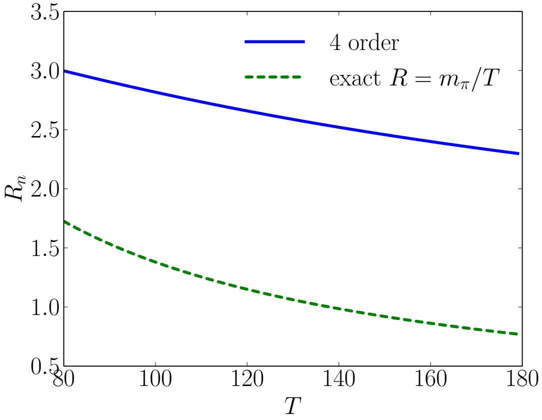
Cumulants at zero μ_Q:

$$\chi_n^Q = \frac{m_\pi^2}{\pi^2 T^2} \sum_{l=1}^{\infty} l^{n-2} K_2(l m_\pi / T)$$

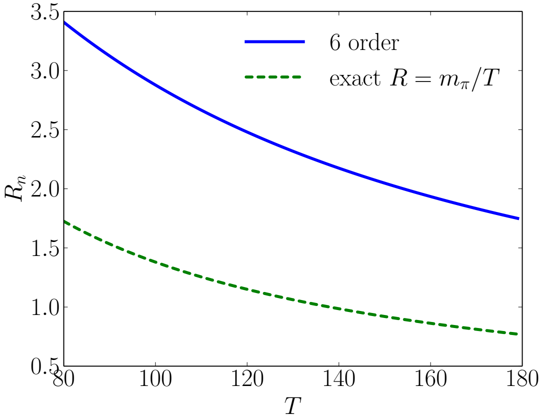
Expansion coefficients

$$c_n = \frac{\chi_n^Q}{n!}$$

Free pions: radius of convergence



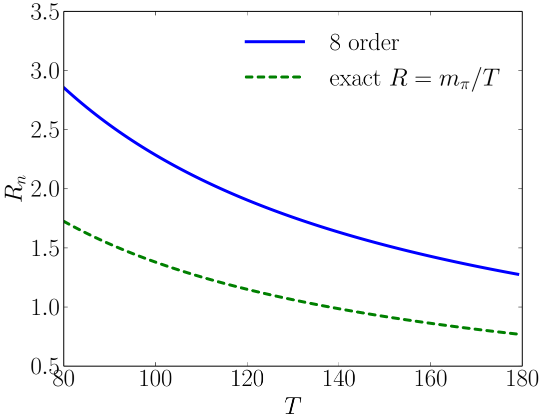
$$R_4 = (c_2/c_4)^{1/2}$$



6 order
exact $R = m_\pi/T$
R_n
3.5
3.0
2.5
2.0
1.5
1.0
0.5
80
100
120
140
160
180
T

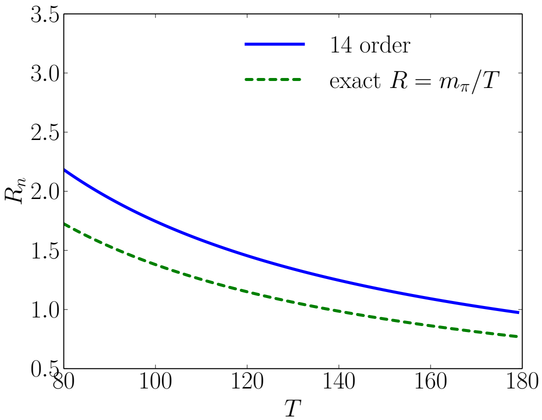
$$R_6 = (c_4/c_6)^{1/2}$$

Free pions: radius of convergence



$$R_8 = (c_6/c_8)^{1/2}$$

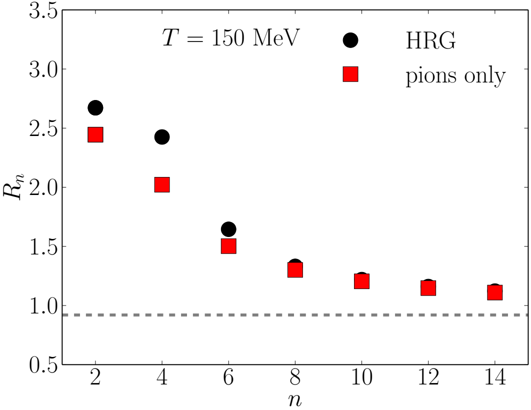
Free pions: radius of convergence



$$R_{14} = (c_{12}/c_{14})^{1/2}$$

14-th order!

Conclusion from this exercise



- at moderate n - significant contribution from doubly charge baryons

see e.g. P. Braun-Munzinger, B. Friman, F. Karsch, K. Redlich and V.S.,Nucl. Phys. A **880***, 48 (2012)*

To describe location of BEC transition, very high order cumulants are required

Can we improve our methods?

Suppose that singularity z_c of $f(z)$ $(= \sum_n f_n z^n)$ is closest to origin, and that behaviour in its neighborhood is given by

$$f(z) = (1 - z/z_c)^{-\nu} r(z) + a(z)$$

$r(t)$ and $a(t)$ are analytic in some circle with center at $z = 0$ and radius $> |z_c|$.
Asymptotic expansion of $f(z) \rightsquigarrow$ pattern in the coefficients

$$f_n \sim \sum_{k=0}^{\infty} \frac{(-1)^k b_k {z_c}^{k-n} \Gamma(n + \nu - k)}{n! \Gamma(\nu - k)} \sim \mathcal{O}(n^{\nu-1})$$

Construct difference

$$f_n z_c - \frac{n + \nu - 1}{n} f_{n-1} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} k b_k {z_c}^{k-n} \Gamma(n + \nu - k - 1)}{n! \Gamma(\nu - k)} \sim \mathcal{O}(n^{\nu-3}) \to 0$$

Consequently $z_c = \frac{f_{n-1}}{f_n}\left(1 + \frac{\nu-1}{n}\right)$, leading term gives ratio method.

C. Hunter and B. Guerrieri 1980

Darboux theorem II

- No improvement if the critical exponent ν is unknown
- One can apply the same analysis to $n-1$ order

$$f_n z_c - \frac{n+\nu-1}{n} f_{n-1} = 0$$

$$f_{n-1} z_c - \frac{(n-1)+\nu-1}{n-1} f_{n-2} = 0$$

- One can solve for z_c and ν
- Applying to the problem at hand

$$\left(\frac{\mu_c}{T}\right)^2 = \frac{\chi_{2(n-1)}\,\chi_{2(n-2)}}{\chi_{2n}\,\chi_{2(n-2)} - \chi_{2(n-1)}^2}$$

Darboux theorem III

Combine adjacent f_n to construct more accurate approximation.
This can be done efficiently by constructing recursive relation (based on pattern in coefficients)

$$\begin{aligned} X_n^0(z_c,\nu) &= f_n; \\ X_n^{m+1}(z_c,\nu) &= z_c X_n^m(z_c,\nu) - (n+\nu-2m-1)X_{n-1}^m(z_c,\nu)/n. \end{aligned}$$

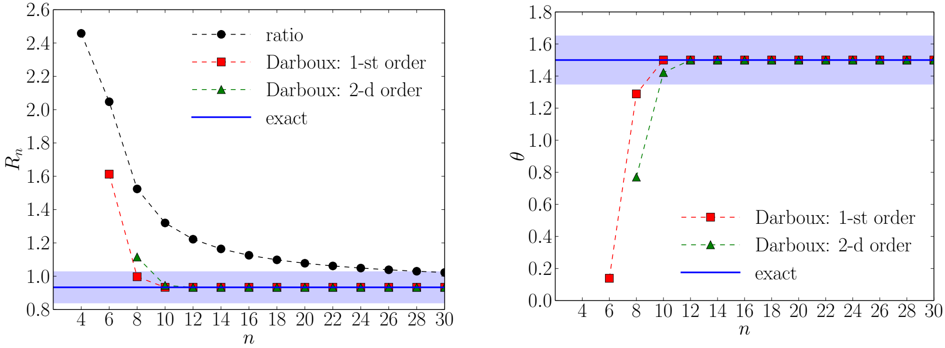
Easy to show that

$$X_n^m \sim O(n^{\nu-2m-1})$$

Two unknowns z_c and ν can be found by solving two equations

$$X_n^m(z_c,\nu) = 0, \quad X_{n-1}^m(z_c,\nu) = 0$$

For simplicity $m = 1, 2$



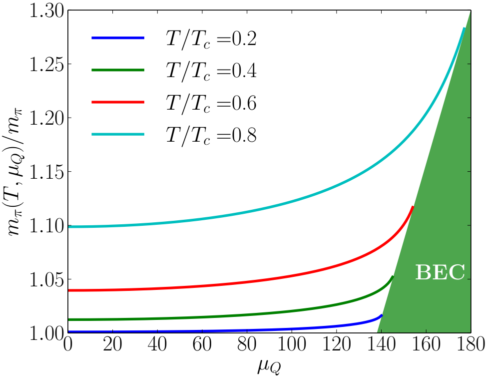
Singular contribution to the pressure

$$\propto \frac{1}{6\pi} T^2 \left(m^2 - \mu^2\right)^{3/2} \rightsquigarrow \theta = 3/2$$

H. Haber and H. Weldon, 1981

Non-perturbative chiral matrix model

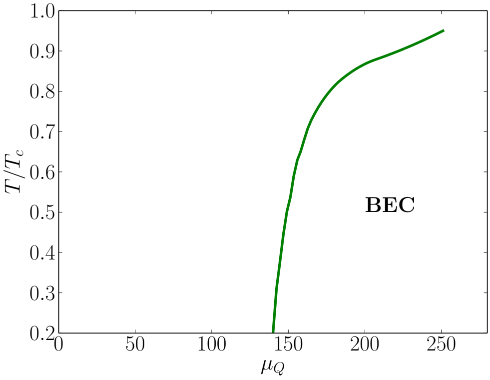
- Only in free theory BE condensation occurs at vacuum pion mass.
- Functional Renormalization Group approach to take into account fluctuations
- Pion mass dependence on μ_Q for different T:



- Details of model: R. Pisarski and V.S. 1604.00022
- Details of FRG solution: V.S., B. Friman, K. Redlich 1108.3231

Non-perturbative chiral matrix model

- The phase diagram in $T - \mu_Q$ plane:

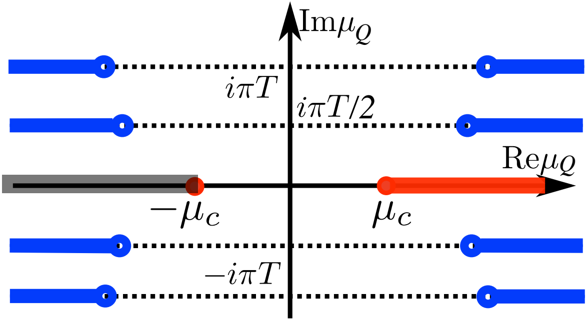


- Details of model:
  R. Pisarski and V.S. 1604.00022
- Details of FRG solution:
  V.S., B. Friman, K. Redlich 1108.3231

- Non-trivial dependence of T_{BEC} on μ_Q.
- Can it be reproduced with cumulant analysis?

- At low T transition is at about $\mu_Q = 140$ MeV as expected.

Thermal singularities



Singularities are located at

$$\mathrm{Im}\ \mu_Q = \pm\pi T \text{ (related to d quark or single charged baryons)}$$

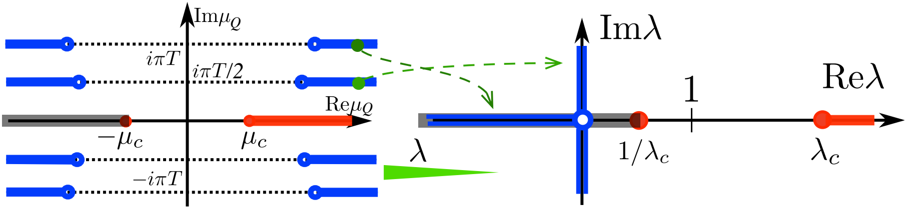
and

$$\mathrm{Im}\ \mu_Q = \pm\pi T/2 \text{ (related to u quark or doubly charged baryons)}$$

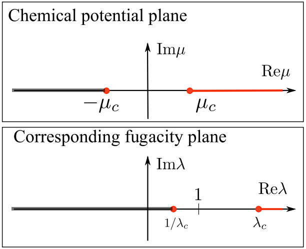
Fugacity

$$\lambda = \exp \mu_Q / T$$

- maps the line Im μ_Q = $i\pi T$ to negative real axis on fugacity plane
- maps the line Im μ_Q = $i\pi T/2$ to imaginary axis on fugacity plane



Mapping to fugacity plane



- Mapping to fugacity plane, $\lambda = \exp\left(\frac{\mu}{T}\right) \rightsquigarrow \mu/T = \ln(\lambda)$
- To simplify notation $\tilde{\lambda} = \lambda - 1$

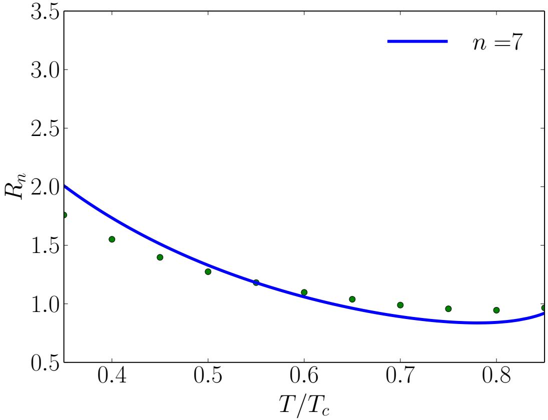
$$\sum_n \frac{1}{n!}\chi_n \left(\frac{\mu}{T}\right)^n =$$

$$\chi_0 + \frac{\tilde{\lambda}^2\chi_2}{2} - \frac{\tilde{\lambda}^3\chi_2}{2} + \tilde{\lambda}^4\left(\frac{11\chi_2}{24} + \frac{\chi_4}{24}\right) + \tilde{\lambda}^5\left(-\frac{5\chi_2}{12} - \frac{\chi_4}{12}\right)$$

$$+ \tilde{\lambda}^6\left(\frac{137\chi_2}{360} + \frac{17\chi_4}{144} + \frac{\chi_6}{720}\right) + \tilde{\lambda}^7\left(-\frac{7\chi_2}{20} - \frac{7\chi_4}{48} - \frac{\chi_6}{240}\right)$$

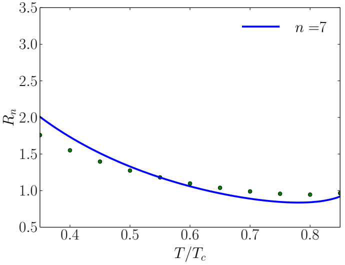
$$+ \ldots$$

Applying Darboux theorem

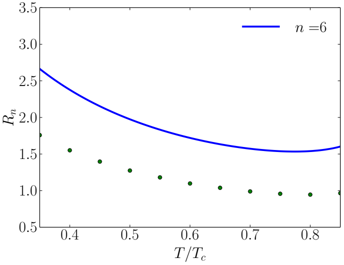


Applying Darboux theorem

Method based on Darboux theorem:



Ratio test:

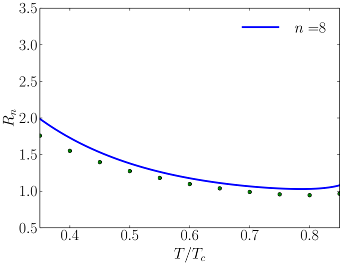


Reminder: 7-th order in fugacity includes only 6-th and lower order cumulants:

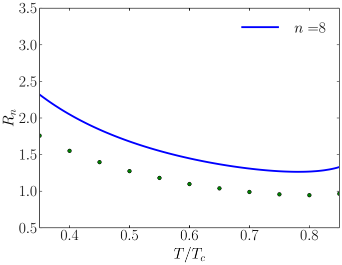
$$\tilde{\lambda}^7 \left(-\frac{7\chi_2}{20} - \frac{7\chi_4}{48} - \frac{\chi_6}{240} \right)$$

Applying Darboux theorem

Method based on Darboux theorem:

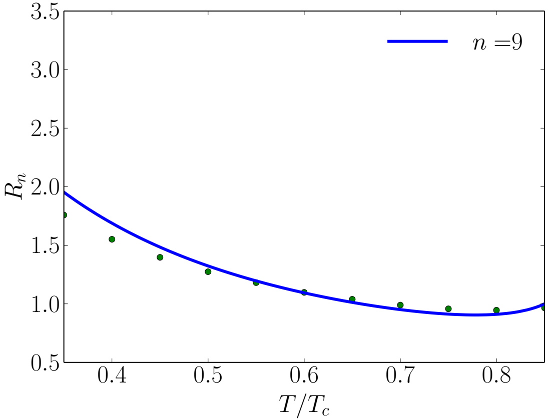


Ratio test:

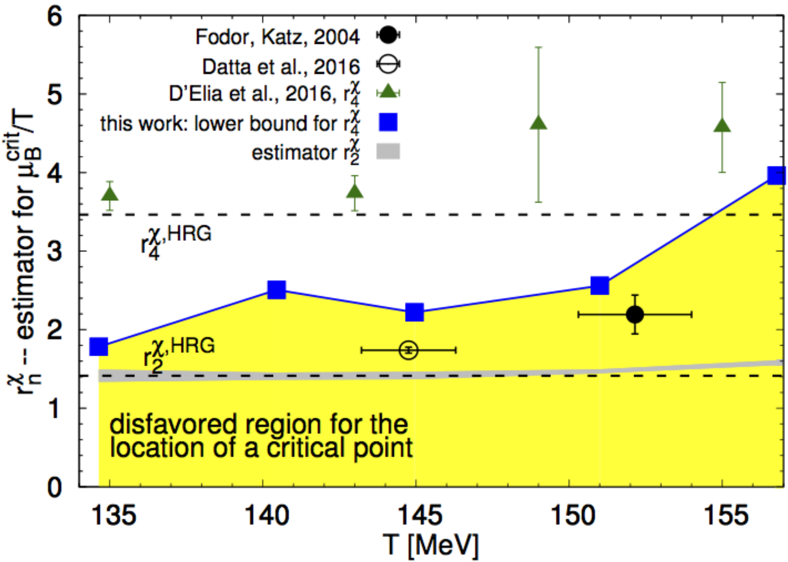


Superior reconstruction of phase boundary

Applying Darboux theorem



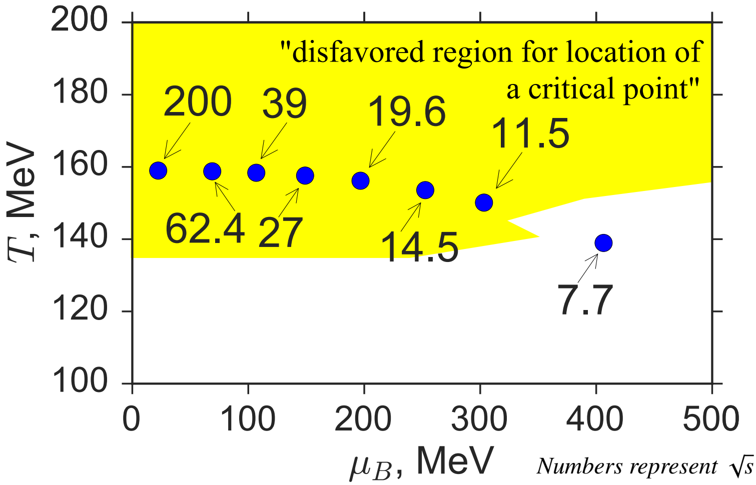
Lattice results for baryon number susceptibilities



BNL-Bi-CCNU Collaboration, arXiv:1701.04325

Lattice results for baryon number susceptibilities

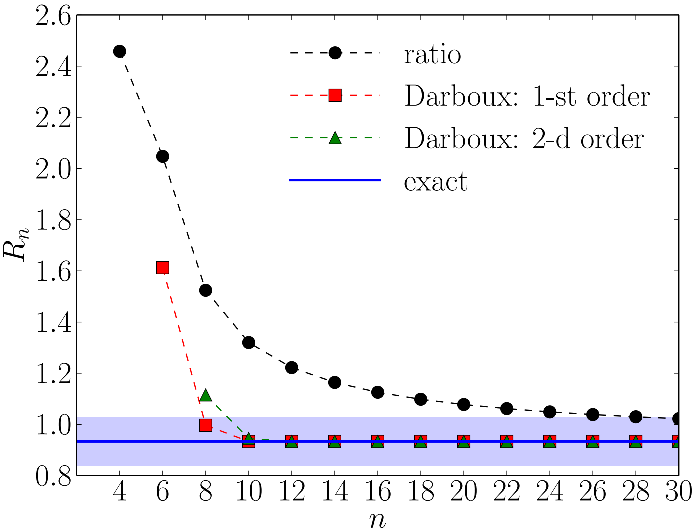
Freeze-out points are from analytical parametrization; for greater precision see A. Andronic, P. Braun-Munzinger, K. Redlich, J. Stachel, arXiv:1611.01347 and LQCD parametrization based on cumulants, BNL-Bi-CCNU Collaboration, arXiv:1701.04325



Numbers represent $\sqrt{s}$ in GeV

Bad news for BES II, good news for CBM!

Words of caution



Ratio test may greatly overestimate the radius of convergence.

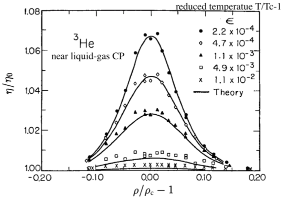
Conclusions and Outlook

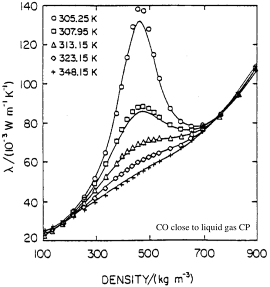
- Cumulant analysis may eventually allow us to reconstruct 3D phase diagram
- In this talk, μ_Q as 3-d axis. Commonly applied ratio test requires very high order cumulants and is very inaccurate
- Using Darboux theorem, with reasonable number of cumulants $n = 6$ it is possible to reconstruct onset of BEC.
- Current lattice analysis based on ratio test my greatly overestimate the radius of convergence.
- A reminder: crossover is not boring; it corresponds to the singularities in the complex plane. These singularities are connected/related to critical point. How to study crossover if you see no phase transitions at $\mu_B/T < \pi$? Map the strip $-\pi < \text{Im}\,\mu_B/T < \pi$ onto the unit circle.

V.S., K. Morita & B. Friman, Phys.Rev. D83 (2011) 071502

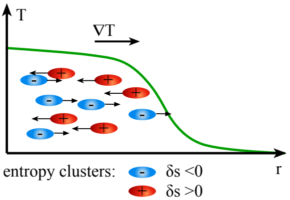

BACKUP

- Dynamic:
  - Divergent shear and bulk viscosities at CP. Model H universality class $\eta \propto \xi^{\frac{1}{19}}$ and $\zeta \propto \xi^{2.8}$, where ξ is correlation length. Cavitation phenomena?
  - Divergent heat and charge conductivities $\lambda \propto \xi^{18/19}$. Possible observable: separation of entropy and charge clusters.
  - Spinodal decomposition at 1-st phase transition J. Randrup 2011; V.S. and D. Voskresensky 2011





Clusters with lower entropy tend to move along the temperature gradient; Clusters with higher entropy tend to move opposite to the temperature gradient.



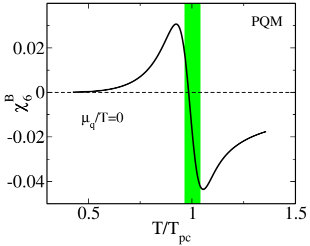
Experimental results from C. Agosta et al J. Low Temp. Phys. 67, 237 (1987). J. Luettmer-Strathmann et al J. Chem. Phys 103, 7482 (1995).

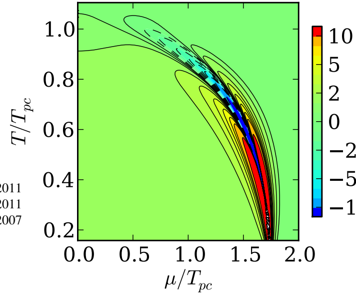
Signatures of phase transition and critical point II

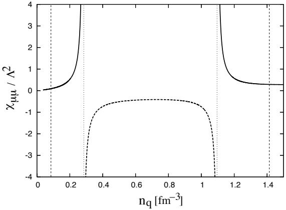
- Static:
  - O(4) crossover: universal sign structure of higher order cumulants, χ_6
  - Divergent cumulants of baryon number fluctuations at CP

$$\chi_n \propto \xi^{3\left(\frac{n\beta\gamma}{2-\alpha}-1\right)}$$

  - In vicinity of CP: universal sign structure of cumulants.
  - Divergent baryon number susceptibility at off-equilibrium first order phase transition



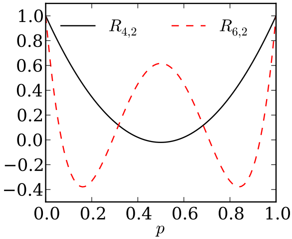




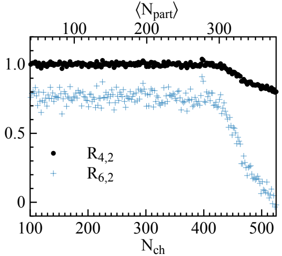
B. Friman, F. Karsch, K. Redlich, V. S. 2011
M. Stephanov 2011
C. Sasaki, K. Redlich, B. Friman 2007

Listed predictions for an infinite static medium.
In HIC:

- Finite lifetime (S. Mukherjee, R. Venugopalan, Y. Yin 2015/2016)
- Conservation laws (A. Bzdak, V. Koch, V.S. 2011)
- Fluctuations not related to interesting physics (V.S., B. Friman, K. Redlich 2012)

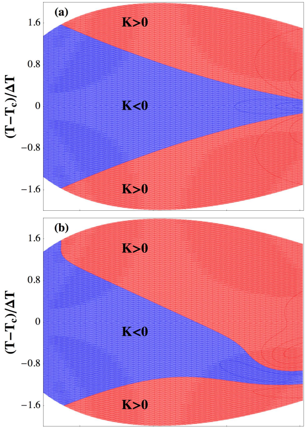


p is a fraction of measured baryons



$\mu = 0$ $R_{n,m} = \chi_n / \chi_m$

Modification due to finite life-time:



S. Mukherjee, R. Venugopalan, Y. Yin 2015