Theory of Light-Meson Decays

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Hadronic and radiative decays of light mesons

Chiral perturbation theory... and its limitations

- $\eta \to 3\pi$: quark masses and Dalitz plot

Dispersion relations and final-state interactions

- pion form factor(s) and $\eta^{(\prime)} \to \pi^+\pi^-\gamma$
- meson transition form factors: $\eta^{(\prime)} \to e^+e^-\gamma$

Dispersion relations for three-body decays

- $\eta' \to \eta\pi\pi$
- $\omega/\phi \to 3\pi$

Summary / Outlook

Light mesons without modeling

Chiral perturbation theory (ChPT) . . .

- Effective field theory: simultaneous expansion in

  quark masses + small momenta

  - ▷ systematically improvable
  - ▷ well-established link to QCD: all symmetry constraints
  - ▷ interrelates many different observables

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. . . and its limitations

- strong final-state interactions render corrections large
- physics of light pseudoscalars (π, K, η) only
  - ▷ (energy) range limited by resonances: $\sigma(500)$, $\rho(770)$ . . .
  - ▷ unitarity is only perturbatively fulfilled
  - ▷ not applicable to decays of (e.g.) vector mesons at all

$\longrightarrow$ find effective ways to resum rescattering / restore unitarity
to apply ChPT where it works best!

Quark masses and $\eta \to 3\pi$ decays

- $\eta \to 3\pi$ isospin violating; two sources in the Standard Model:

$$m_u \neq m_d \qquad\qquad e^2 \neq 0$$

- electromagnetic contribution small

Sutherland 1967

Baur, Kambor, Wyler 1996; Ditsche, BK, Meißner 2009

$$\eta \to \pi^+\pi^-\pi^0 : \quad \mathcal{A}_c^{\mathsf{LO}}(s,t,u) = \frac{B(m_u - m_d)}{3\sqrt{3}F_\pi^2}\left\{1 + \frac{3(s-s_0)}{M_\eta^2 - M_\pi^2}\right\}$$

$$s = (p_{\pi^+} + p_{\pi^-})^2 \ , \ 3s_0 \doteq M_\eta^2 + 3M_\pi^2$$

- $\Delta I = 1$ relation between charged and neutral decay amplitudes:

$$\eta \to 3\pi^0 : \quad \mathcal{A}_n(s,t,u) \ = \ \mathcal{A}_c(s,t,u) + \mathcal{A}_c(t,u,s) + \mathcal{A}_c(u,s,t)$$

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- relevance: (potentially) clean access to $m_u - m_d$

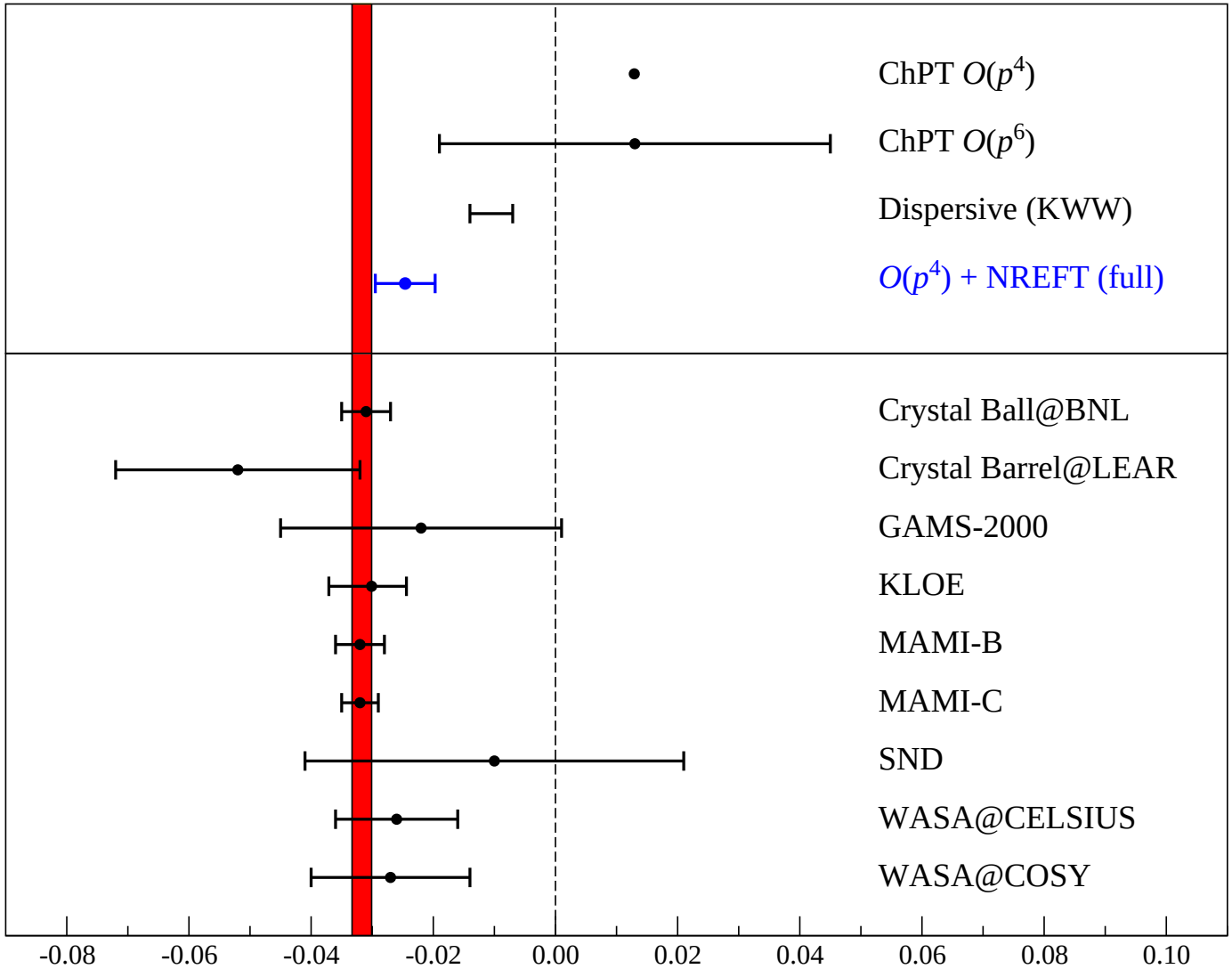
  but: large higher-order / final-state interactions

$\longrightarrow$ require good theoretical Dalitz-plot description to extract normalisation

Kampf et al. 2011, P. Guo et al. 2015, 2016, Colangelo et al. 2016...

$\eta \to 3\pi^0$ Dalitz plot parameter α

$$|\mathcal{A}_n(x,y)|^2 \;=\; |\mathcal{N}_n|^2\{1 + 2\,\alpha\, z + \ldots\} \qquad z \propto (s-s_0)^2 + (t-s_0)^2 + (u-s_0)^2$$



$10^3 \times \alpha$
$+13$
$+13 \pm 32$
$-7 \ldots -14$
-25 ± 5

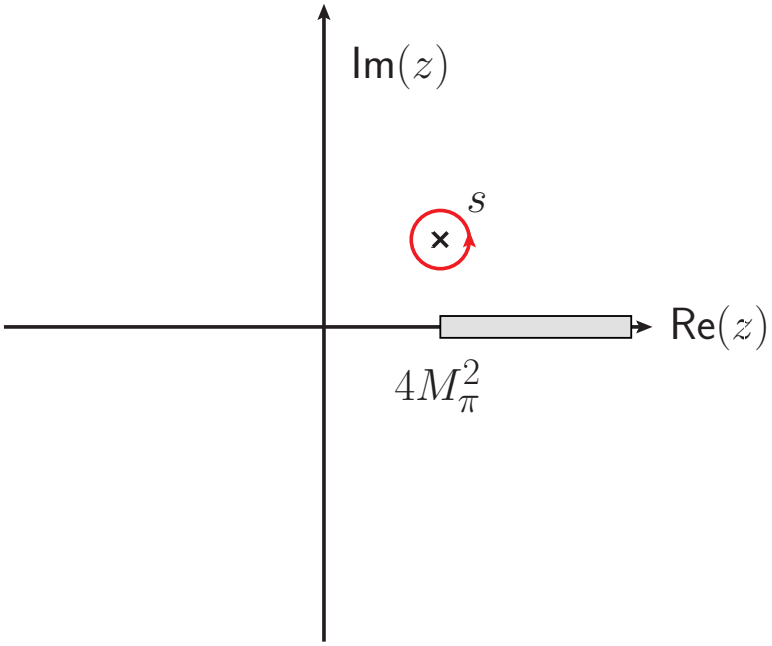
(resums rescatt. to 2 loops)

Schneider, BK, Ditsche 2011

PDG average:

-31.7 ± 1.6

Dispersion relations for pedestrians



analyticity ($\simeq$ causality)

& Cauchy's theorem:

$$T(s) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{T(z)dz}{z-s}$$

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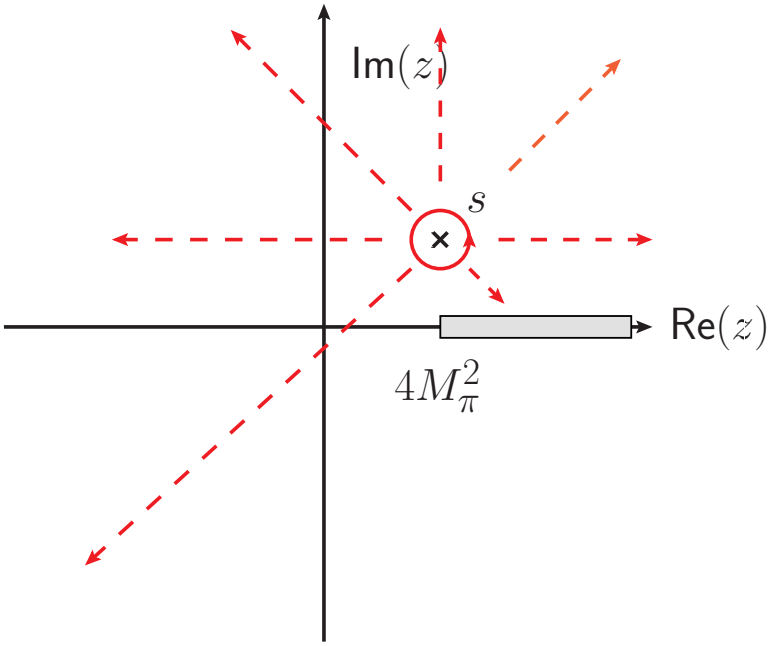

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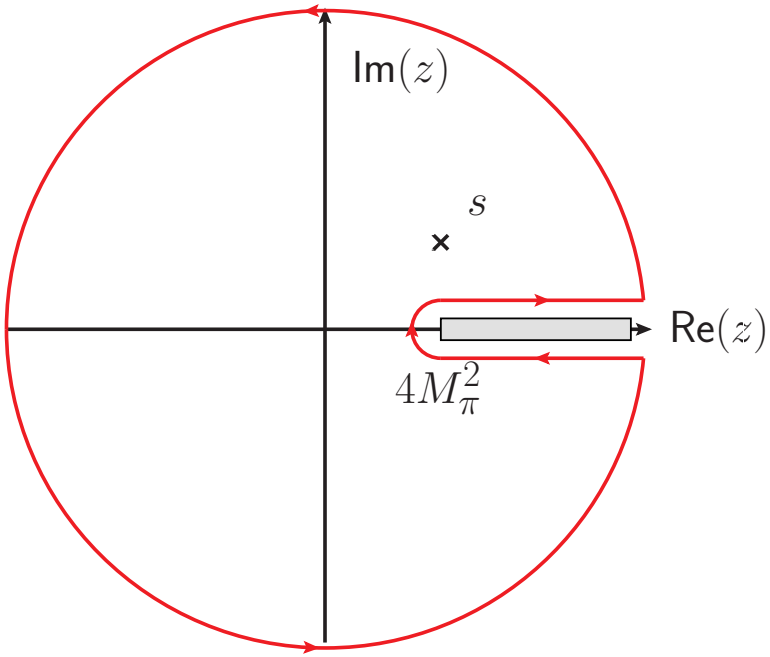

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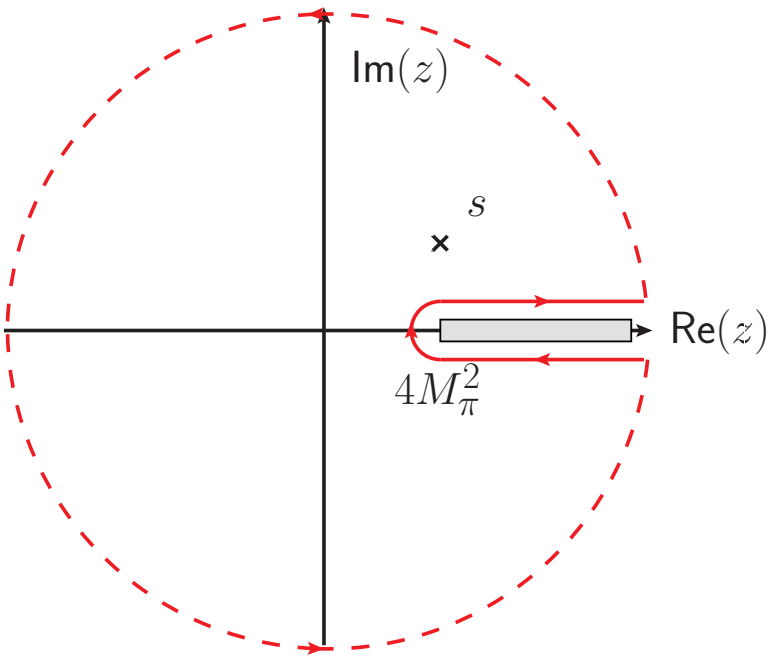

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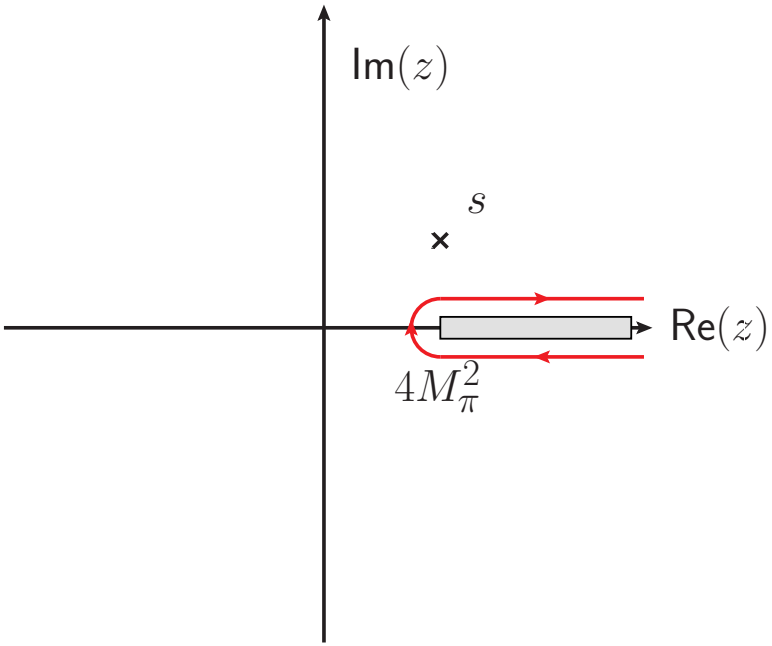

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$$T(s) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{T(z)dz}{z-s}$$

$$\longrightarrow \frac{1}{2\pi i} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{disc}\, T(z)dz}{z-s}$$

$$= \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{Im}\, T(z)dz}{z-s}$$

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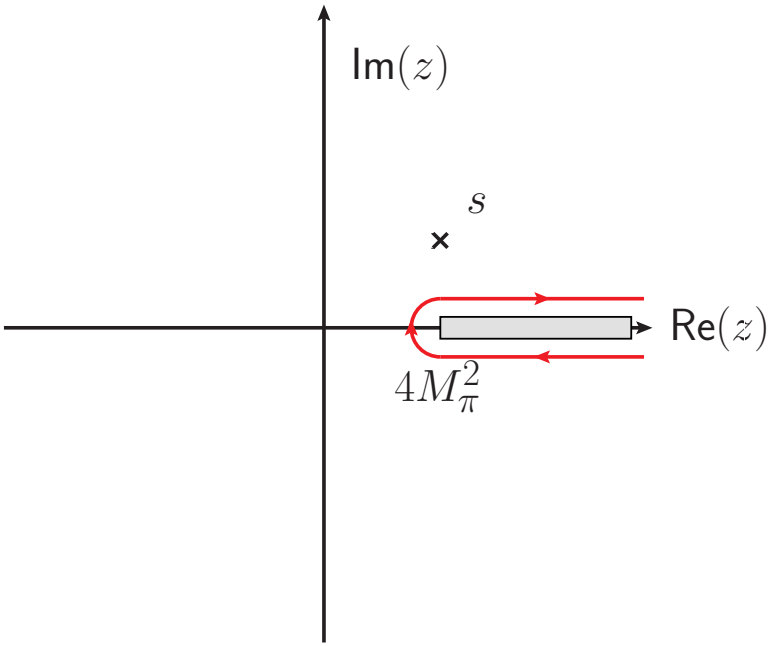

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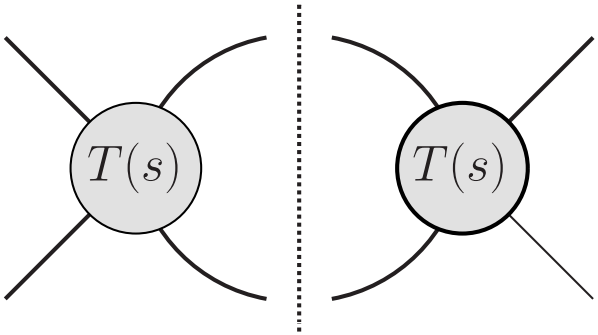
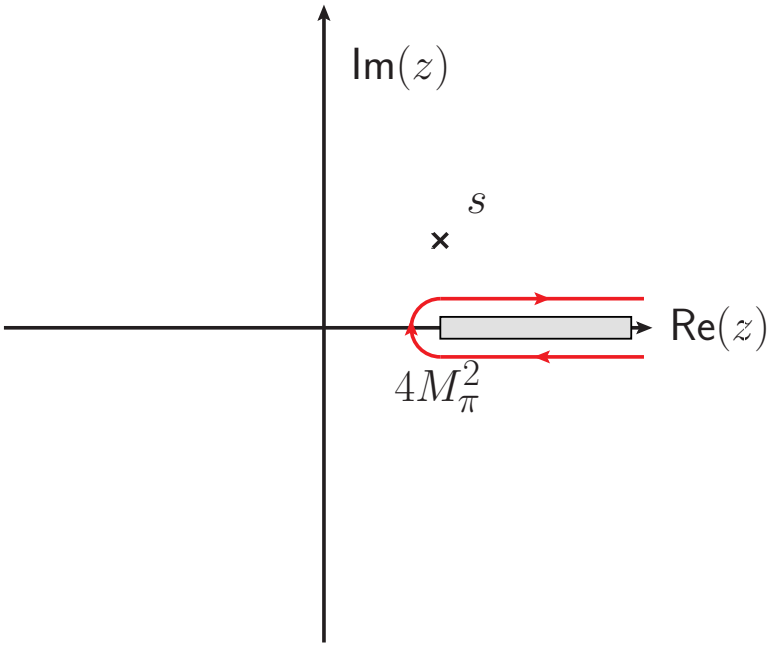


e.g. if $T(s)$ is a $\pi\pi$ partial wave $\longrightarrow$

$$\frac{\operatorname{disc} T(s)}{2i} = \operatorname{Im} T(s) = \frac{2q_\pi}{\sqrt{s}} \theta(s - 4M_\pi^2) |T(s)|^2$$

Dispersion relations for pedestrians



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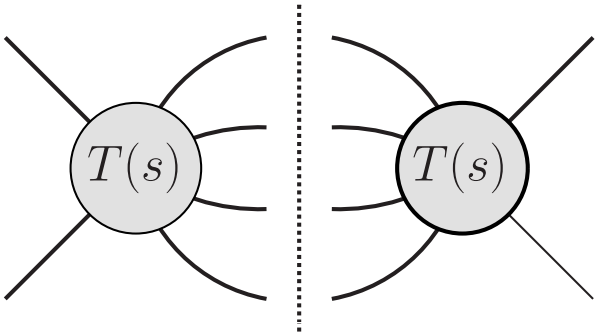
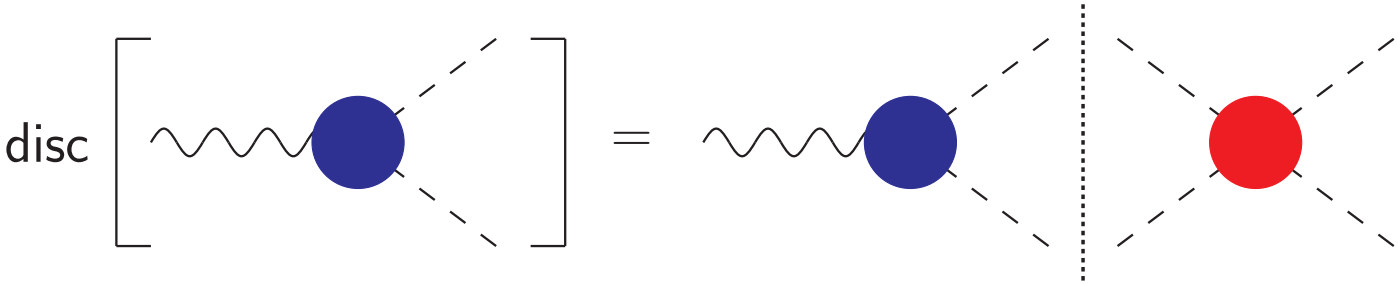


inelastic intermediate states ($K\bar{K}$, 4π)
suppressed at low energies
$\longrightarrow$ will be neglected in the following

Dispersion relation for the pion form factor

- final-state interactions of two particles: form factor



$$\frac{1}{2i}\,\text{disc}\,F_I(s) \;=\; \text{Im}\,F_I(s) \;=\; F_I(s)\times\theta(s-4\,M_\pi^2)\times\sin\delta_I(s)\,e^{-i\delta_I(s)}$$

$\longrightarrow$ final-state theorem: phase of $F_I(s)$ is just $\delta_I(s)$ Watson 1954

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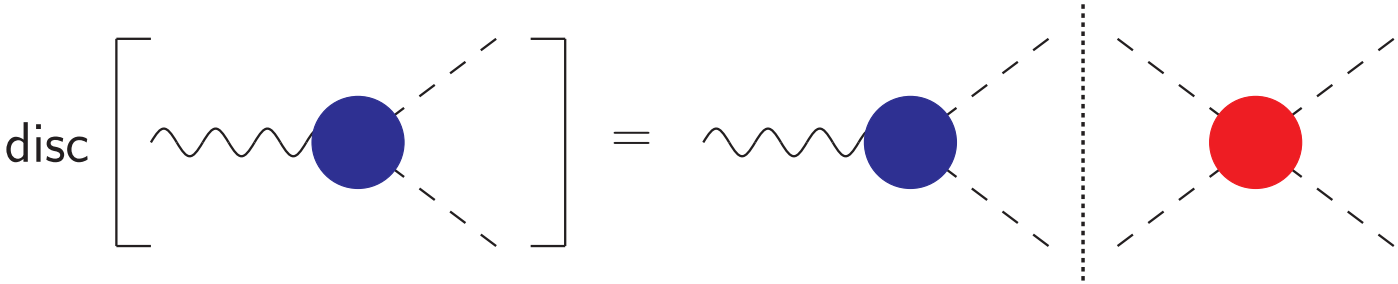

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- solution to this homogeneous integral equation known:

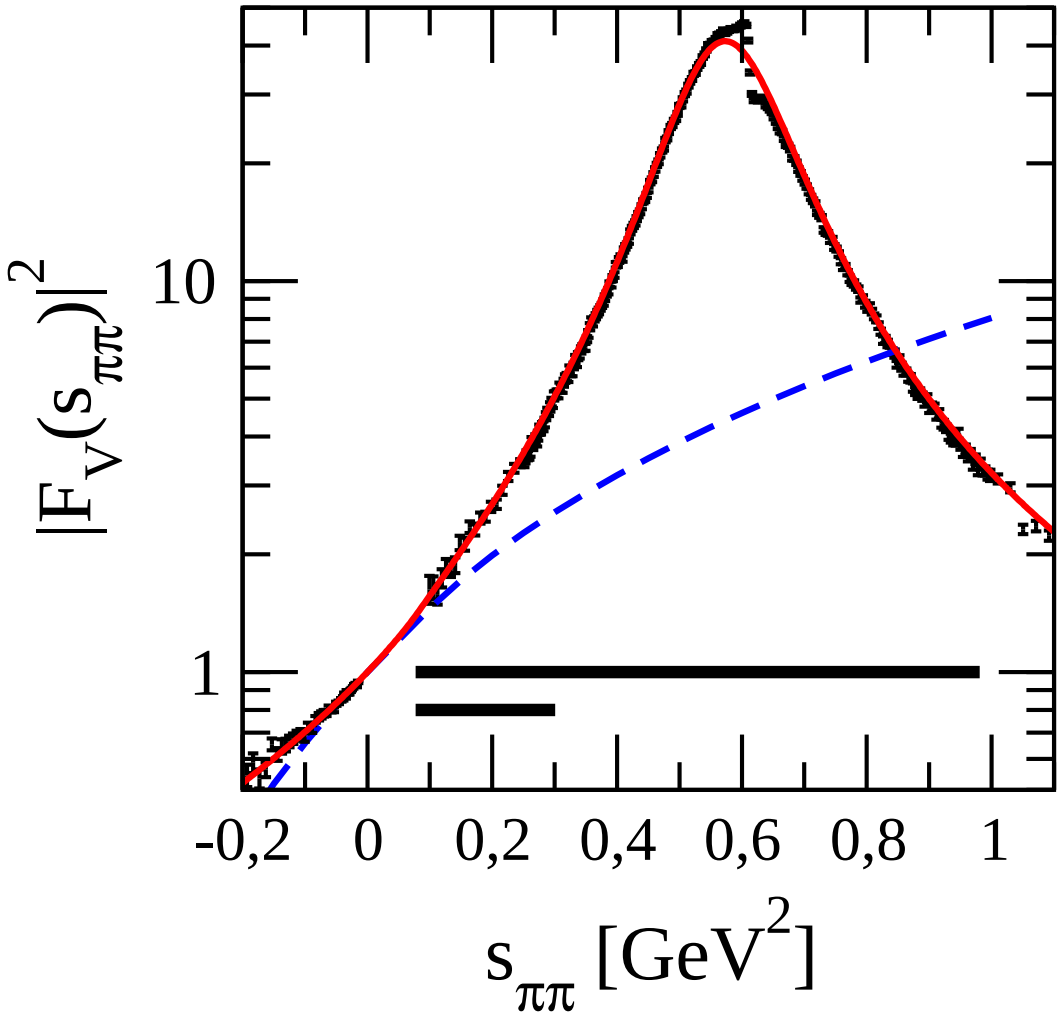
$$F_I(s) = P_I(s)\Omega_I(s)\;, \quad \Omega_I(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds'\,\frac{\delta_I(s')}{s'(s'-s)}\right\}$$

$P_I(s)$ polynomial, $\Omega_I(s)$ Omnès function Omnès 1958

- constrain polynomial using symmetries / chiral perturbation theory (normalisation/derivatives at $s=0$)

Pion vector form factor

- pion vector form factor clearly non-perturbative: ρ resonance



ChPT at one loop

data on $e^+e^- \to \pi^+\pi^-$

Omnès representation

Stollenwerk et al. 2012

$\longrightarrow$ Omnès representation vastly extends range of applicability

Final-state universality: $\eta,\ \eta' \to \pi^+\pi^-\gamma$

- $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ driven by the chiral anomaly, $\pi^+\pi^-$ in P-wave
  $\longrightarrow$ final-state interactions the same as for vector form factor
- ansatz: $\mathcal{A}^{\eta^{(\prime)}}_{\pi\pi\gamma} = A \times P(t) \times F^V_\pi(t)$, $P(t) = 1 + \alpha^{(\prime)} t$

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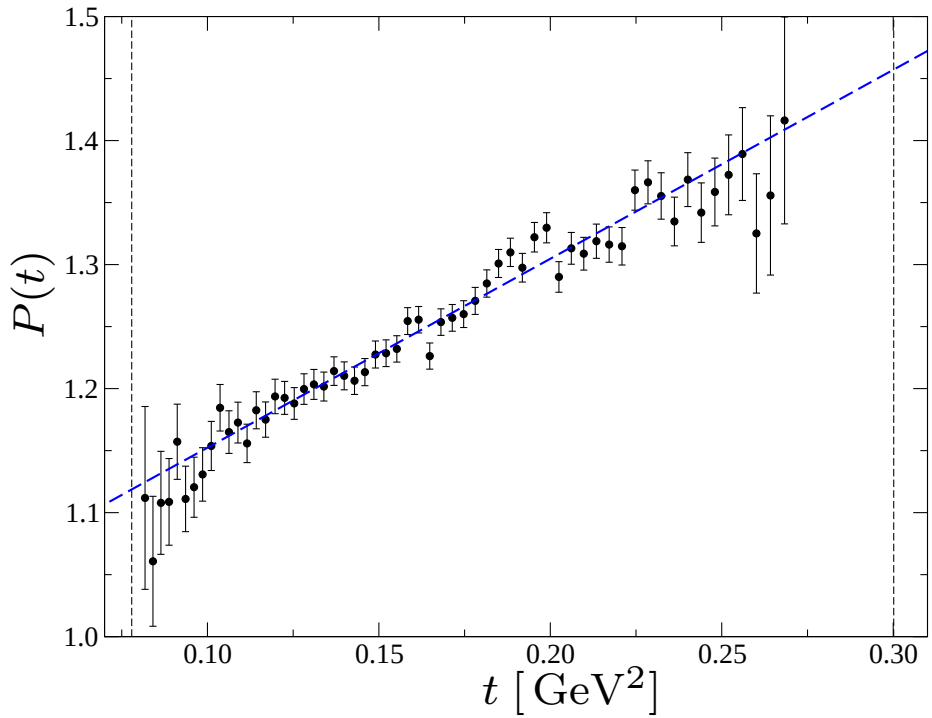
- divide data by Omnès function $\longrightarrow P(t)$

Stollenwerk et al. 2012

BK, Plenter 2015

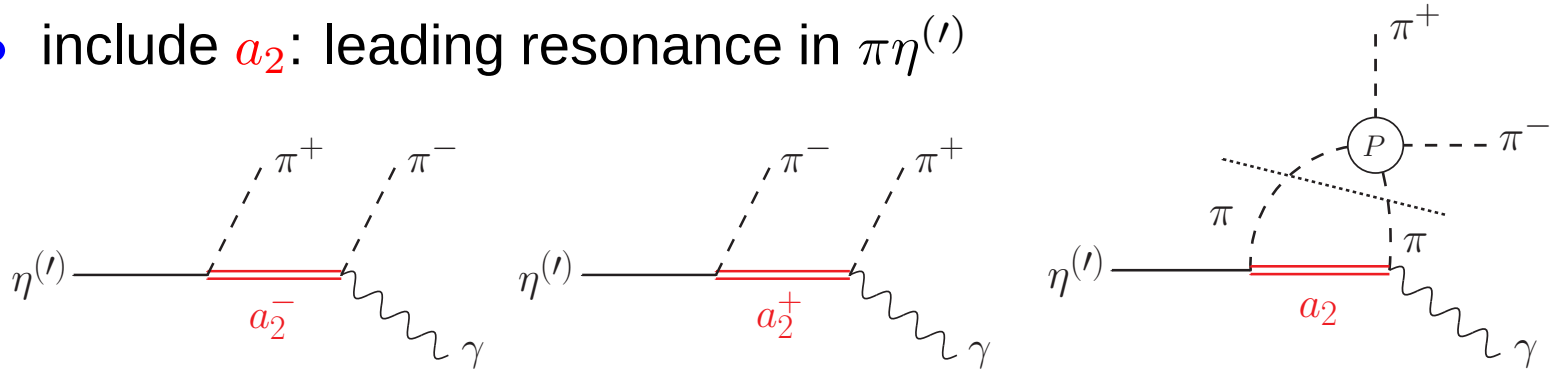


$\longrightarrow$ exp.: $\alpha_{\text{WASA}} = (1.89 \pm 0.64)\,\text{GeV}^{-2}$, $\alpha_{\text{KLOE}} = (1.31 \pm 0.08)\,\text{GeV}^{-2}$

$\longrightarrow$ interpret α by matching to chiral perturbation theory

$\eta,\ \eta' \to \pi^+\pi^-\gamma$ with left-hand cuts

- include a_2: leading resonance in $\pi\eta^{(\prime)}$



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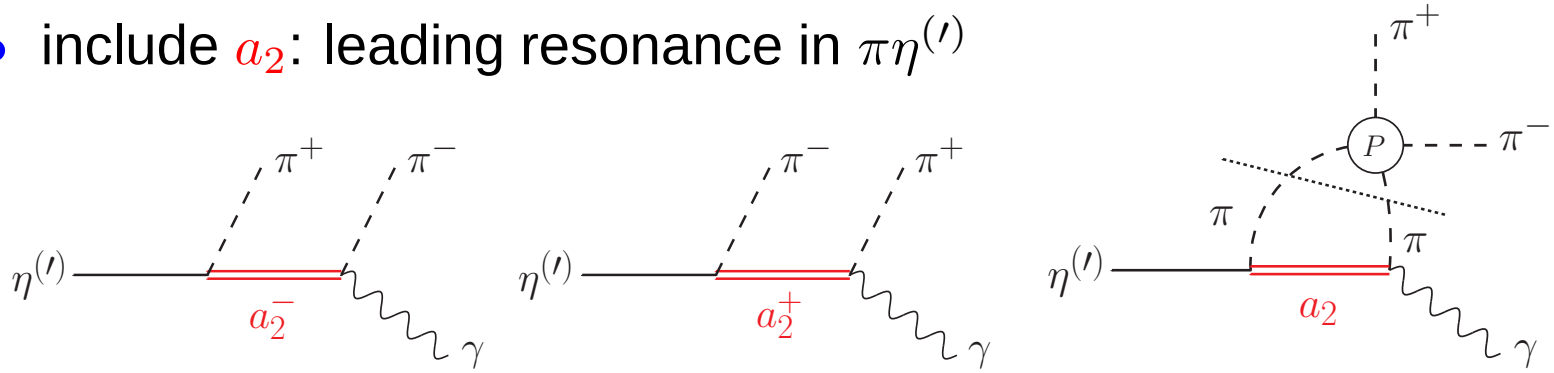

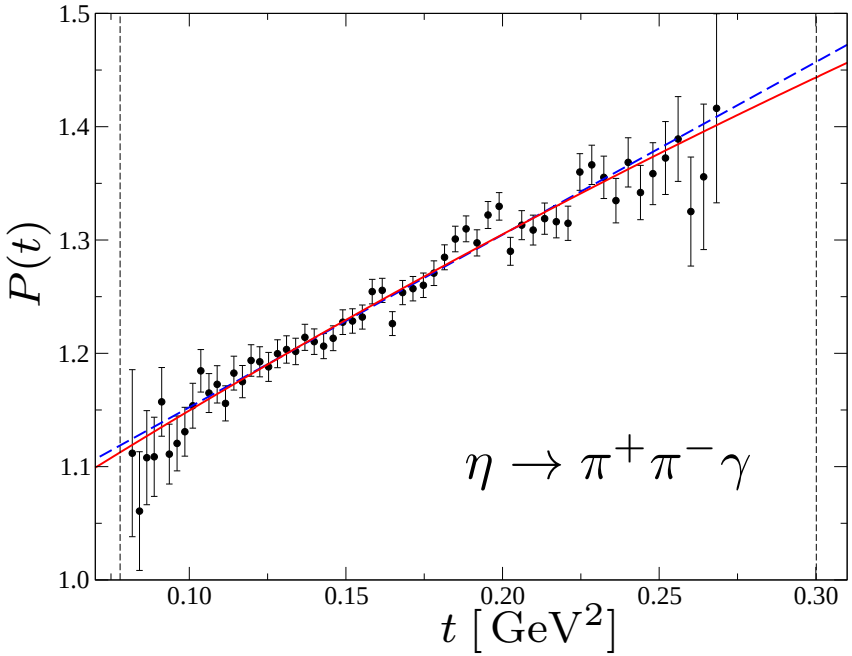

KLOE 2013; BK, Plenter 2015

- induces curvature in $P(t)$

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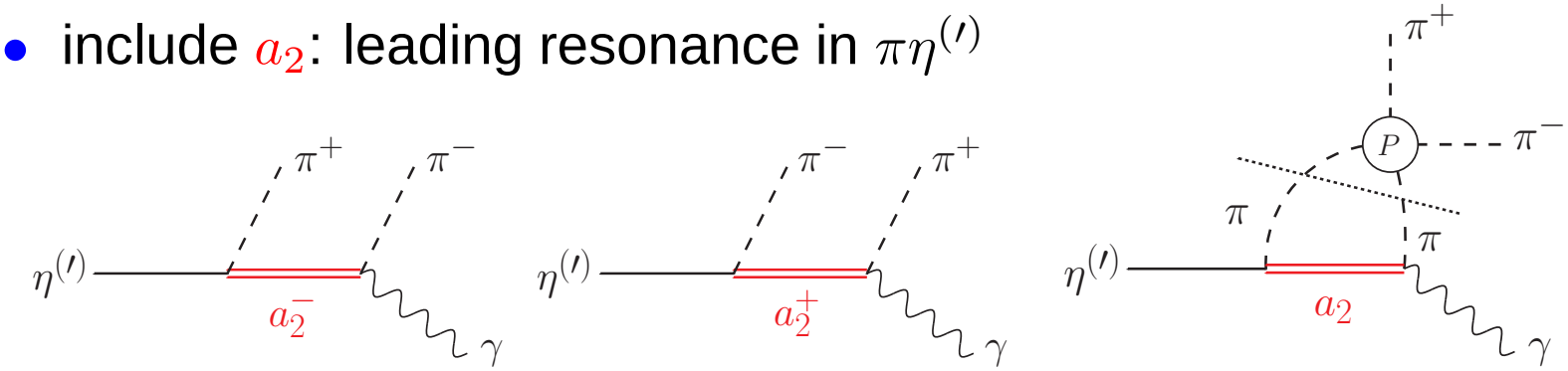

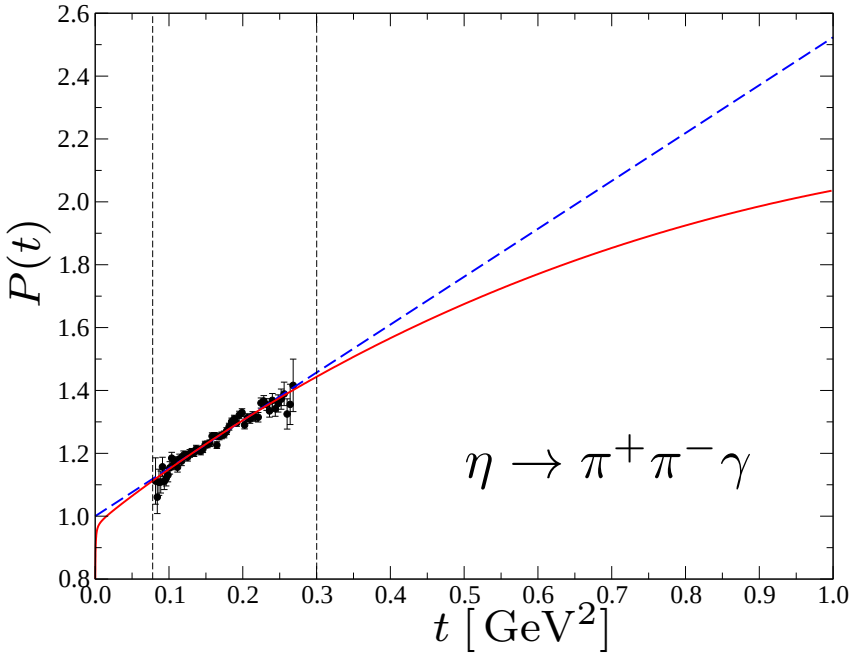

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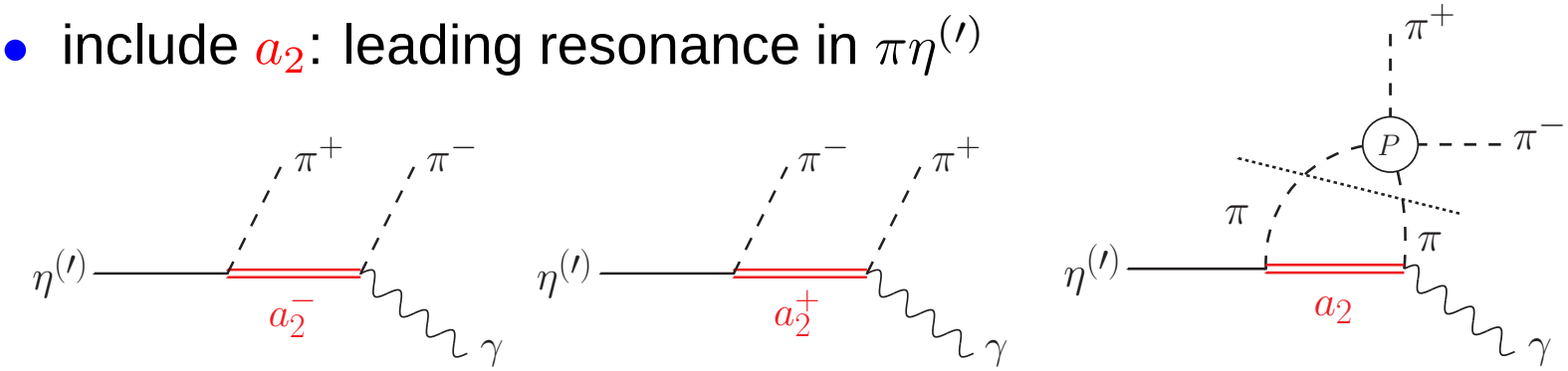

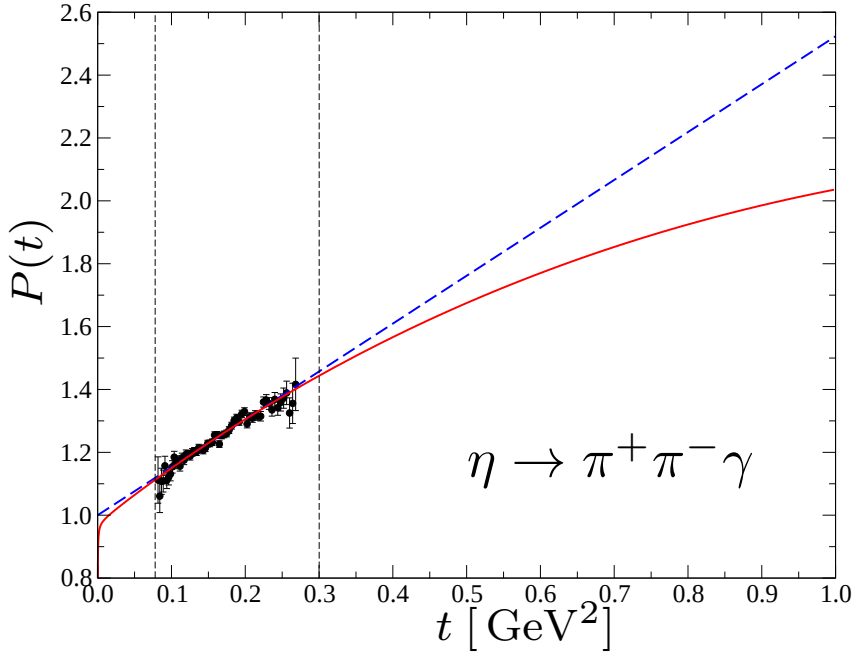

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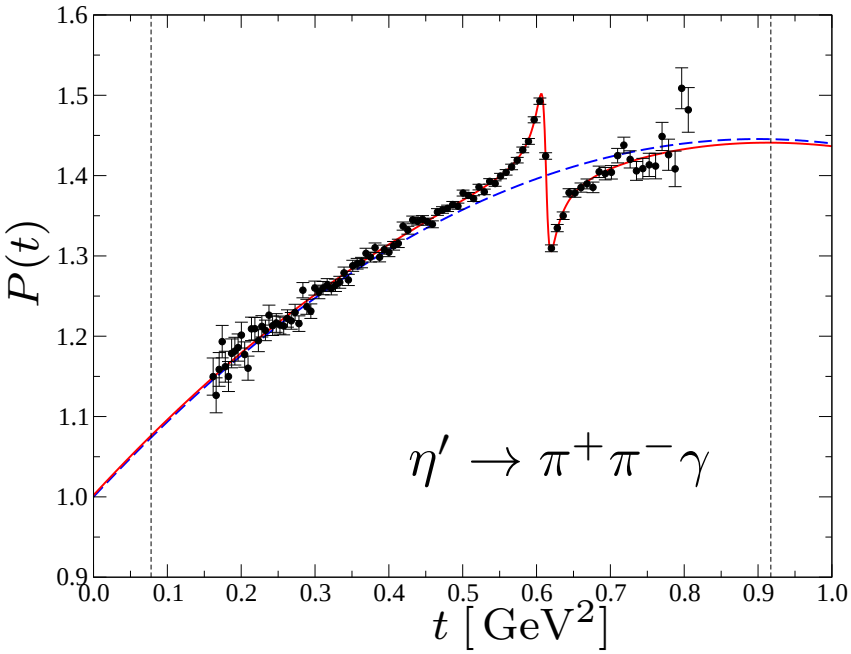
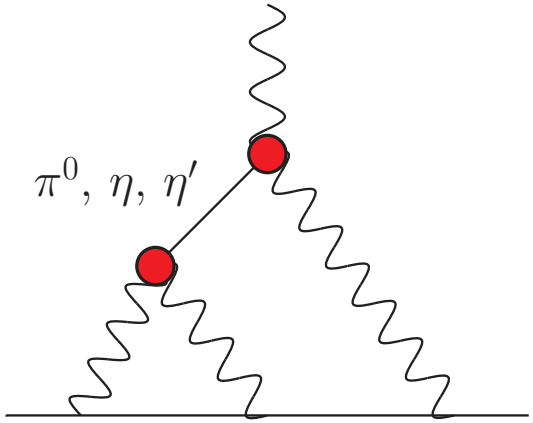


BESIII prel.; Hanhart et al. 2017

- induces curvature in $P(t)$
- curvature, plus ρ–ω mixing

Pseudoscalar transition form factors and $(g-2)_\mu$

- largest individual contribution
  to hadronic light-by-light scattering:
  pseudoscalar pole terms
  singly / doubly virtual form factors
  $F_{P\gamma\gamma^*}(q^2,0)$ and $F_{P\gamma^*\gamma^*}(q_1^2,q_2^2)$



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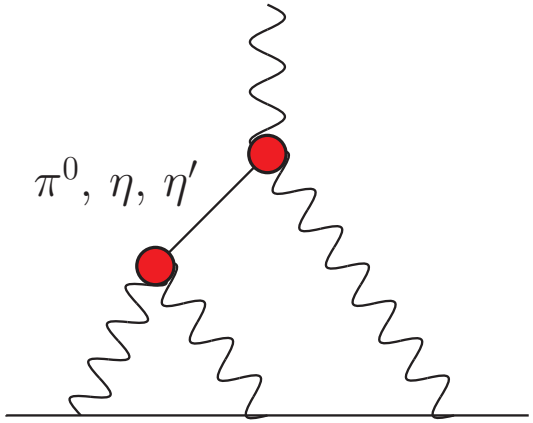

- normalisation fixed by Wess–Zumino–Witten anomaly, e.g.:

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F_π: pion decay constant $\longrightarrow$ measured at 1.5% level PrimEx 2011

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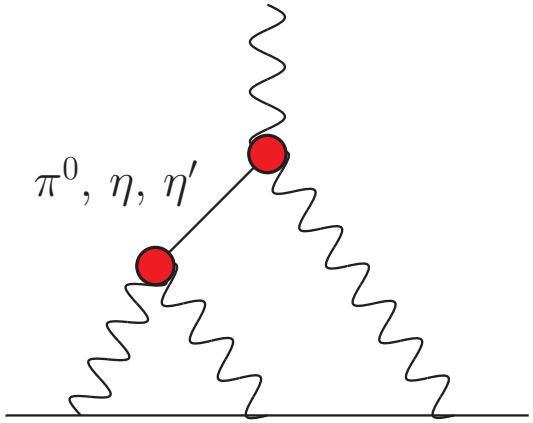

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- q_i^2-dependence: often modelled by vector-meson dominance
  $\longrightarrow$ what can we learn from analyticity and unitarity constraints?
  $\longrightarrow$ what experimental input sharpens these constraints?

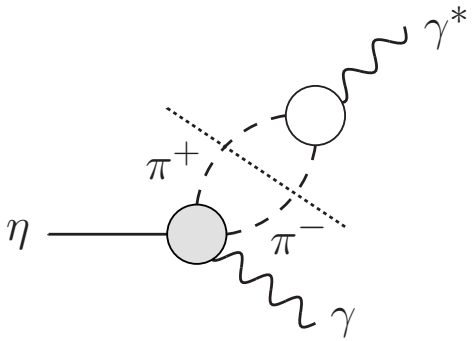
Transition form factor $\eta \to \gamma^*\gamma$

- another disp. rel.: $\eta \to \pi^+\pi^-\gamma$ + $e^+e^- \to \pi^+\pi^- \longrightarrow \eta \to e^+e^-\gamma$

Hanhart et al. 2013

$$\bar{F}_{\eta\gamma^*\gamma}(q^2,0) = 1 + \frac{\kappa_\eta q^2}{96\pi^2 F_\pi^2}\int_{4M_\pi^2}^{\infty} ds\sigma(s)^3 P(s)\frac{|F_\pi^V(s)|^2}{s-q^2}$$

$$+ \Delta F^{I=0}_{\eta\gamma^*\gamma}(q^2,0) \; [\longrightarrow \text{VMD}]$$



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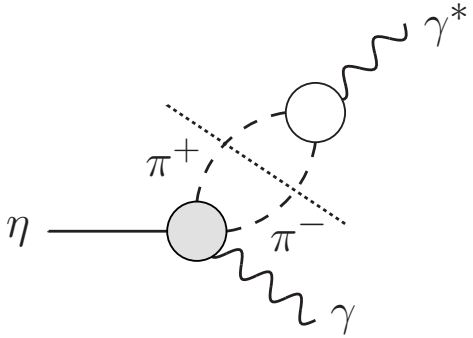
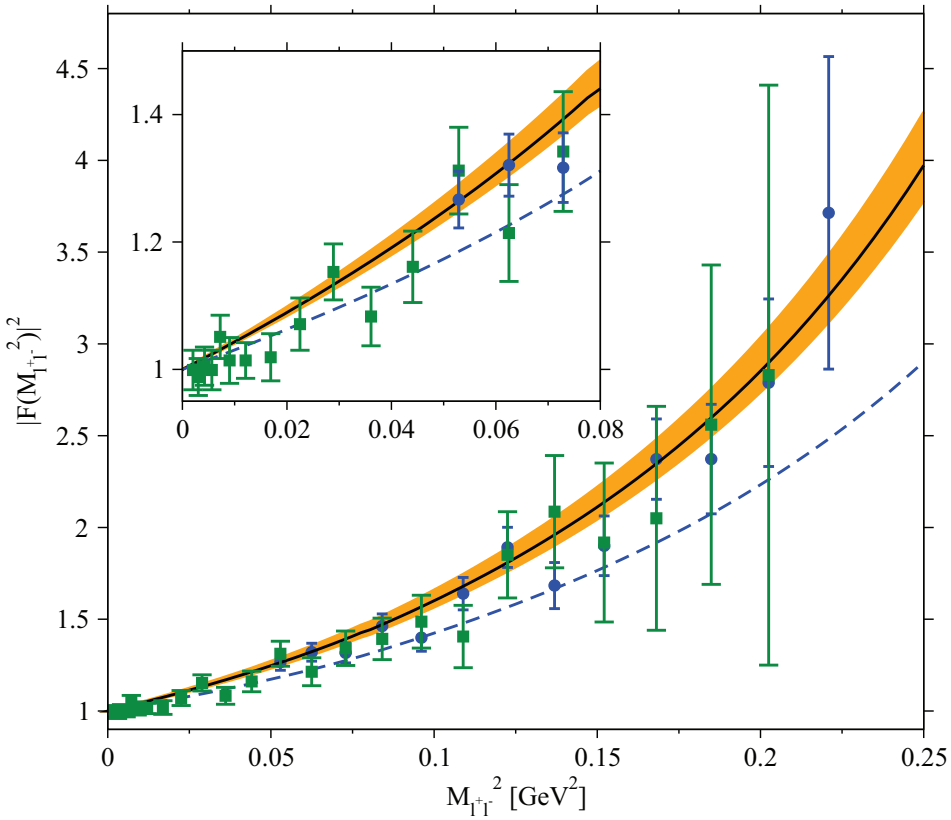




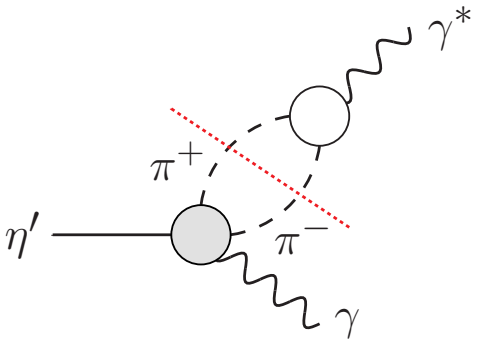
$\longrightarrow$ huge statistical advantage of using hadronic input for $\eta \to \pi^+\pi^-\gamma$ over direct measurement of $\eta \to e^+e^-\gamma$ (rate suppressed by α^2_{QED})

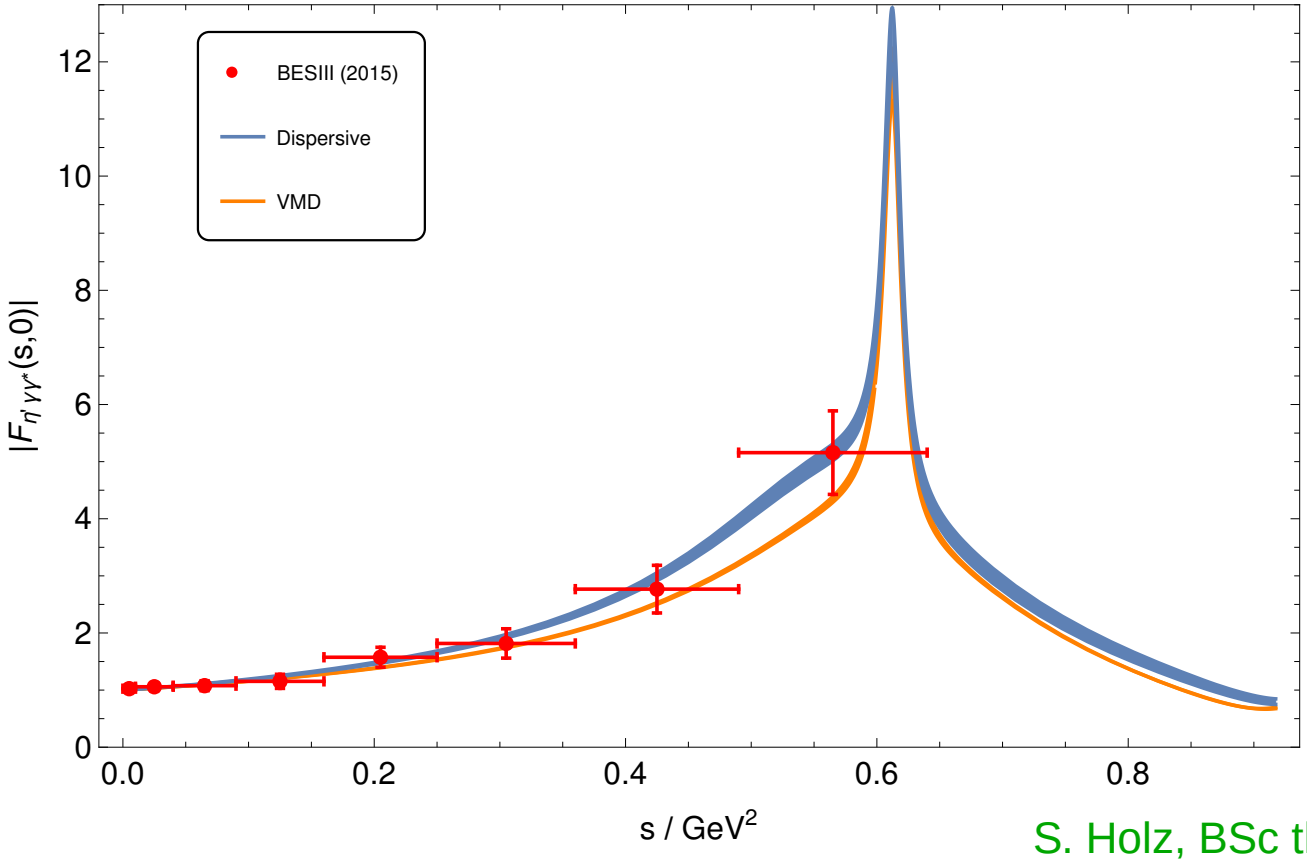
figure courtesy of C. Hanhart

data: NA60 2011, A2 2014

Prediction for η' transition form factor

- isovector: combine high-precision data on $\eta' \to \pi^+\pi^-\gamma$ and $e^+e^- \to \pi^+\pi^-$
- isoscalar: VMD, couplings fixed from $\eta' \to \omega\gamma$ and $\phi \to \eta'\gamma$





S. Holz, BSc thesis 2016

Dispersion relations for three-body decays: $\eta' \to \eta\pi\pi$

- solve Khuri–Treiman equations Isken, BK, Schneider, Stoffer 2017

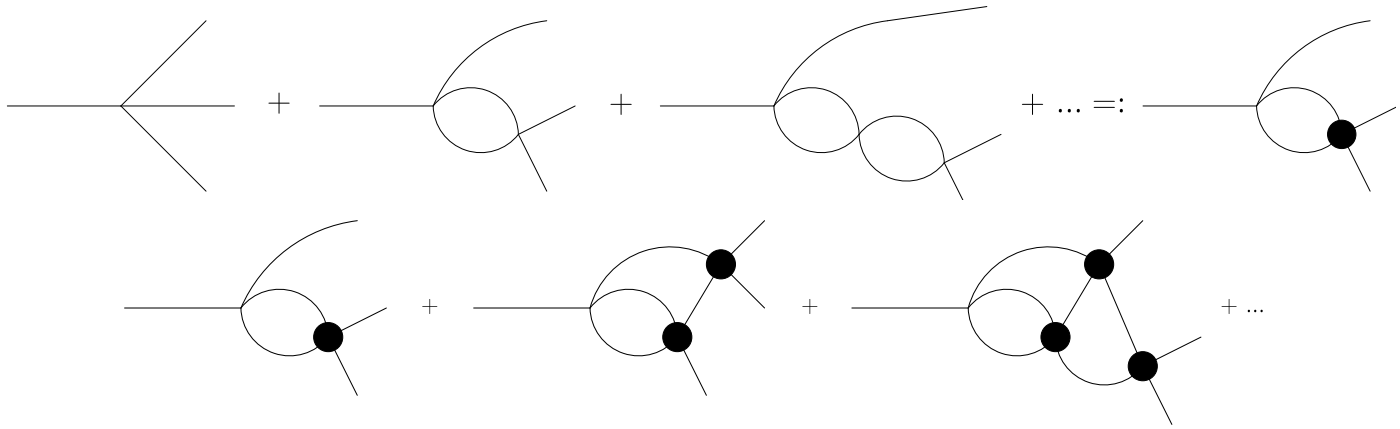
  input: S-wave phase shifts $\delta_0 \equiv \delta_{\pi\pi}$ and $\delta_1 \equiv \delta_{\pi\eta}$

$$\mathcal{A}(s,t,u) = \mathcal{A}_0(s) + \mathcal{A}_1(t) + \mathcal{A}_1(u),$$

$$\mathcal{A}_0(s) = \Omega_0(s)\left\{\alpha + \beta s + \frac{s^2}{\pi}\int_{s_{\rm thr}}^{\infty}\frac{dx}{x^2}\frac{\hat{\mathcal{A}}_0(x)\sin\delta_0(x)}{|\Omega_0(x)|(x-s)}\right\}$$

$$\mathcal{A}_1(t) = \Omega_1(t)\left\{\gamma t + \frac{t^2}{\pi}\int_{t_{\rm thr}}^{\infty}\frac{dx}{x^2}\frac{\hat{\mathcal{A}}_1(x)\sin\delta_1(x)}{|\Omega_1(x)|(x-t)}\right\}$$

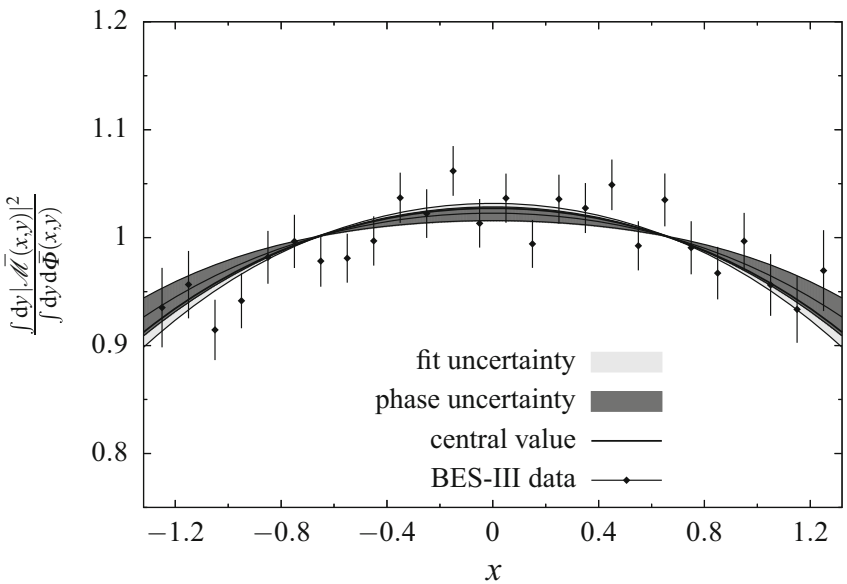
- $\hat{\mathcal{A}}_{0/1}$: partial-wave projections of crossed-channel amplitudes:

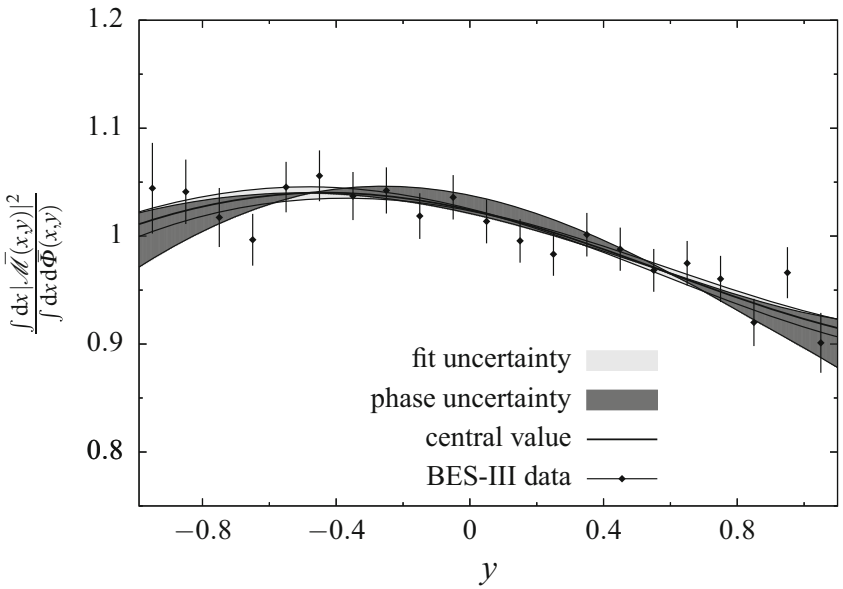


$\longrightarrow$ crossed-channel rescattering fully taken into account

$\eta' \to \eta\pi\pi$ Dalitz plot

- 3 or 4 subtraction constants: Isken, BK, Schneider, Stoffer 2017
  more predictive vs. less dependence on phase shift uncertainty
- one-dimensional projections vs. data: BESIII 2010





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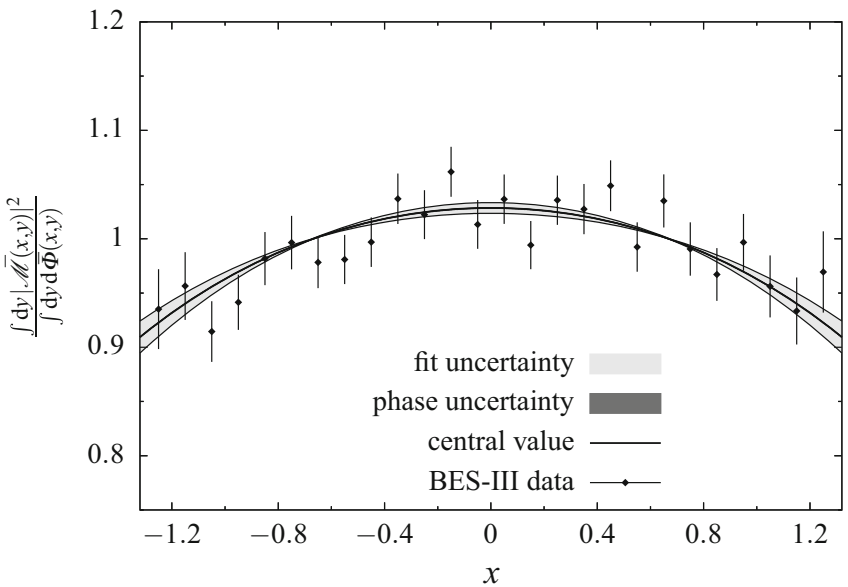

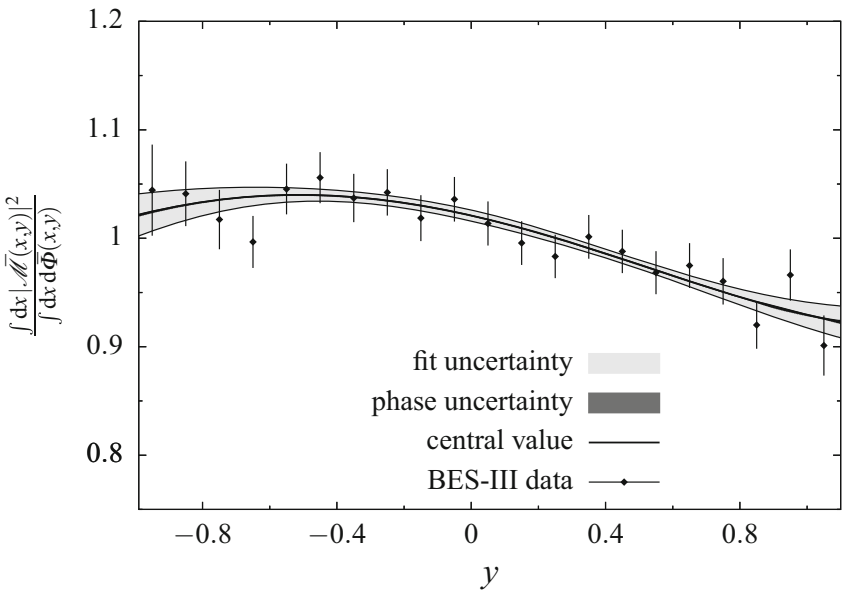

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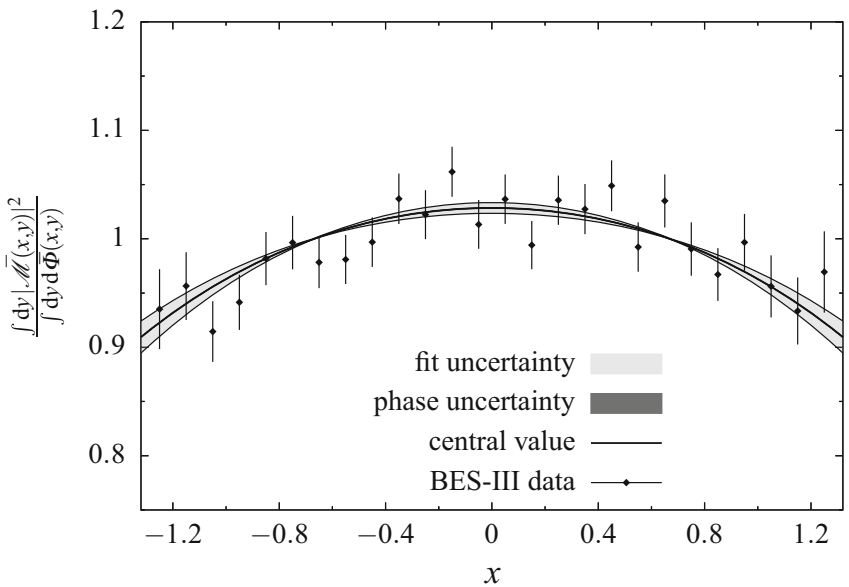

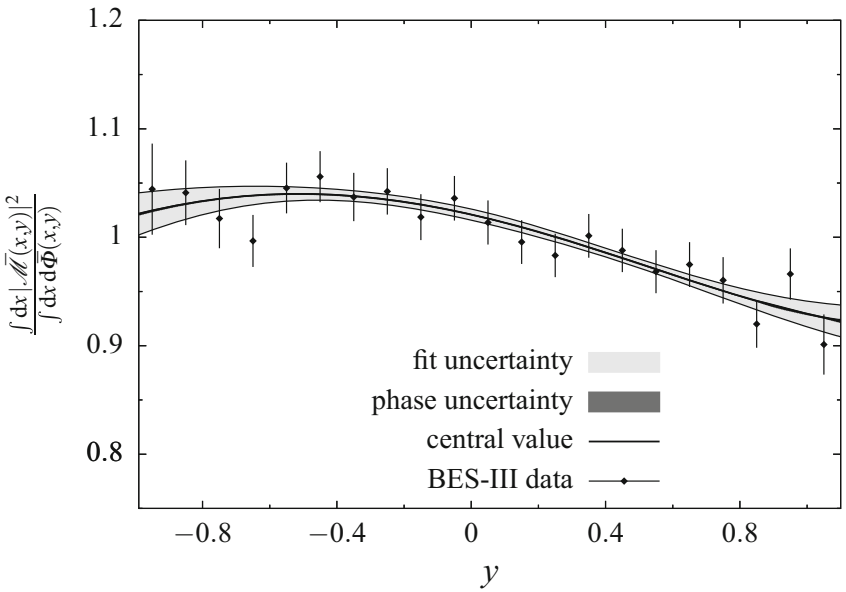

- Dalitz plot parameters well reproduced, higher ones predicted
- analysis tool for new high-prec. Dalitz plots A2@MAMI, BESIII 2017
- ingredient for forthcoming $\eta' \to 3\pi$ analysis BESIII 2016

Dispersion relations for three-body decays: vector mesons

$\omega/\phi \to 3\pi$

- beyond ChPT: copious efforts to develop EFT for vector mesons

  Bijnens et al.; Bruns, Meißner; Lutz, Leupold; Gegelia et al.; Kampf et al.. . .

- vector mesons highly important for (virtual) photon processes

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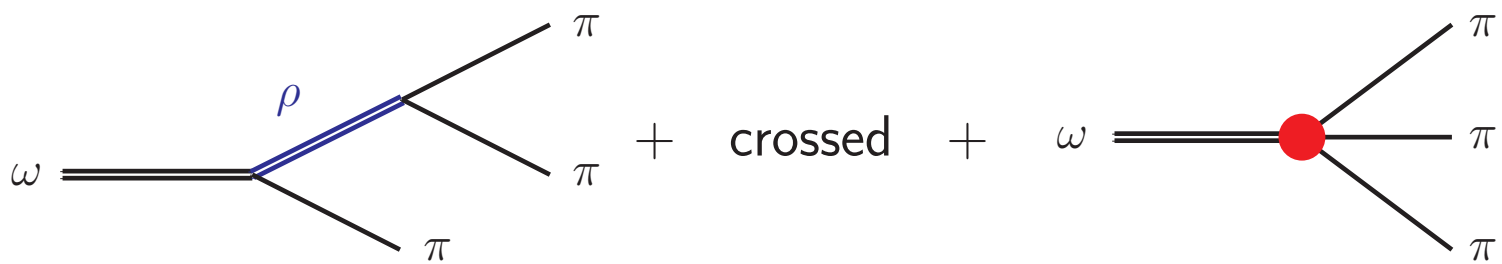
- $\omega/\phi \to 3\pi$ analyzed in terms of KLOE 2003, CMD-2 2006

  sum of 3 Breit–Wigners (ρ^+, ρ^-, ρ^0)

  + constant background term



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  sum of 3 Breit–Wigners (ρ^+, ρ^-, ρ^0)

  + constant background term

Problem:

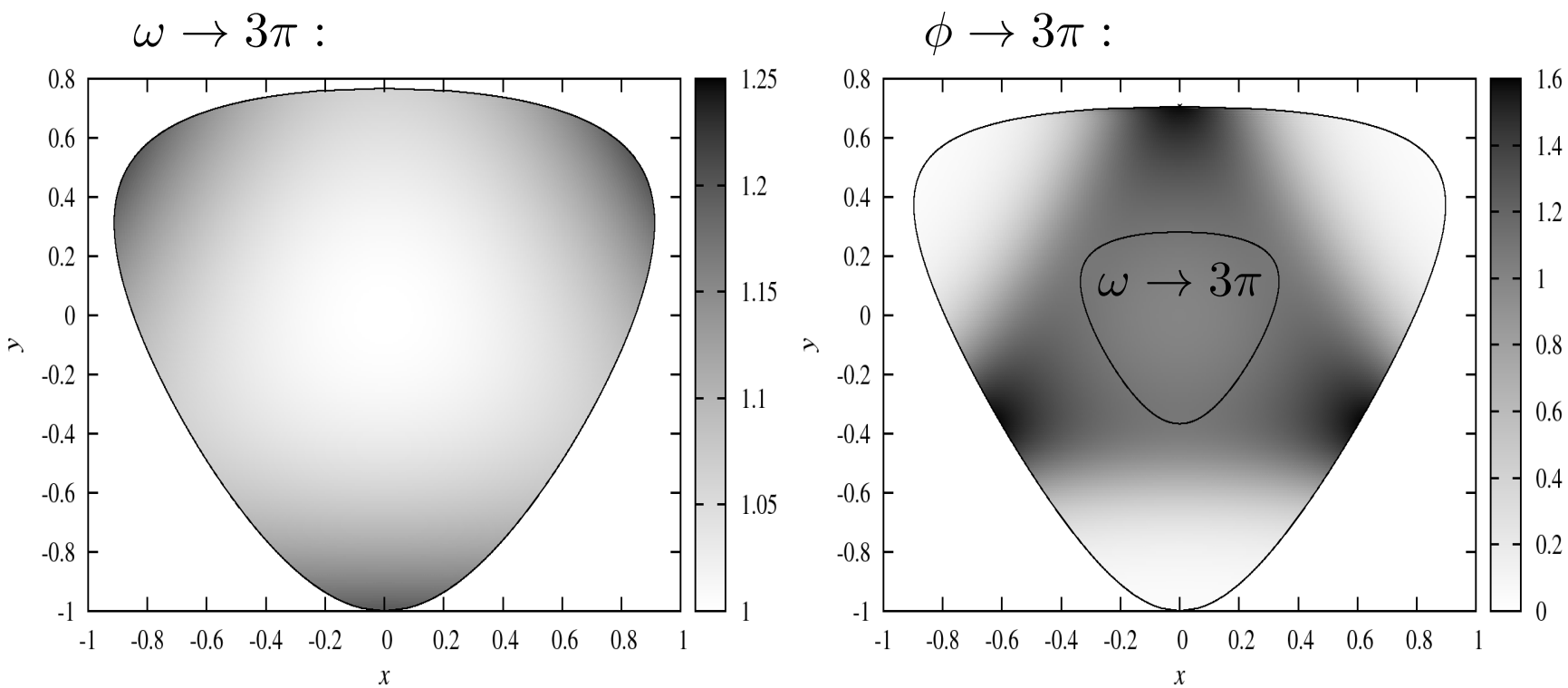
$\longrightarrow$ unitarity fixes Im/Re parts

$\longrightarrow$ adding a contact term destroys this relation

$\longrightarrow$ reconcile data with dispersion relations? Niecknig, BK, Schneider 2012

$\omega/\phi \to 3\pi$ Dalitz plots

- only one subtraction constant $a \longrightarrow$ fix to partial width
- normalised Dalitz plot in $y \propto s - s_0\,, \quad x \propto t - u$:

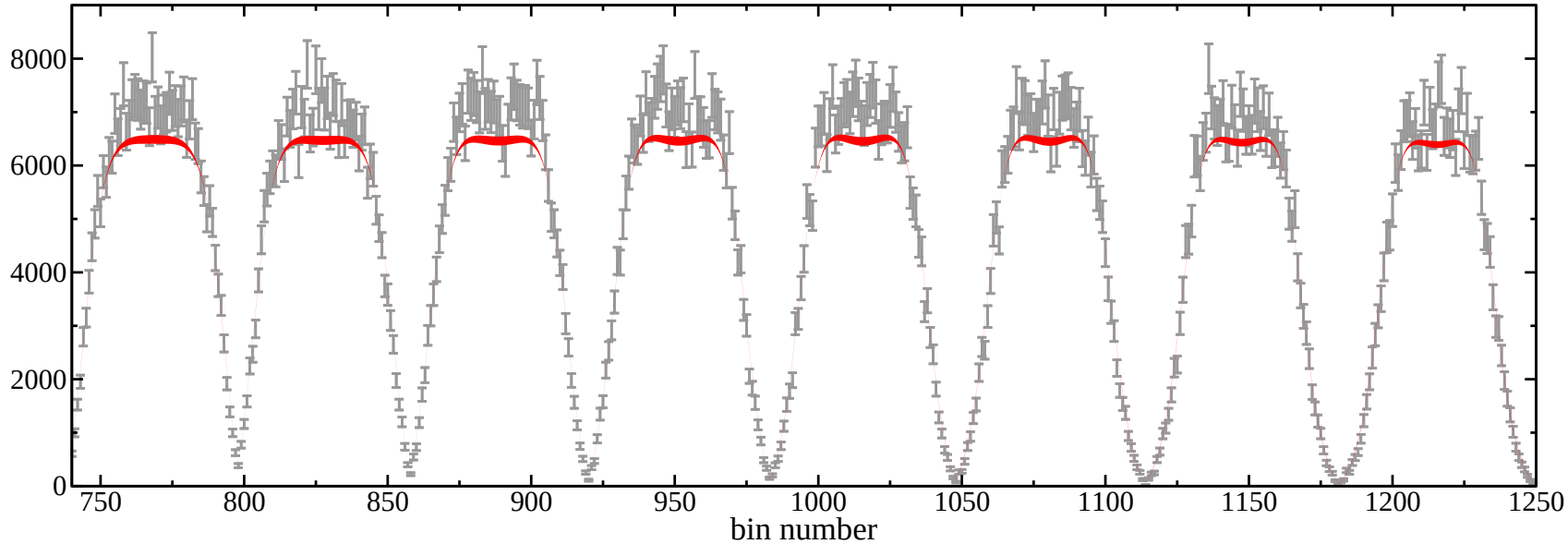


- ω Dalitz plot is relatively smooth
- ϕ Dalitz plot clearly shows ρ resonance bands

Niecknig, BK, Schneider 2012

Experimental comparison to $\phi \to 3\pi$

KLOE Dalitz plot: $2 \cdot 10^6$ events, 1834 bins Niecknig, BK, Schneider 2012



	$\hat{\mathcal{F}} = 0$
χ^2/ndof	$1.71 \ldots 2.06$

$$\mathcal{F}(s) = a\,\Omega(s) = a\,\exp\left[\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\delta_1^1(s')}{s'-s}\right]$$

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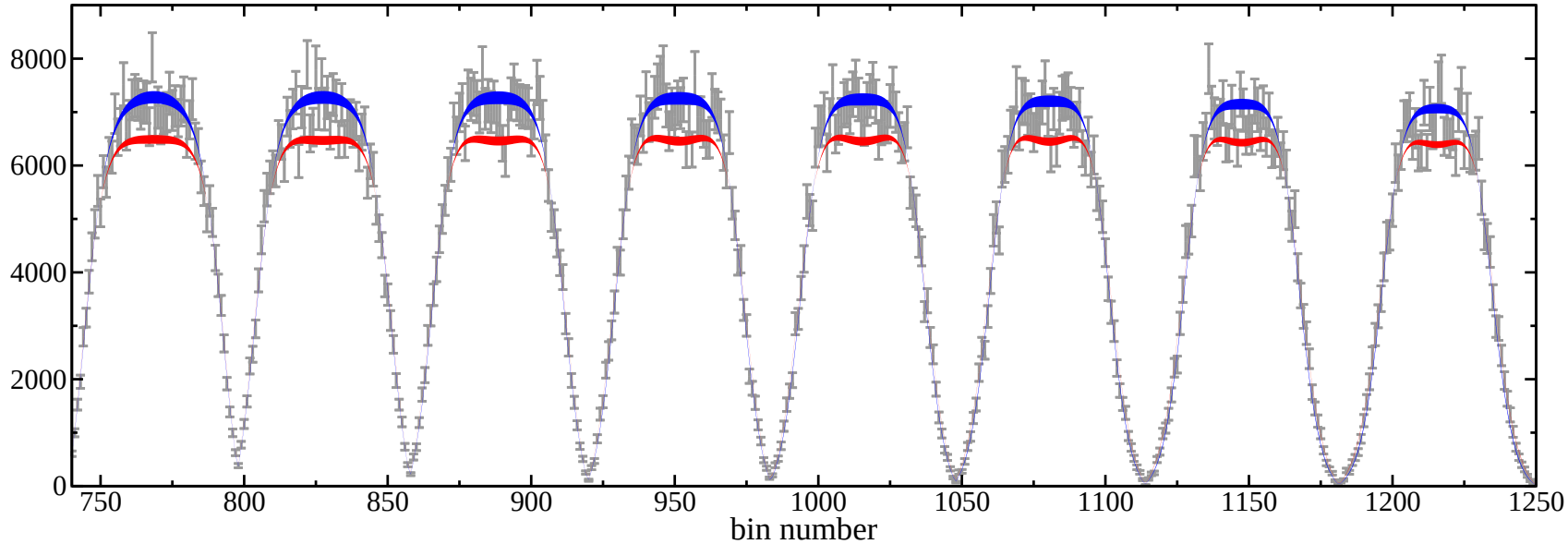

	$\hat{\mathcal{F}} = 0$	once-subtracted
χ^2/ndof	$1.71 \ldots 2.06$	$1.17 \ldots 1.50$

$$\mathcal{F}(s) = a\,\Omega(s)\left[1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\hat{\mathcal{F}}(s')\sin\delta_1^1(s')}{|\Omega(s')|(s'-s-i\epsilon)}\right]$$

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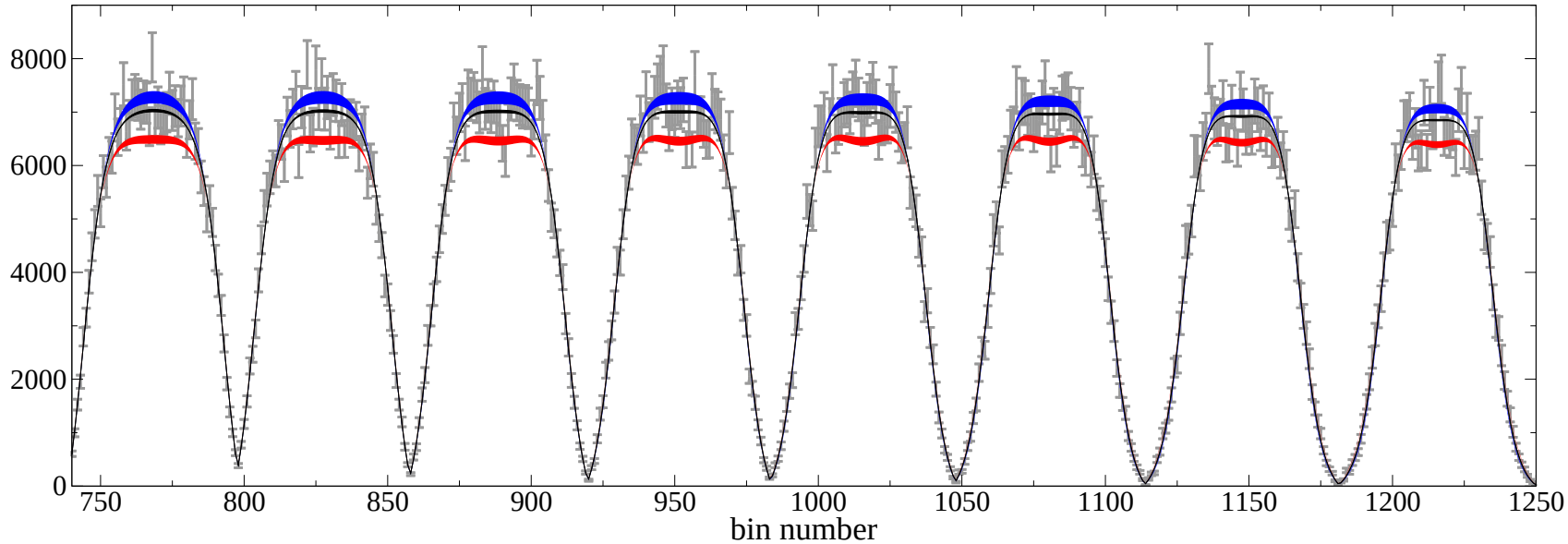

	$\hat{\mathcal{F}} = 0$	once-subtracted	twice-subtracted
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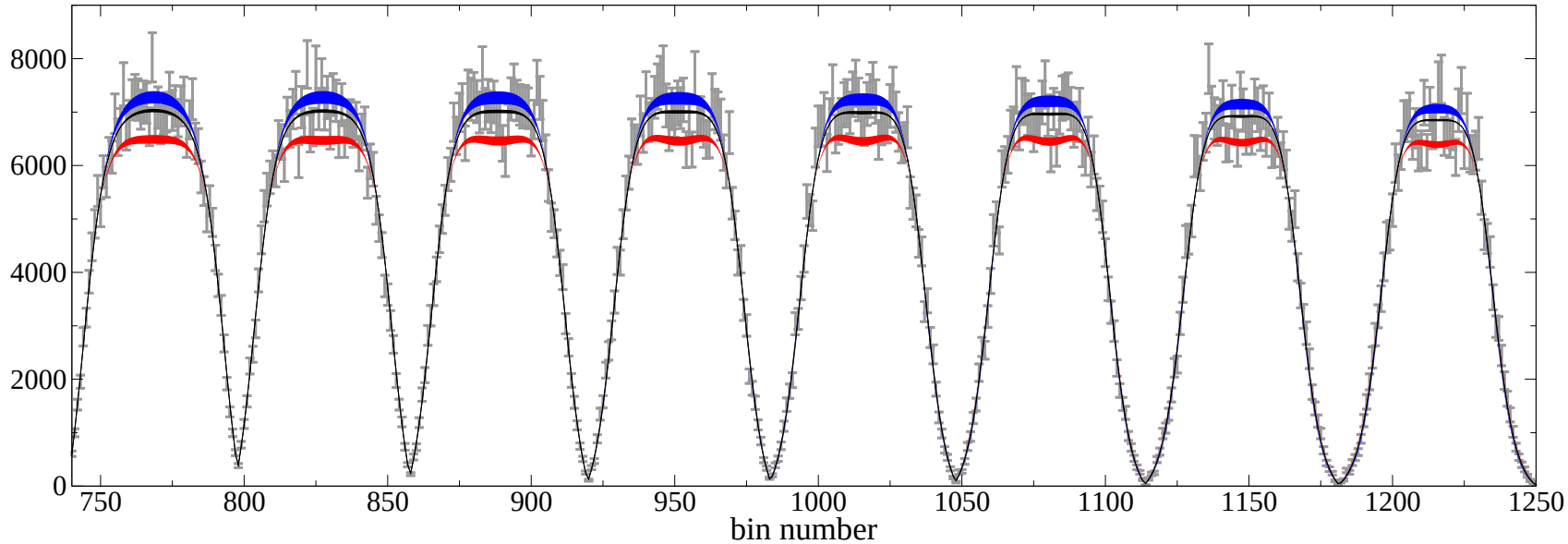

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χ^2/ndof	$1.71 \ldots 2.06$	$1.17 \ldots 1.50$	$1.02 \ldots 1.03$

- perfect fit respecting analyticity and unitarity possible
- contact term emulates neglected rescattering effects
- no need for "background" — inseparable from "resonance"

Predictions for experiment: $\omega \longrightarrow 3\pi$ Dalitz plot parameters

- $\omega \to 3\pi$ Dalitz plot smooth $\longrightarrow$ polynomial parameterisation

$$|\mathcal{F}_{\rm pol}(z,\phi)|^2 = |\mathcal{N}|^2 \Big\{ 1 + 2\alpha z + 2\beta z^{3/2} \sin 3\phi + 2\gamma z^2 + 2\delta z^{5/2} \sin 3\phi + \mathcal{O}\big(z^3\big) \Big\}$$

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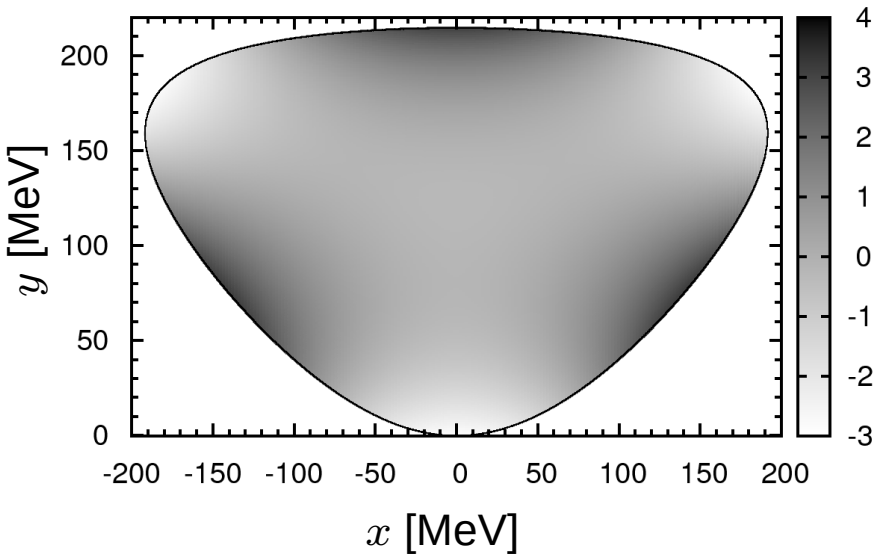
$\big(|\mathcal{F}_{\rm pol}(z,\phi)|^2 / |\mathcal{F}(z,\phi)|^2 - 1\big)$[%]



$\alpha \times 10^3$	$\beta \times 10^3$	$\gamma \times 10^3$	$\delta \times 10^3$
$84 \ldots 96$	—	—	—
$74 \ldots 84$	$24 \ldots 28$	—	—
$73 \ldots 81$	$24 \ldots 28$	$3 \ldots 6$	—
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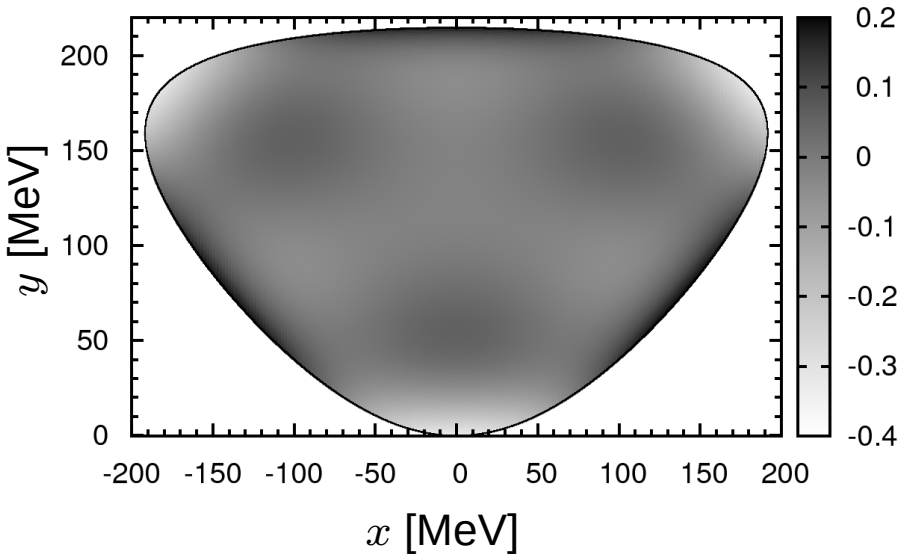

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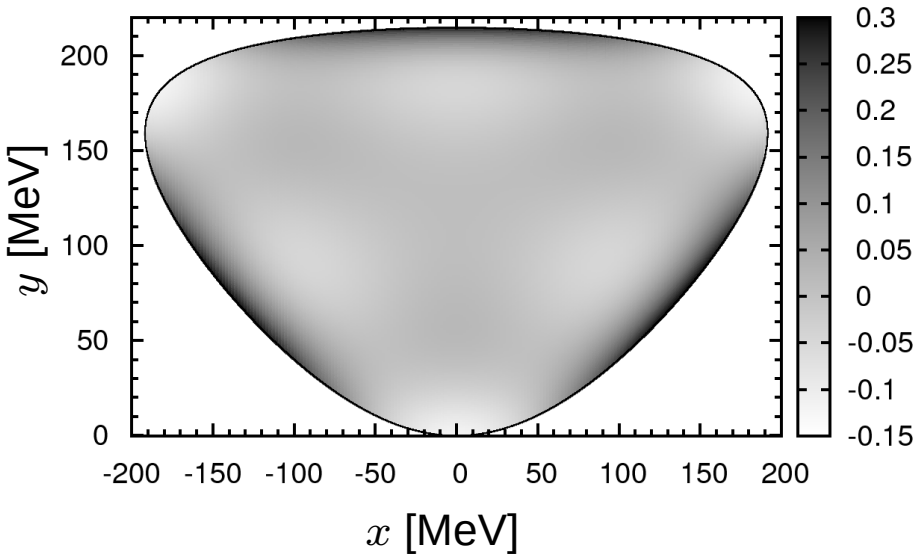

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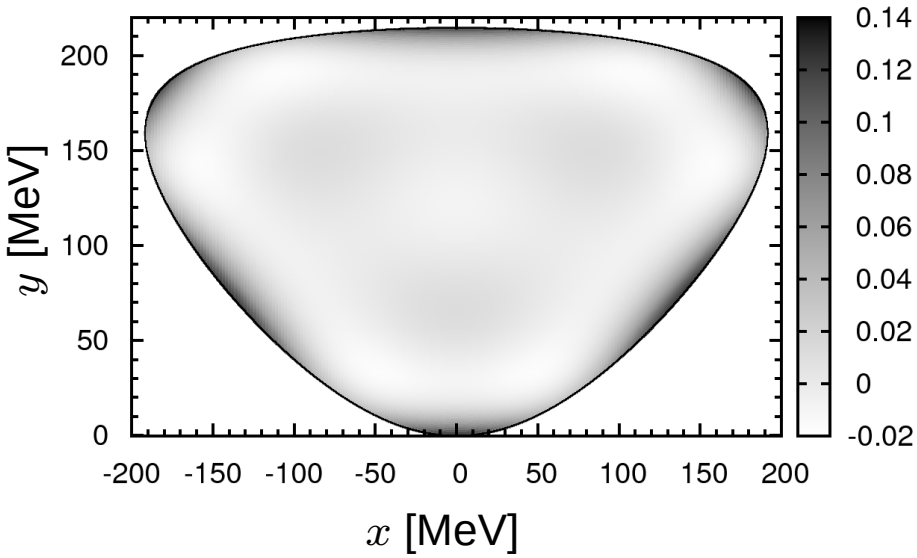

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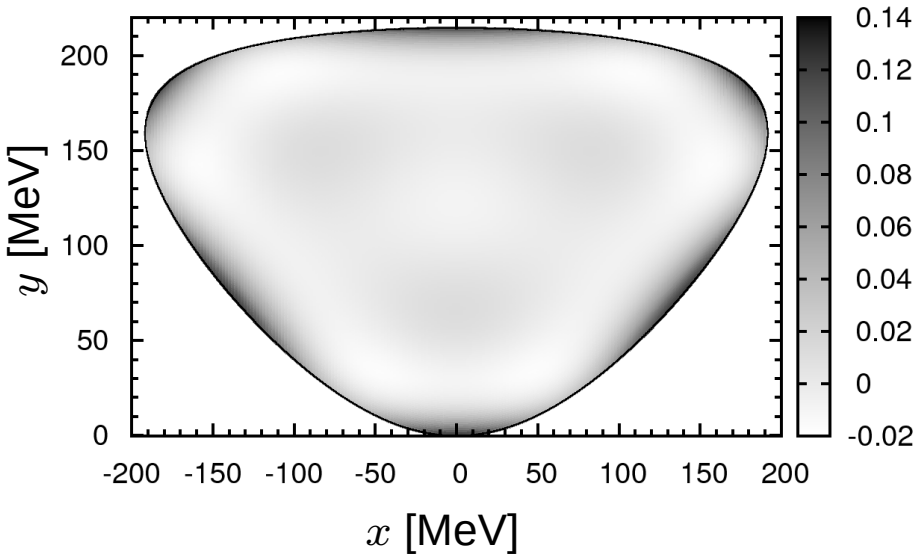

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$\longrightarrow$ 2 Dalitz plot parameters sufficient at 1% accuracy

$\longrightarrow$ first experimental measurement:

$$\alpha = (147 \pm 36) \times 10^{-3}$$

WASA-at-COSY 2016

β not yet significant

Summary / Outlook

Dispersion relations for meson decays

- based on unitarity, analyticity, crossing symmetry
- rigorous treatment of two- and three-hadron final states
- matching to ChPT where it works best:

  (sub)threshold, normalisation, slopes...
- relates hadronic to radiative decays / transition form factors

  not covered here: $\omega/\phi \to 3\pi \quad \longrightarrow \quad \omega/\phi \to \pi^0 \ell^+ \ell^-$

  Schneider, BK, Niecknig 2012

Where do we go from here?

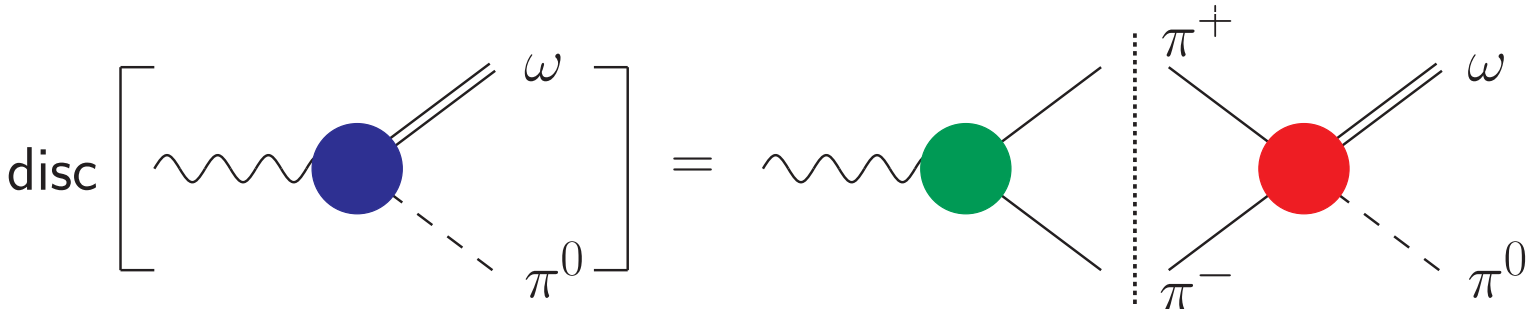
- radiative corrections in $\eta \to 3\pi$ via matching to EFTs

  $\longrightarrow$ improved light quark masses — Colangelo et al.
- $\eta' \to 3\pi$ from $\eta' \to \eta\pi\pi$ + $\eta\pi \to \pi\pi$ — Isken et al.
- comprehensive π^0, η, η' transition form factor program:

  doubly virtual $\longrightarrow$ $(g-2)_\mu$ theory initiative — Bonn, Jülich...

Spares

Transition form factor $\omega(\phi) \to \pi^0 \ell^+ \ell^-$



- ω transition form factor related to

  pion vector form factor $\times$ $\omega \to 3\pi$ decay amplitude

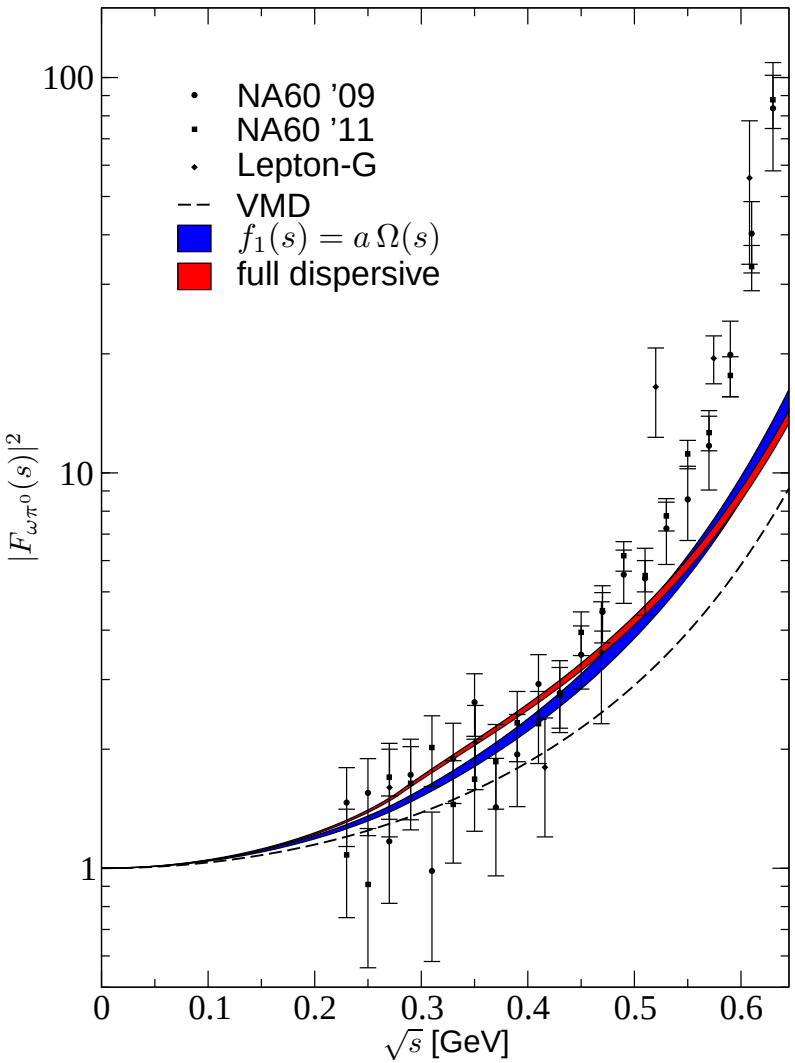
- form factor normalization yields rate $\Gamma(\omega \to \pi^0 \gamma)$

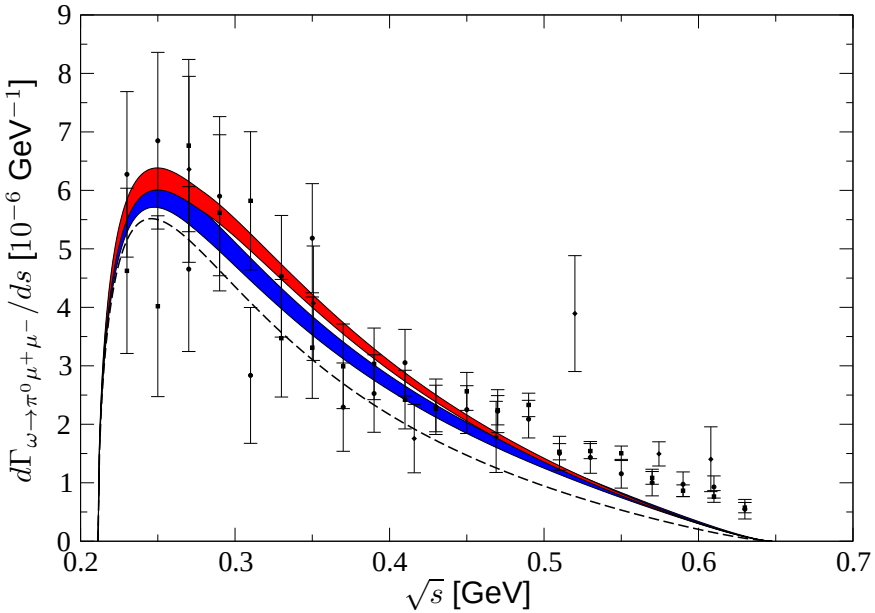
  (2nd most important ω decay channel)

  $\longrightarrow$ works at 95% accuracy

Schneider, BK, Niecknig 2012

Numerical results: $\omega \to \pi^0 \mu^+ \mu^-$

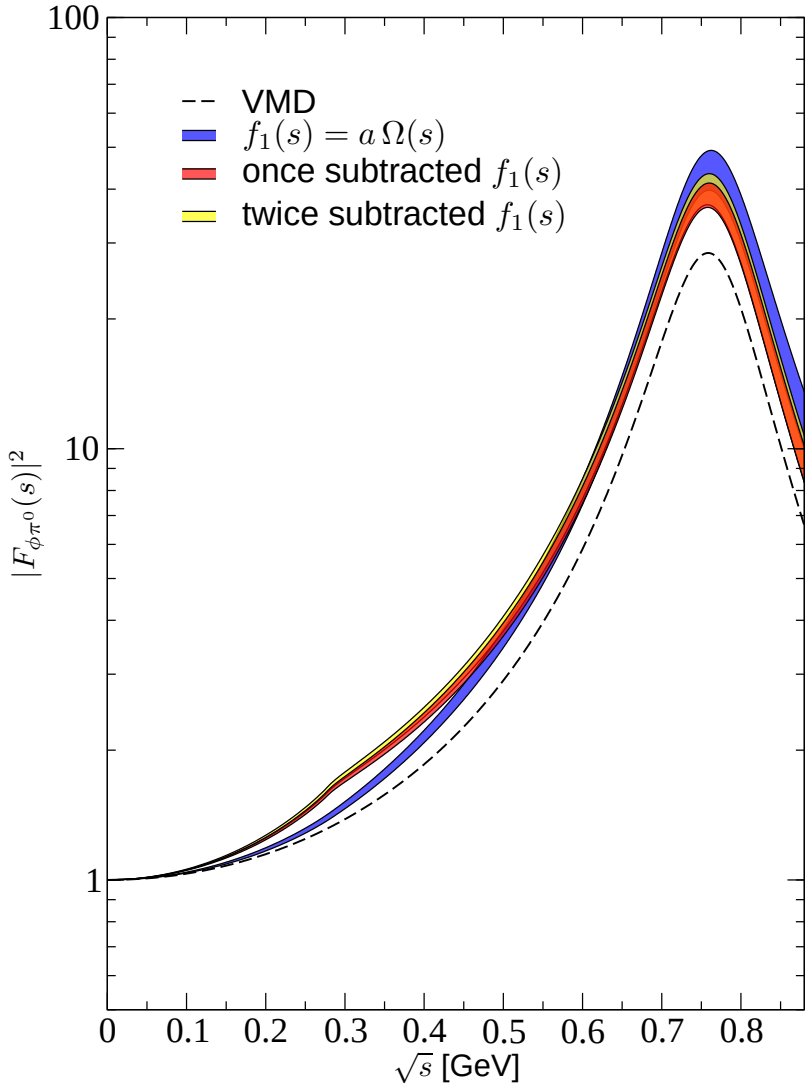




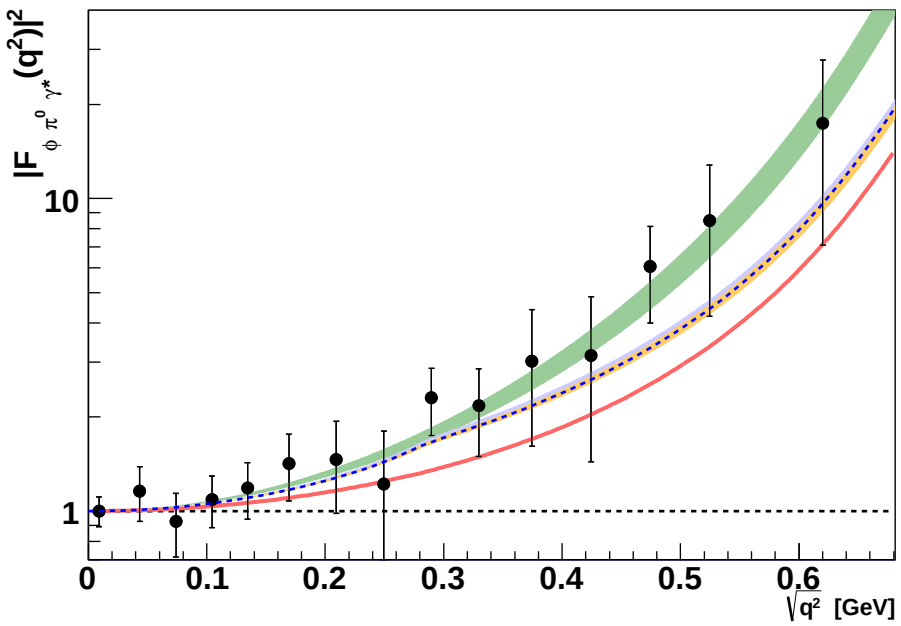
- clear enhancement vs. VMD
  cf. also Danilkin et al. 2015
- incompatible with data from heavy-ion coll. NA60 2009, 2011
- $\omega \to \pi^0 e^+ e^-$ data: no tension (but less precise) A2 2016

- NA60 data potentially in conflict with unitarity bounds
  Ananthanarayan, Caprini, BK 2014, Caprini 2015

Numerical results: $\phi \to \pi^0 \ell^+ \ell^-$



Schneider, BK, Niecknig 2012



KLOE 2016

- measurement in ρ peak region would be extremely helpful
- $\phi \to 3\pi$ partial-wave amplitude backed up by experiment

Niecknig, BK, Schneider 2012

$\eta \to 3\pi$: final-state interactions

- strong final-state interactions among pions
  - ▷ tree level: $\Gamma(\eta \to \pi^+\pi^-\pi^0) = 66$ eV — Cronin 1967
  - ▷ one-loop: $\Gamma(\eta \to \pi^+\pi^-\pi^0) = 160 \pm 50$ eV — Gasser, Leutwyler 1985
  - ▷ experimental: $\Gamma(\eta \to \pi^+\pi^-\pi^0) = 296 \pm 16$ eV — PDG
- major source: large S-wave final-state rescattering $\longrightarrow$
  use dispersion relations to resum those beyond loop expansion
- similar formalism to $\omega/\phi \to 3\pi$, but more partial waves
  (S waves $I = 0,\, 2$, P wave $I = 1$)
  match subtraction constants to ChPT and/or to data

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  match subtraction constants to ChPT and/or to data
- on the other hand: consider $r = \dfrac{\Gamma(\eta \to 3\pi^0)}{\Gamma(\eta \to \pi^+\pi^-\pi^0)}$

  ChPT: $r_{\text{tree}} = 1.54$, $r_{\text{1-loop}} = 1.46$, $r_{\text{2-loop}} = 1.47$
  PDG: $r = 1.432 \pm 0.026$ (fit) , $r = 1.48 \pm 0.05$ (average)

  $\longrightarrow$ agrees rather well — Bijnens, Ghorbani 2007

From unitarity to integral equations

Decay amplitude can be decomposed into single-variable functions

$$\mathcal{M}(s,t,u) = i\epsilon_{\mu\nu\alpha\beta} n^\mu p^\nu_{\pi^+} p^\alpha_{\pi^-} p^\beta_{\pi^0} \left\{ \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u) \right\}$$

Unitarity relation for $\mathcal{F}(s)$:

$$\text{disc}\, \mathcal{F}(s) = 2i \Big\{ \underbrace{\mathcal{F}(s)}_{\text{right-hand cut}} + \underbrace{\hat{\mathcal{F}}(s)}_{\text{left-hand cut}} \Big\} \times \theta(s - 4\,M_\pi^2) \times \sin\,\delta_1^1(s)\, e^{-i\delta_1^1(s)}$$

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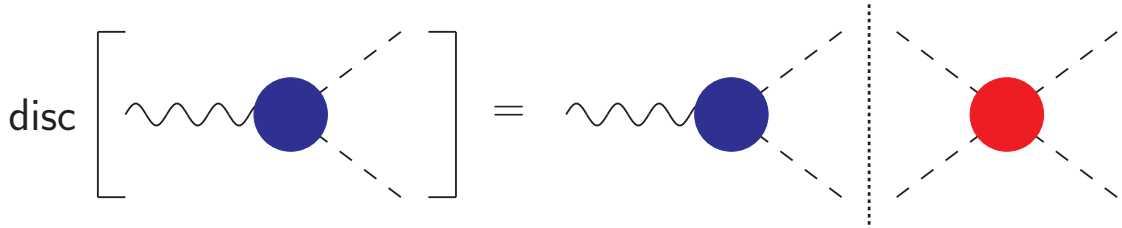
From unitarity to integral equations

Unitarity relation for $\mathcal{F}(s)$:

$$\text{disc}\,\mathcal{F}(s) = 2i\Big\{\underbrace{\mathcal{F}(s)}_{\text{right-hand cut}}\Big\} \times \theta(s-4\,M_\pi^2) \times \sin\,\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

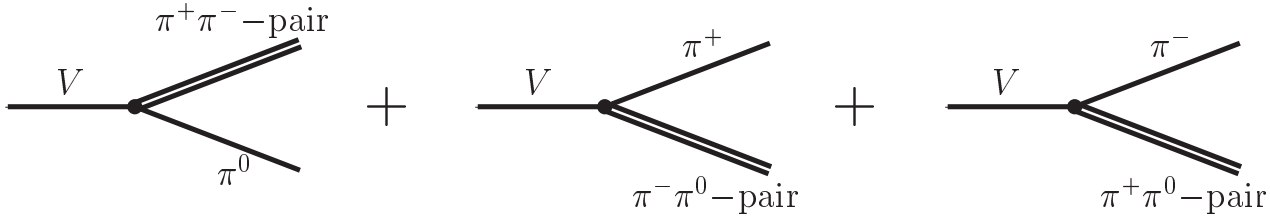


- right-hand cut only $\longrightarrow$ Omnès problem

$$\mathcal{F}(s) = a\,\Omega(s)\ , \qquad \Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\delta_1^1(s')}{s'-s-i\epsilon}\right\}$$

$\longrightarrow$ amplitude given in terms of pion vector form factor

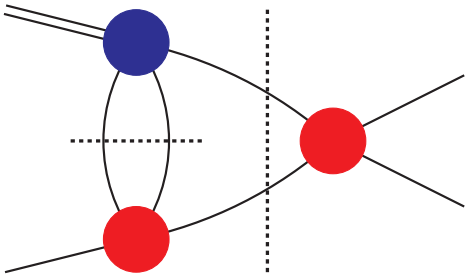
$\mathcal{F}(s,t,u) =$



From unitarity to integral equations

Unitarity relation for $\mathcal{F}(s)$:

$$\text{disc}\,\mathcal{F}(s) = 2i\Big\{\underbrace{\mathcal{F}(s)}_{\text{right-hand cut}} + \underbrace{\hat{\mathcal{F}}(s)}_{\text{left-hand cut}}\Big\} \times \theta(s-4\,M_\pi^2) \times \sin\,\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

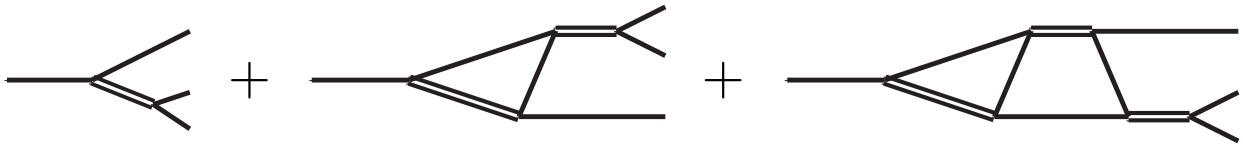


- inhomogeneities $\hat{\mathcal{F}}(s)$: angular averages over the $\mathcal{F}(t)$, $\mathcal{F}(u)$

$$\mathcal{F}(s) = a\,\Omega(s)\left\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\,\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s-i\epsilon)}\right\}$$

$$\hat{\mathcal{F}}(s) = \frac{3}{2}\int_{-1}^{1} dz\,(1-z^2)\mathcal{F}\big(t(s,z)\big)$$

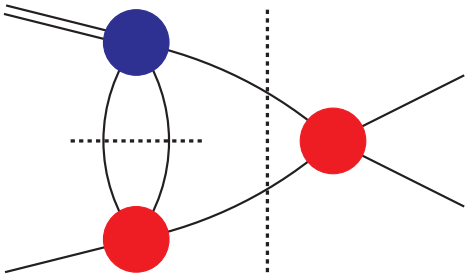
Anisovich, Leutwyler 1998

$\mathcal{F}(s) =$  $+ \cdots$

From unitarity to integral equations

Unitarity relation for $\mathcal{F}(s)$:

$$\text{disc}\,\mathcal{F}(s) = 2i\Big\{\underbrace{\mathcal{F}(s)}_{\text{right-hand cut}} + \underbrace{\hat{\mathcal{F}}(s)}_{\text{left-hand cut}}\Big\} \times \theta(s-4\,M_\pi^2) \times \sin\,\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$



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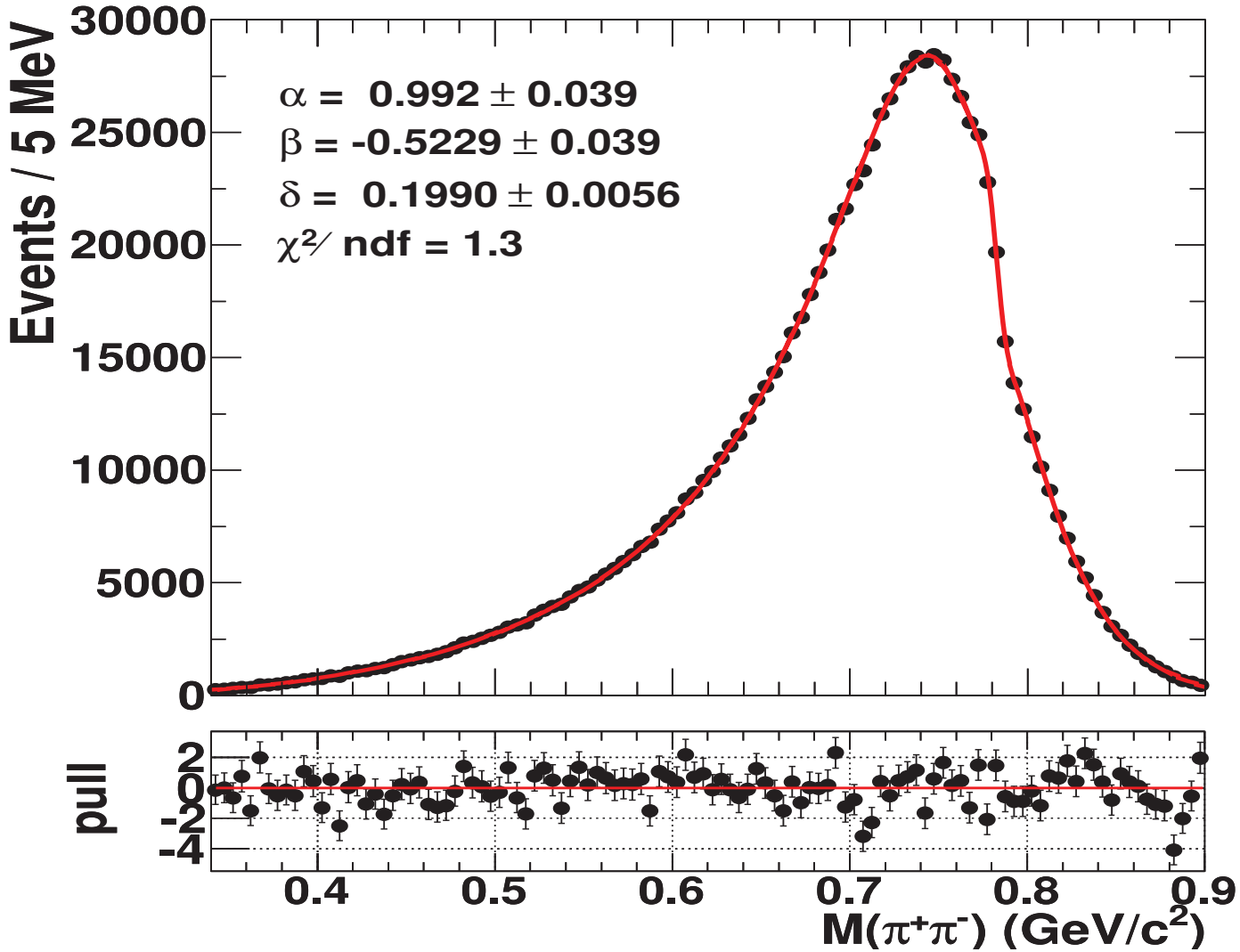
Anisovich, Leutwyler 1998

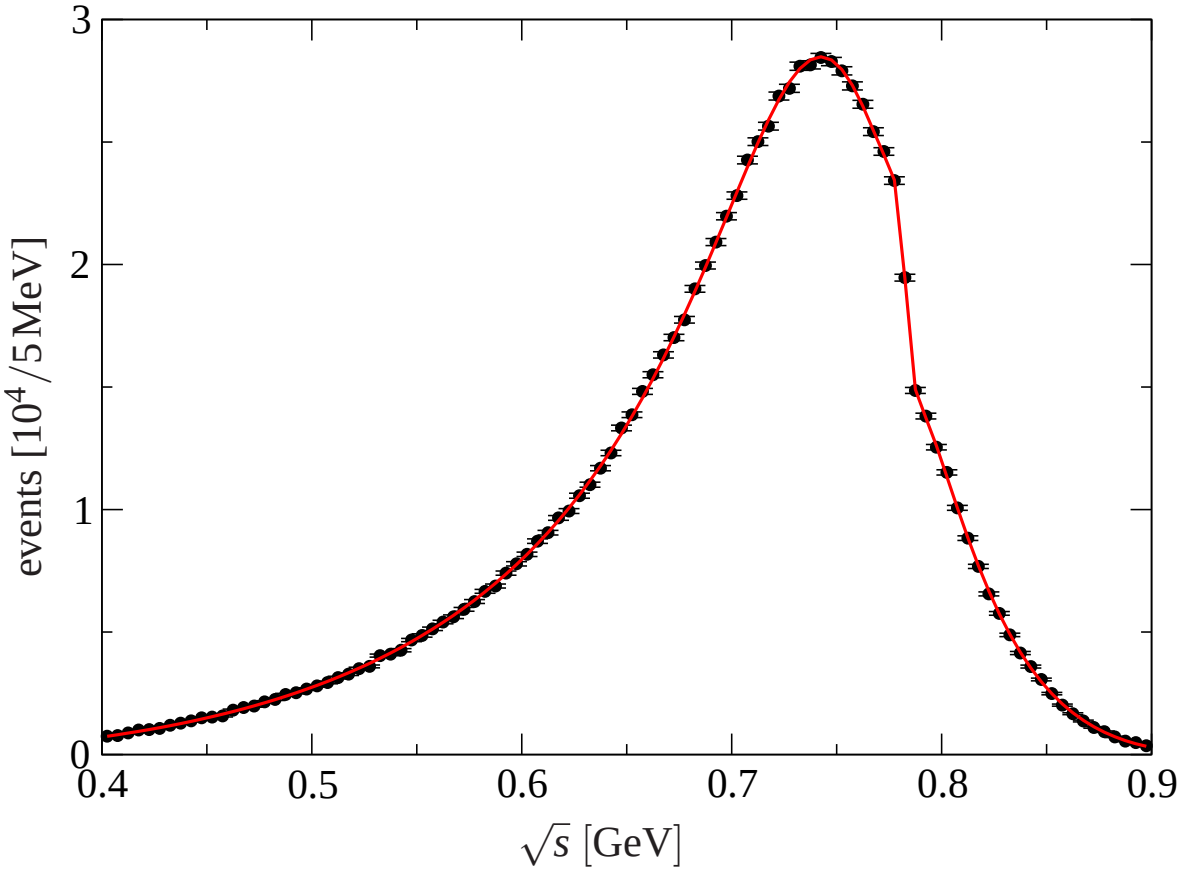
$\longrightarrow$ crossed-channel scattering between s-, t-, and u-channel

New data on $\eta' \to \pi^+\pi^-\gamma$



BESIII preliminary, Fang 2015

New data on $\eta' \to \pi^+\pi^-\gamma$



fit to pseudodata after BESIII preliminary

- fit form $$\left[A(1+\alpha t+\beta t^2)+\frac{\kappa}{m_\omega^2-t-im_\omega\Gamma_\omega}\right]\times\Omega(t)$$

  $\longrightarrow$ curvature $\propto \beta t^2$ essential (smaller than a_2 prediction)

  $\longrightarrow$ even ρ–ω mixing clearly visible

Hanhart et al. 2017