Coalescence, flow, HBT, and all that . . .

Ulrich Heinz

THE OHIO STATE UNIVERSITY

2nd EMMI Workshop on

Antimatter, Hyper-Matter, and Exotica Production at the LHC

Università degli Studi di Torino, Turin, Nov. 6-10, 2017

Coalescence, flow, HBT, and all that . . .

Prologue

- Coalescence model for the production of light nuclei in high energy hadron-hadron and hadron-nucleus collisions (cosmic rays) first introduced in the 1960s:

  Hagedorn 1960,1962,1965; Butler & Pearson 1963; Schwarzschild & Zupančič 1963

- Further development in the 1970s and 80s motivated by first experimental results with heavy-ion collisions at the BEVALAC (Gutbrod et al. 1976):

  Bond, Johansen, Koonin, Garpmann 1977; Mekjian 1977, 1978; Kapusta 1980, Sato & Yazaki 1981; Remler 1981, Gyulassy, Frankel & Remler 1983; Csernai & Kapusta 1986; Mrówczynski 1987; Dover et al. 1991

- Long initial discussions about the interpretation of the "invariant coalescence factor" B_A defined by

$$E_A \frac{dN_A}{d^3P_A} = B_A \left(E_\mathrm{p} \frac{dN_\mathrm{p}}{d^3P_\mathrm{p}} \right)^Z \left(E_\mathrm{n} \frac{dN_\mathrm{n}}{d^3P_\mathrm{n}} \right)^N \Bigg|_{P_\mathrm{p}=P_\mathrm{n}=P_A/A} .$$

  (1) "momentum-space coalescence volume" (Butler & Pearson, Schwarzschild & Zupančič, Gutbrod et al.);
  (2) "inverse fireball volume" $B_A \sim V^{A-1}$ (Bond et al, Mekjian).

Prologue

- The 1980s saw an increased focus on the phase-space and quantum mechanical aspects of nuclei formation through coalescence. An important paper by Danielewicz & Schuck 1992 used quantum kinetic theory to allow for scattering by a 3rd body to account for energy conservation in deuteron formation. Scheibl & Heinz 1999 used their work to derive a generalized Cooper-Frye formula for nuclear cluster spectra from coalescence,

$$E\frac{d^3N_A}{d^3P} = \frac{2J_A+1}{(2\pi)^3}\int_{\Sigma_\mathrm{f}} P\cdot d^3\sigma(R)\, f_\mathrm{p}^Z(R,P/A)\, f_\mathrm{n}^N(R,P/A)\, \mathcal{C}_A(R,P),$$

  where the "quantum mechanical correction factor" $\mathcal{C}_A(R,P)$, first introduced by Hagedorn 1960, accounts for the suppression of the coalescence probability in small or rapidly expanding fireballs where the cluster wave function may not fit inside the "homogeneity volume" of nucleons with similar momenta that contribute to the coalescence.

- A connection between deuteron coalescence and femtoscopic 2-particle correlations (intensity interferometry) was first noted in Mrówczynski 1993.
  Working it out in detail in a semi-realistically parametrized expanding fireball model, Scheibl & Heinz 1999 found the following main results:

Coalescence, flow, HBT, and all that . . .

Main results: 1. The quantum mechanical correction factor

The quantum mechanical correction factor (approximately independent of position) averaged over the freeze-out surface is given by

$$\langle \mathcal{C}_A \rangle (P) \equiv \langle \mathcal{C}_A(R,P) \rangle_\Sigma = \frac{\int_\Sigma P\cdot d^3\sigma(R)\, f_n^{A-Z}(R,P/A)\, f_p^Z(R,P/A)\, \mathcal{C}_A(R,P)}{\int_\Sigma P\cdot d^3\sigma(R)\, f_n^{A-Z}(R,P/A)\, f_p^Z(R,P/A)},$$

$$\approx e^{-B/T} \Big/ \left[\left(1 + \frac{2}{3} \frac{r^2_{A,\mathrm{rms}}}{\mathcal{R}^2_\perp(M_\perp/A)} \right) \left(1 + \frac{2}{3} \frac{r^2_{A,\mathrm{rms}}}{\mathcal{R}^2_\parallel(M_\perp/A)} \right)^{1/2} \right]$$

$B = M_A - Am < 0$ is binding energy of the nuclear cluster; $M_\perp/A \approx m_\perp$ is transverse mass of the coalescing nucleons. $\mathcal{C}_A(R,P)$ obtained by folding the internal Wigner density of the cluster with the phase-space densities of the coalescing nucleons; for example, for deuterons

$$\mathcal{C}_\mathrm{d}(R,P) = \int \frac{d^3q\, d^3r}{(2\pi)^3} \mathcal{D}(\boldsymbol{r},\boldsymbol{q}) \frac{f_p(R_+,P_+)\, f_n(R_-,P_-)}{f_p(R,P/2)\, f_n(R,P/2)}$$

$$\approx \int d^3r \left|\phi_d(\boldsymbol{r})\right|^2 \frac{f_p(R_+,P/2)\, f_n(R_-,P/2)}{f_p(R,P/2)\, f_n(R,P/2)}$$

where $\mathcal{D}(\boldsymbol{r},\boldsymbol{q}) = 8\exp(-\boldsymbol{r}^2/d^2 - \boldsymbol{q}^2 d^2)$, with $d = \sqrt{8/3}\, r_{d,\mathrm{rms}} = 3.2\,\mathrm{fm}$, is the deuteron internal Wigner density in its rest frame, $R^0_\pm = R^0_d \pm \boldsymbol{u}_d \cdot \boldsymbol{r}$, $\boldsymbol{R}_\pm = \boldsymbol{R} \pm \frac{1}{2}\left(\boldsymbol{r} + \frac{\boldsymbol{u}_d\cdot\boldsymbol{r}}{1+u^0_d}\boldsymbol{u}_d\right)$, $\boldsymbol{u}_d = \boldsymbol{P}/m_d$, and similarly for $P_\pm$.

Main results: 2. The invariant coalescence factor

By dividing the invariant cluster spectrum by the appropriate powers of the invariant nucleon spectra one obtains

$$B_A(P) = \frac{2J_A+1}{2^A}\,\langle \mathcal{C}_A \rangle\,\frac{M_\perp V_{\mathrm{eff}}(A,M_\perp)}{m_\perp V_{\mathrm{eff}}(1,m_\perp)}\left(\frac{(2\pi)^3}{m_\perp V_{\mathrm{eff}}(1,m_\perp)}\right)^{A-1} e^{(M_\perp - Am)(1/T_p^* - 1/T_A^*)}$$

where T_p^*, T_A^* are the inverse slope parameters ("effective temperatures") of the nucleon and cluster spectra, and the effective volume V_{eff} is given by

$$V_{\mathrm{eff}}(A,M_\perp) = \frac{V_{\mathrm{eff}}(1,m_\perp)}{A^{3/2}} = \left(\frac{2\pi}{A}\right)^{3/2} V_{\mathrm{hom}}(m_\perp) \quad \Longrightarrow \quad \frac{M_\perp V_{\mathrm{eff}}(A,M_\perp)}{m_\perp V_{\mathrm{eff}}(1,m_\perp)} = A^{3/2}$$

in terms of the **homogeneity volume** $V_{\mathrm{hom}}(m_\perp) = \mathcal{R}_\perp^2(m_\perp)\,\mathcal{R}_\parallel(m_\perp)$ where $\mathcal{R}_\perp(m_\perp)$ and $\mathcal{R}_\parallel(m_\perp)$ are the transverse ("sideward") and longitudinal HBT radii measured for particle pairs with transverse pair mass $m_\perp$:

$$\mathcal{R}_\perp(m_\perp) = \frac{\Delta\rho}{\sqrt{1+(m_\perp/T)\eta_f^2}}, \qquad \mathcal{R}_\parallel(m_\perp) = \frac{\tau_0\,\Delta\eta}{\sqrt{1+(m_\perp/T)(\Delta\eta)^2}}.$$

Here $\Delta\rho$, $\Delta\eta$ are the geometric (Gaussian) fireball widths in transverse (radial) and longitudinal (space-time rapidity) directions, τ_0 is the nucleon kinetic freeze-out time, and η_f and $\Delta\eta = (\tau_0\Delta\eta)/\tau_0$ are the transverse and longitudinal flow velocity gradients.

Coalescence, flow, HBT, and all that . . .

The model emission function (Csörgő and Lörstad 1996)

Assumption: simultaneous **kinetic** freeze-out of pions, kaons, and nucleons and coalescence of nuclei on a common "last scattering surface" Σ_f characterized by a position-dependent freeze-out time $t_\mathrm{f}(\boldsymbol{x})$.

Coordinate system: Milne (τ, η) and transverse polar (ρ, ϕ) coordinates:

$$R^\mu = (\tau \cosh \eta, \rho \cos \phi, \rho \sin \phi, \tau \sinh \eta)$$

Additional simplifications:

1. azimuthal symmetry ($b = 0$ collisions);
2. boost-invariant longitudinal flow rapidity $\eta_l(\tau, \rho, \eta) = \eta$ (Bjorken scaling);
3. linear transverse flow rapidity profile $\eta_\perp(\tau, \rho, \eta) = \eta_f \frac{\rho}{\Delta\rho}$;
   $$u^\mu(R) = \cosh \eta_\perp (\cosh \eta, \tanh \eta_\perp \cos \phi, \tanh \eta_\perp \sin \phi, \sinh \eta);$$
4. sudden freeze-out at constant longitudinal proper time τ_0 and temperature T:
   $$P \cdot d^3\sigma(R) = \tau_0 m_\perp \cosh(\eta - Y)\, \rho\, d\rho\, d\phi\, d\eta;$$
5. Boltzmann approximation for nucleons and nuclei:

$$f_i(R, P) = e^{\mu_i/T}\, e^{-P\cdot u(R)/T}\, H(R), \quad i = p, n;$$

$$H(R) = H(\eta, \rho) = \exp\left(-\frac{\rho^2}{2(\Delta\rho)^2} - \frac{\eta^2}{2(\Delta\eta)^2} \right).$$

Cluster spectra: thermal emission vs. coalescence

Thermal cluster emission: $\mu_A = Z\mu_p + (A{-}Z)\mu_n$

$$E\frac{d^3N_A}{d^3P} = \frac{2J_A+1}{(2\pi)^3} e^{\mu_A/T} \int_{\Sigma_\mathrm{f}} P\cdot d^3\sigma(R)\, e^{-P\cdot u(R)/T}\, H(R)$$

Classical coalescence (pointlike nucleons, ignoring cluster binding energy):

$$E\frac{d^3N_A}{d^3P} = \frac{2J_A+1}{(2\pi)^3} e^{\mu_A/T} \int_{\Sigma_\mathrm{f}} P\cdot d^3\sigma(R)\, e^{-P\cdot u(R)/T}\, \big(H(R)\big)^A$$

Quantum coalescence:

$$E\frac{d^3N_A}{d^3P} = \frac{2J_A+1}{(2\pi)^3} e^{\mu_A/T} \int_{\Sigma_\mathrm{f}} P\cdot d^3\sigma(R)\, e^{-P\cdot u(R)/T}\, \big(H(R)\big)^A \mathcal{C}_A(R,P)$$

$$\approx \frac{2J_A+1}{(2\pi)^3} e^{\mu_A/T} \langle\mathcal{C}_A\rangle(P) \int_{\Sigma_\mathrm{f}} P\cdot d^3\sigma(R)\, e^{-P\cdot u(R)/T}\, \big(H(R)\big)^A$$

For freeze-out at constant energy density, temperature and chemical potential: $H(R) = \mathrm{const.} = 1 = \big(H(R)\big)^A \implies$ **thermal emission and classical coalescence give identical results** while quantum coalescence gives slightly (15-20%) smaller yields.

Gaussian $H(R)$: thermal cluster emission spectrum

Using saddle point integration one obtains for the Gaussian profile function $H(R)$ (Scheibl & Heinz 1999)

$$E\frac{d^3N_A}{d^3P} = \frac{2J_A+1}{(2\pi)}\, e^{(\mu_A-M)/T} M_\perp V_{\rm eff}(1,M_\perp) \exp\left(-\frac{M_\perp - M}{T_A^*} - \frac{Y^2}{2(\Delta\eta)^2}\right)$$
$$= \frac{2J_A+1}{(2\pi)^{3/2}}\, e^{(\mu_A-M)/T} M_\perp V_{\rm hom}(M_\perp) \exp\left(-\frac{M_\perp - M}{T_A^*} - \frac{Y^2}{2(\Delta\eta)^2}\right)$$

with an inverse slope parameter ("effective temperature") that increases linearly with the cluster mass:

$$T_A^* = T + M\,\eta_f^2.$$

Gaussian $H(R)$: spectrum from classical coalescence

Using saddle point integration one obtains for the Gaussian profile function $H(R)$

$$E\frac{d^3N_A}{d^3P} = \frac{2J_A+1}{(2\pi)}\, e^{(\mu_A - Am)/T}\, M_\perp\, V_{\rm eff}(A, M_\perp)\, \exp\left(-\frac{M_\perp - Am}{T_A^*} - \frac{AY^2}{2(\Delta\eta)^2}\right)$$
$$= \frac{2J_A+1}{(2\pi)^{3/2}}\, e^{(\mu_A - Am)/T}\, M_\perp\, \frac{V_{\rm hom}(m_\perp)}{A^{3/2}}\, \exp\left(-\frac{M_\perp - Am}{T_A^*} - \frac{AY^2}{2(\Delta\eta)^2}\right)$$

where $M_\perp \equiv \sqrt{(Am)^2 + P_\perp^2}$, with an inverse slope parameter ("effective temperature") independent of cluster size:

$$T_A^* = T + \frac{Am}{A}\eta_f^2 = T + m\,\eta_f^2 = T_p^*.$$

Gaussian $H(R)$: spectrum from quantum coalescence

Using saddle point integration one obtains for the Gaussian profile function $H(R)$

$$E\frac{d^3N_A}{d^3P} = \frac{2J_A+1}{(2\pi)}\, e^{(\mu_A - Am)/T} \langle \mathcal{C}_A\rangle(P)\, M_\perp\, V_{\rm eff}(A, M_\perp) \exp\left(-\frac{M_\perp - Am}{T_A^*} - \frac{AY^2}{2(\Delta\eta)^2}\right)$$

$$= \frac{2J_A+1}{(2\pi)^{3/2}}\, e^{(\mu_A - Am)/T} \langle \mathcal{C}_A\rangle(P)\, M_\perp \frac{V_{\rm hom}(m_\perp)}{A^{3/2}} \exp\left(-\frac{M_\perp - Am}{T_A^*} - \frac{AY^2}{2(\Delta\eta)^2}\right)$$

where $M_\perp \equiv \sqrt{(Am)^2 + P_\perp^2}$, with an inverse slope parameter ("effective temperature") independent of cluster size:

$$T_A^* = T + \frac{Am}{A}\eta_f^2 = T + m\,\eta_f^2 = T_p^*.$$

Clearly, this last feature is incompatible with experimental observations which show clusters flowing as if they were thermally emitted. This problem does not persist for a constant density profile $H(R) = 1$.

Coalescence, flow, HBT, and all that . . .

Cluster spectra from wave function overlap

Danielewicz and Schuck 1992:

In the cluster rest frame, coalescence is a non-relativistic process. Starting from the square of the overlap matrix element between the deuteron wave function and those of a proton and a neutron and rewriting it in terms of density matrices and ultimately Wigner densities, Danielewicz and Schuck showed that in the deuteron rest frame the deuteron momentum spectrum can be calculated as

$$\frac{dN_d}{d^3P_d} = \frac{-3i}{(2\pi)^3}\int d^4r_\mathrm{d}\, d^3r \int \frac{d^4p_1}{(2\pi)^4}\,\frac{d^3p_2}{(2\pi)^3}\,(2\pi)^4\delta^4(P_d - p_1^* - p_2)$$
$$\times\mathcal{D}\left(\boldsymbol{r}, \tfrac{\boldsymbol{p}_1-\boldsymbol{p}_2}{2}\right)\left[\Sigma_p^<(p_1^*, r_+)\, f_n^W(p_2, r_-) + \Sigma_n^<(p_1^*, r_+)\, f_p^W(p_2, r_-)\right],$$

where p^* denotes an off-shell momentum, due to a preceding collision of the off-shell particle with a third body. The energy-momentum conserving δ-function can only be satisfied if either the neutron or the proton is off-shell.

Cluster spectra from wave function overlap

The off-shell nucleon self energy is given by

$$
\begin{aligned}
-i\Sigma_N^<(p^*,x) &= \sum_j \int \frac{d^3q}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3}(2\pi)^4\delta^4(p^*+q-p'-q') \\
&\qquad \times |M_{Nj\to Nj}|^2 f_N^W(p',x)\, f_j^W(q',x)\left(1\pm f_j^W(q,x)\right) \\
&\approx f_N(p^*,x)\sum_j \int \frac{d^3q}{(2\pi)^3}\, f_j(q,x) \\
&\qquad \times \left[\int \frac{d^3p'}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3}(2\pi)^4\delta^4(p^*+q-p'-q')|M_{Nj\to Nj}|^2\left(1\pm f_j(q',x)\right)\right] \\
&= \frac{f_N(p^*,x)}{\tau^N_{\mathrm{scatt}}(\boldsymbol{p},x)}\,.
\end{aligned}
$$

Deuterons have twice the scattering rate of their constituent nucleons. Since any scattering is likely to break up the deuteron, the integration over $t_d = r_d^0$ in the deuteron rest frame should only go from $t_f - \frac{1}{2}\tau^N_{\mathrm{scatt}}$ to t_f. Assuming the scattering time to be sufficiently short to neglect any change in the distribution functions during this time interval the factors of τ^N_{scatt} cancel, and ...

Cluster spectra from wave function overlap

... and we get

$$\frac{dN_d}{d^3P_d} = \frac{3}{(2\pi)^3}\int d^3r_d \int \frac{d^3r\, d^3q}{(2\pi)^3}\,\mathcal{D}(\boldsymbol{r},\boldsymbol{q}) f_p(q_+, r_+)\, f_n(q_-^*; r_-)$$

where

$$q_+^\mu = \left(\sqrt{m^2+\boldsymbol{q}^2},\, \boldsymbol{q}\right), \qquad q_-^{*\mu} = \left(M_d - \sqrt{m^2+\boldsymbol{q}^2},\, -\boldsymbol{q}\right).$$

Lorentz transforming this to the global frame by Lorentz-boosting the rest-frame positions and momenta with the four-velocity of the deuteron, we can use $E_d d^3r_d = P_d \cdot d^3\sigma(R_d)$ and write this as

$$E_d\frac{dN_d}{d^3P_d} = \frac{3}{(2\pi)^3}\int_{\Sigma_\mathrm{f}} P_d\cdot d^3\sigma(R_d)\, f_p(R_d, P_d/2)\, f_n(R_d, P_d/2)\, \mathcal{C}_d(R_d, P_d)$$

which defines the previously listed quantum mechanical correction factor $\mathcal{C}_d(R_d, P_d)$.

The quantum mechanical correction factor

Some typical values for the quantum mechanical correction factor for deuterons from the Gaussian emission function model are listed in the following Table:

TABLE I. The quantum-mechanical correction factor $\mathcal{C}_d^0$ for Hulthen and harmonic oscillator wave functions calculated with Eq. (3.21), for different fireball parameters at nucleon freeze-out (for details see text).

τ_0 [fm/c]		9.0			6.0	
T [MeV]	168	130	100	168	130	100
η_f	0.28	0.35	0.43	0.28	0.35	0.43
Hulthen	0.86	0.84	0.80	0.80	0.78	0.74
harm. osc.	0.84	0.81	0.76	0.76	0.72	0.66

Note that the more realistic Hulthen wave function, which (in spite of the same $r_{\rm rms}$) peaks at a smaller value of r than the Gaussian, has better overlap with the "homogeneity factor" $f(R_+, P/2)f(R_-, P/2)/f^2(R, P/2)$ than the Gaussian one, because the latter peaks strongly at $r = 0$.

Coalescence, flow, HBT, and all that . . .

Summary (in the form of qualitative predictions):

- For kinetic freeze-out at constant temperature, chemical potential and thus constant particle and energy density, **classical coalescence produces the same particle yields and spectra as the thermal model**, **independent of the value of the kinetic freeze-out temperature**. Just as the chemical temperature extracted from elementary hadron yield ratios provides no information about their kinetic freeze-out temperature, the chemical temperature extracted from particle ratios involving nuclear clusters provides no information about the temperature at which the coalescence process took place.

- Quantum mechanical effects, which scale with the ratio of the intrinsic cluster volume divided by the homogeneity volume of the coalescing nucleons (which can be extracted from femtoscopic measurements), **suppress** the cluster yields by 15-25% in collisions between large nuclei and by larger factors in smaller and more rapidly expanding systems. Binding energy correction effects are typically small.

- The invariant coalescence factors B_A are proportional to $A^{1/3}/\left(m_\perp V_{\rm hom}(m_\perp)\right)^{A-1}$ and thus **increase** with $m_\perp$ (due to the corresponding **decrease** of the HBT homogeneity volume) and **decrease** with $\sqrt{s}$ (due to the corresponding **increase** of the HBT homogeneity volume), qualitatively consistent with observations.