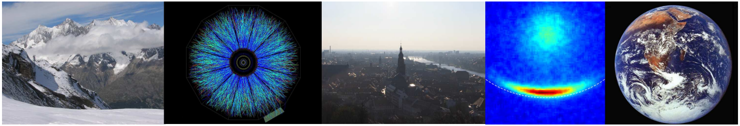
Functional renormalization and ultracold quantum gases

Stefan Flörchinger (Heidelberg)

EMMI Workshop, Riezlern 2009

How should we describe the world?



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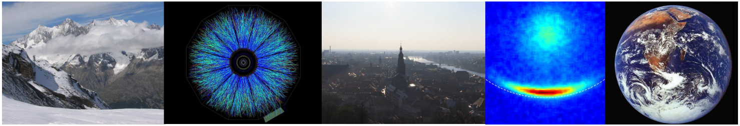

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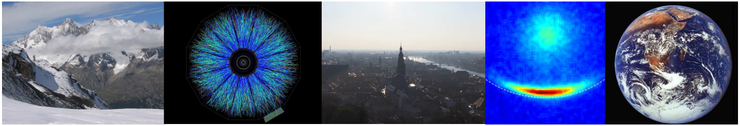

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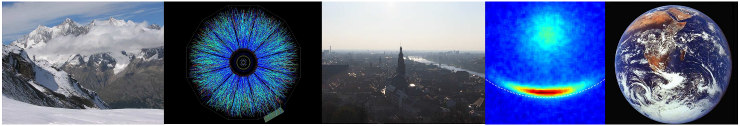

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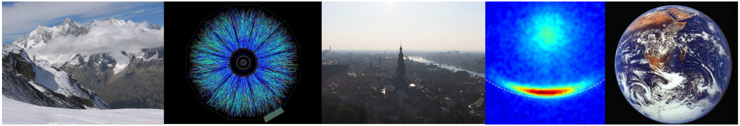

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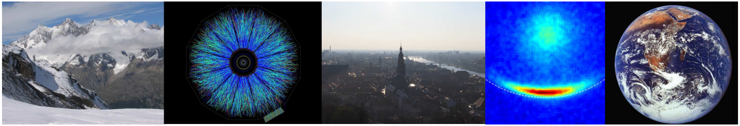

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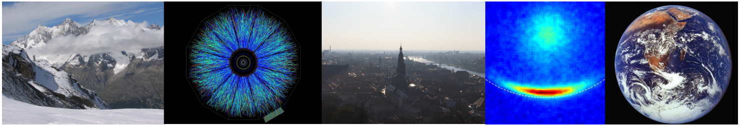

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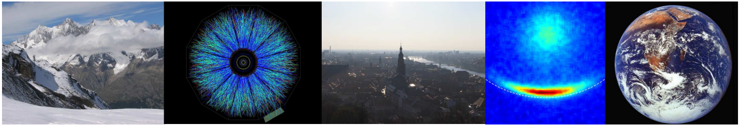

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QUANTUM FIELD THEORY.

Classical field theory

- Describes electro-magnetic fields, waves, ... ($\hbar \to 0$).
- Crucial object: classical action

$$S[\phi] = \int dt \int d^d x \ \mathcal{L}(\phi, \partial_t \phi, \vec{\nabla}\phi, \dots)$$

- Classical field equations from $\frac{\delta S}{\delta \phi} = 0$.
- Symmetries of S lead to conserved currents.
- All physical observables are easily obtained from S.

Quantum field theory

- Describes also electrons, atoms, quarks, gluons, protons,...
- Crucial object: quantum effective action

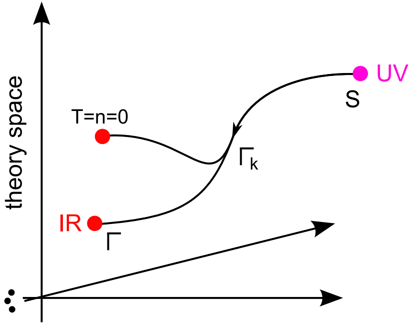
$$\Gamma[\phi] = \int dt \int d^d x \; U(\phi) + \ldots$$

- Quantum field equations from $\frac{\delta \Gamma}{\delta \phi} = 0$.
- Symmetries of Γ lead to conserved currents.
- All physical observables are easily obtained from Γ.
- Γ is generating functional of 1-PI Feynman diagrams and depends on external parameters like T, μ, or $\vec{B}$.

How do we obtain the quantum effective action $\Gamma[\phi]$?

Idea of functional renormalization: $\Gamma[\phi] \to \Gamma_k[\phi]$

- k is additional infrared cutoff parameter.
- $\Gamma_k[\phi] \to \Gamma[\phi]$ for $k \to 0$.
- $\Gamma_k[\phi] \to S[\phi]$ for $k \to \infty$.
- Dependence on T, μ or $\vec{B}$ trivial for $k \to \infty$.



$\Gamma[\phi]$ *and the grand canonical ensemble*

Functional integral representation of the partition function

$$Z = e^{-\beta\Omega_G} = \mathsf{Tr}\, e^{-\beta(H-\mu N)} = \int D\chi\, e^{-S[\chi]}.$$

Generalization with $J = \frac{\delta}{\delta\phi}\Gamma_k[\phi]$

$$e^{-\Gamma_k[\phi]} = \int D\chi\, e^{-S[\phi+\chi]+J\chi-\frac{1}{2}\chi\, R_k\, \chi}.$$

- R_k is an infrared cutoff function
  - suppresses all fluctuations $R_k \to \infty$ for $k \to \infty$.
  - is removed $R_k \to 0$ for $k \to 0$.
- $\Gamma_k[\phi]$ is the *average action* or *flowing action*.
- Grand canonical potential is obtained from $\beta\Omega_G = \Gamma_k[\phi]$ for $k = 0$ and $J = 0$.

How the flowing action flows

Simple and exact flow equation (Wetterich 1993)

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \mathsf{STr} \left(\Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \partial_k R_k .$$

- Differential equation for a functional.
- For most cases not solvable exactly.
- Approximate solutions can be found from Truncations.
  - Ansatz for Γ_k with a finite number of parameters.
  - Derive ordinary differential equations for this parameters or couplings from the flow equation for Γ_k.
  - Solve these equations numerically.

Lagrangians

We use a local field theory to describe the microscopic model. Examples:

1. Bose gas with pointlike interaction

$$\mathcal{L} = \varphi^* \left(\partial_\tau - \vec{\nabla}^2 - \mu\right) \varphi + \frac{1}{2}\lambda \left(\varphi^* \varphi\right)^2 .$$

2. Fermions in the BCS-BEC-Crossover

$$\begin{aligned}\mathcal{L} \;&=\; \psi^\dagger(\partial_\tau - \vec{\nabla}^2 - \mu)\psi + \varphi^*(\partial_\tau - \frac{1}{2}\vec{\nabla}^2 - 2\mu + \nu)\varphi \\ &\quad -h(\varphi^*\psi_1\psi_2 + h.c.).\end{aligned}$$

These are effective theories on the length scale of the Bohr radius or van-der-Waals length.

Symmetries of nonrelativistic field theories

- U(1) for particle number conservation.
- Translations and Rotations.
- Galilean boost transformations.
- Possibly conformal symmetries.
- U(1) and Galilean invariance are broken spontaneously by a Bose-Einstein condensate.
- Galilean invariance is broken explicitely for $T > 0$.

Truncations

For many purposes *derivative expansions* are suitable approximations. For example we use for the BCS-BEC Crossover

$$\begin{aligned}\Gamma_k &= \int_{\tau,\vec{x}} \Big\{ \psi^\dagger(\partial_\tau - \vec{\nabla}^2 - \mu)\psi + \varphi^*(Z_\varphi \partial_\tau - A_\varphi \frac{1}{2}\vec{\nabla}^2)\varphi \\ &\quad -h(\varphi^*\psi_1\psi_2 + h.c.) + \frac{1}{2}\lambda_\psi(\psi^\dagger\psi)^2 + U_k(\varphi^*\varphi, \mu) \Big\}\end{aligned}$$

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- The coefficients Z_φ, A_φ, λ_ψ, h and the effective potential U_k are scale-dependent.
- The effective potential U_k contains no derivatives - describes homogeneous fields.
- Wave-function renormalization and self-energy corrections for fermions can be included as well.

The effective potential

- We use a Taylor expansion around the minimum ρ_0

$$U_k(\varphi^* \varphi) = -p + m^2 \left(\varphi^* \varphi - \rho_0\right) + \frac{1}{2} \lambda \left(\varphi^* \varphi - \rho_0\right)^2.$$

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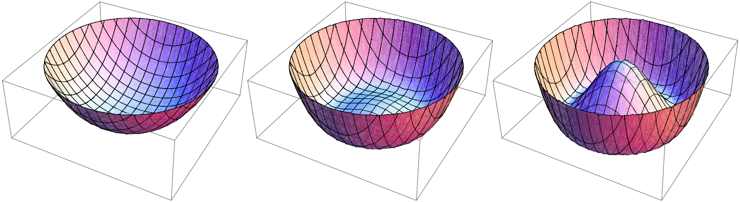
- Symmetry breaking:



$T > T_c$ $T < T_c$ $T \ll T_c$

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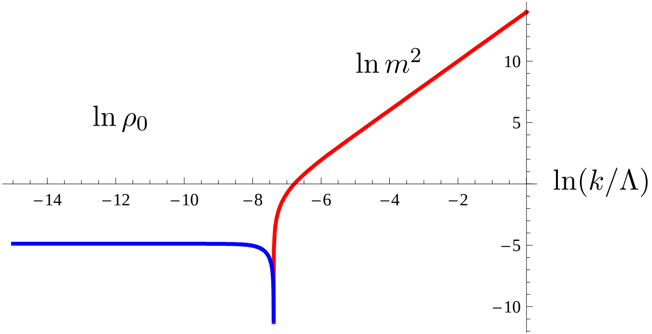
- Typical flow:



Solving the flow equation - Phase diagram

- Information on phase diagram is contained in form of the effective potential $U(\rho, \mu, T)$ at macroscopic scale.

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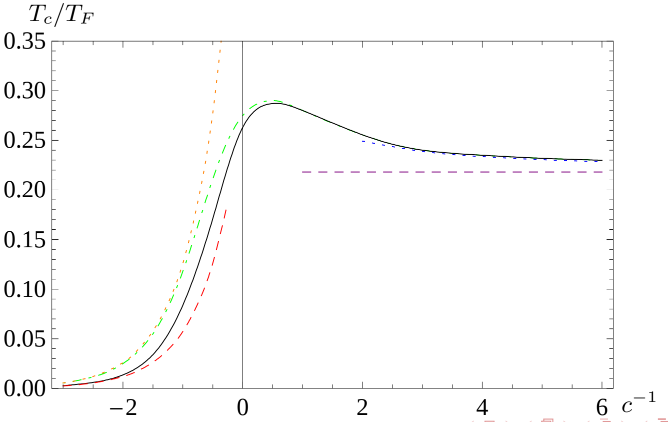
- Very nice generalization of Landau's theory!
- Examples: BCS-BEC Crossover

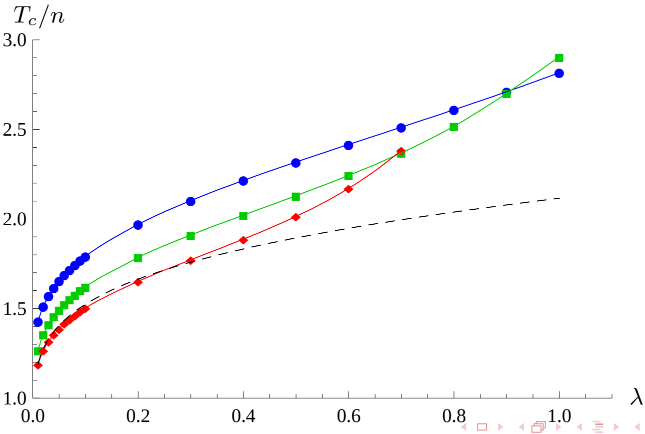
(Floerchinger, Scherer, Diehl and Wetterich, PRB 78, 174528 (2008).)



Solving the flow equation - Phase diagram

- Information on phase diagram is contained in form of the effective potential $U(\rho, \mu, T)$ at macroscopic scale.
- Very nice generalization of Landau's theory!
- Examples: Superfluid Bose gas in $d = 2$

Floerchinger and Wetterich, PRA 79, 013601 (2009).



Solving the flow equation - Thermodynamic observables

From grand canonical potential

$$dU = -dp = -s\,dT - n\,d\mu$$

take derivatives e. g. for Bose gas in $d = 3$

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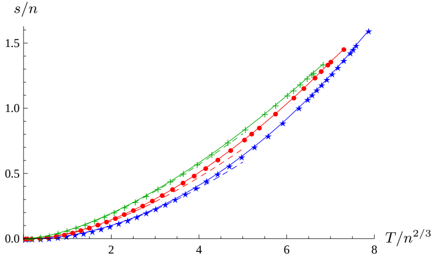

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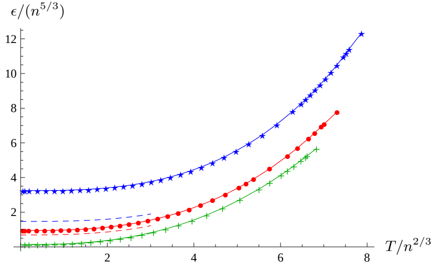

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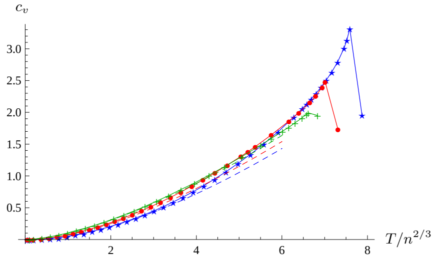

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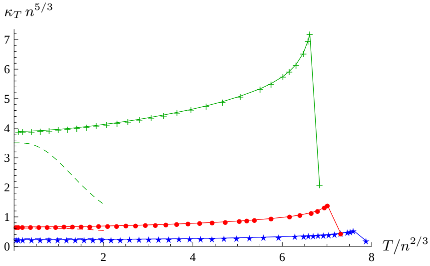

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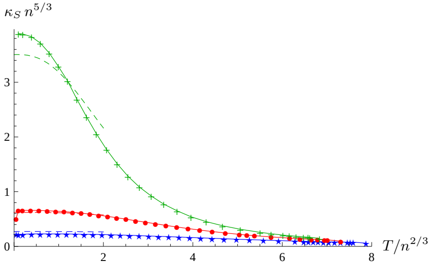

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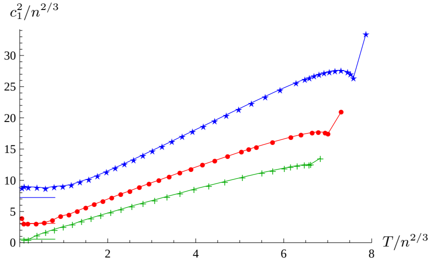

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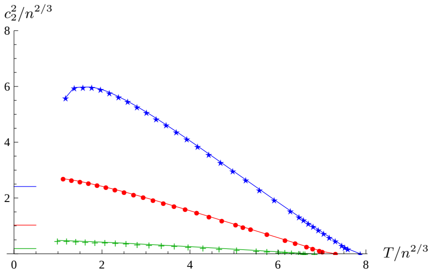

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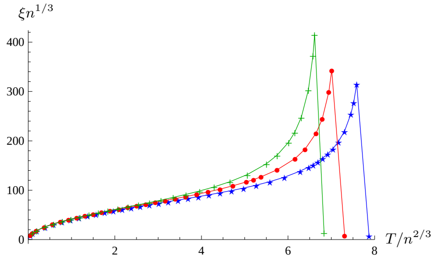

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Flow equations for "new physics"

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- 3 component Fermi gas - ??
  - Three-body problem: Efimov effect
    (Efimov 1970, Review: Braaten and Hammer 2006)
  - On the lattice: Trion formation
    (Rapp, Zarand, Honerkamp and Hofstetter 2007)

Three component Fermi gas

- For equal masses, densities etc. global SU(3) symmetry

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \to u \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad u \in \mathsf{SU(3)}.$$

Similar to flavor symmetry in the Standard model!

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- For small scattering length $|a| \to 0$
  - BCS ($a < 0$) or BEC ($a > 0$) superfluidity at small T.
  - order parameter is conjugate triplet $\bar{\mathbf{3}}$ under SU(3)

$$\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix} \sim \begin{pmatrix}\psi_2\psi_3\\ \psi_3\psi_1\\ \psi_1\psi_2\end{pmatrix}.$$

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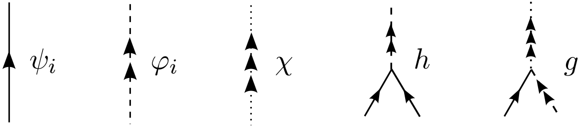
  - SU(3) symmetry is broken spontaneously for $\varphi \neq 0$.
- What happens for large $|a|$?

Simple truncation for fermions with three components

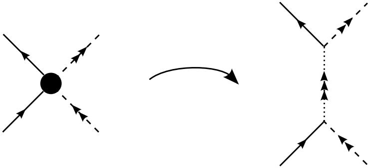
$$\Gamma_k = \int_x \psi^\dagger(\partial_\tau - \vec{\nabla}^2 - \mu)\psi + \varphi^\dagger(\partial_\tau - \frac{1}{2}\vec{\nabla}^2 + m_\varphi^2)\varphi$$
$$+\chi^*(\partial_\tau - \frac{1}{3}\vec{\nabla}^2 + m_\chi^2)\chi$$
$$+h\epsilon_{ijk}(\varphi_i^*\psi_j\psi_k + h.c.) \quad + g(\varphi_i\psi_i^*\chi + h.c.).$$

- Units are such that $\hbar = k_B = 2M = 1$
- Wavefunction renormalization for ψ, φ and χ is implicit.
- Γ_k contains terms for
  - fermion field $\psi = (\psi_1, \psi_2, \psi_3)$
  - bosonic field $\varphi = (\varphi_1, \varphi_2, \varphi_3) \sim (\psi_2\psi_3, \psi_3\psi_1, \psi_1\psi_2)$
  - trion field $\chi \sim \psi_1\psi_2\psi_3$

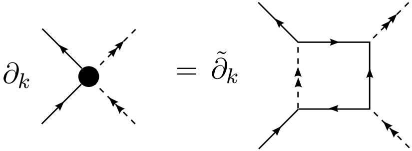


"Refermionization"

- Trion field is introduced via a generalized Hubbard-Stratonovich transformation



- Fermion-boson coupling is regenerated by the flow

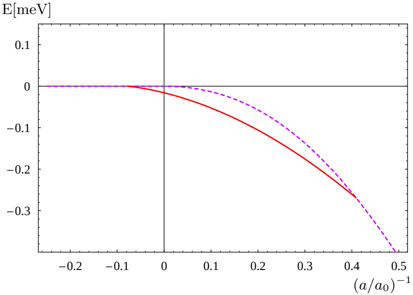


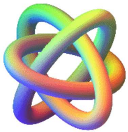
- Use scale dependent field $\partial_k \chi \sim \psi_i \varphi_i$ to express this again by trion exchange.

  (Gies and Wetterich 2002, Pawlowski 2008, Floerchinger and Wetterich 2009)

Binding energies

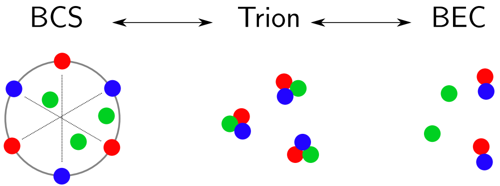
- Vacuum limit $T \to 0, \quad n \to 0.$





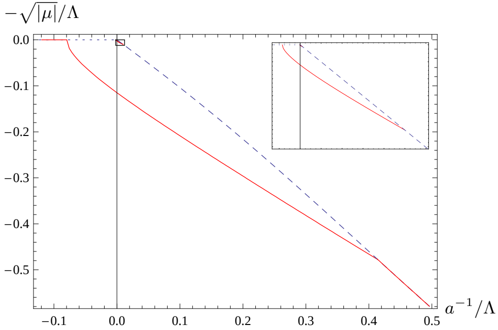
- Binding energy per atom for
  - molecule or dimer φ (dashed line)
  - trion or trimer χ (solid line)
- For large scattering length a trion is energetically favorable!
- Three-body bound state even for $a < 0$.

Quantum phase diagram



- BCS-Trion-BEC transition
  (Floerchinger, Schmidt, Moroz and Wetterich, PRA **79**, 013603 (2009)).
  - $a \to 0_-$: Cooper pairs, $SU(3) \times U(1) \to SU(2) \times U(1)$.
  - $a \to 0_+$: BEC of molecules, $SU(3) \times U(1) \to SU(2) \times U(1)$.
  - $a \to \pm\infty$: Trion phase, $SU(3)$ unbroken.
- Quantum phase transitions
  - from BCS to Trion phase
  - from Trion to BEC phase.

Efimov effect



- Self-similarity in energy spectrum.
- Efimov trimers become more and more shallow. At $a = \infty$

$$E_{n+1} = e^{-2\pi/s_0} E_n.$$

- Simple truncation: $s_0 \approx 0.82$.
- Advanced truncation: $s_0 \approx 1.006$

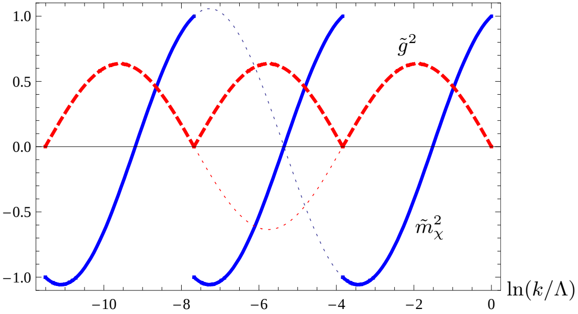
(Moroz, Floerchinger, Schmidt and Wetterich, PRA, **79**, 042705 (2009)).

Renormalization group limit cycle

- For $\mu = 0$ and $a^{-1} = 0$ flow equations for rescaled couplings

$$k \frac{\partial}{\partial k} \begin{pmatrix} \tilde{g}^2 \\ \tilde{m}_\chi^2 \end{pmatrix} = \begin{pmatrix} 7/25 & -13/25 \\ 36/25 & 7/25 \end{pmatrix} \begin{pmatrix} \tilde{g}^2 \\ \tilde{m}_\chi^2 \end{pmatrix}.$$

- Solution is log-periodic in scale.



- Every zero-crossing of $\tilde{m}_\chi^2$ corresponds to a new bound state.

Conclusions

- Functional renormalization is a useful method to describe ultracold quantum gases.
- Quantitative precision seems reachable.
- Unified description of
    - Bosons and Fermions,
    - Weak and strong coupling,
    - Few-Body and Many-Body physics.