Solitonic ground states in (color) superconductivity

Michael Buballa (TU Darmstadt), Dominik Nickel (MIT)

TECHNISCHE UNIVERSITÄT DARMSTADT

Massachusetts Institute of Technology

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EMMI

Hirschegg 2010:

Strongly Interacting Matter under Extreme Conditions

http://crunch.ikp.physik.tu-darmstadt.de/hirschegg/

Motivation

- We have already heard about:
  - imbalanced Fermi liquids (*M. Zwierlein*)
  - asymmetric nuclear matter (*X.-G. Huang*)
  - FFLO in one-dimensional optical lattices (*R. Bakhtiari*)
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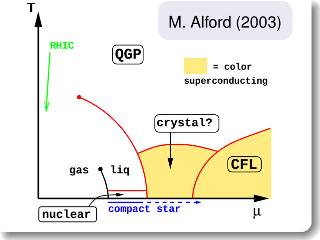
- This talk is about:

  LOFF (= FFLO) phases
  in color superconducting
  asymmetric quark matter



based on: D. Nickel, M.B., Phys. Rev. D79, 054009 (2009) [arXiv:0811.2400].

Outline

1. Introduction
   - *imbalanced pairing in CSC*
2. Model
   - *NJL model for inhomogeneous pairing*
3. Numerical Results
   - *one-dimensional solutions*
4. Outlook

Color superconductivity

- diquark condensates: $\langle q^T \, \mathcal{O} \, q \rangle$
- Pauli principle:

  $\mathcal{O} = \mathcal{O}_{spin} \otimes \mathcal{O}_{color} \otimes \mathcal{O}_{flavor} =$ totally antisymmetric
- most attractive channel:
  - spin 0 $\quad (=$ antisymmetric)
  - color $\bar{3}$ $\quad (=$ antisymmetric)

  ➜ antisymmetric in flavor

  ➜ pairing between unequal flavors

$$example: \quad \langle q^T \, C\gamma_5 \, \tau_2 \, \lambda_2 \, q \rangle \sim (\uparrow\downarrow - \downarrow\uparrow) \otimes (r\,g - g\,r) \otimes (ud - du)$$

Neutral quark matter

- constraints in compact stars:
  - color neutrality: $n_r = n_g = n_b$
  - electric neutrality: $n_Q = \frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s - n_e = 0$
  - β equilibrium: $\mu_e = \mu_d - \mu_u$ ➜ $n_e \ll n_{u,d}$

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  - electric neutrality:
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$$\left.\begin{array}{l}\text{electric neutrality} \\ \beta \text{ equilibrium}\end{array}\right\} \quad \frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s \approx 0$$

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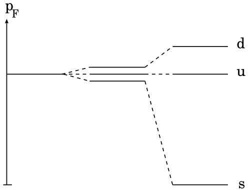
➜ all flavors have different Fermi momenta

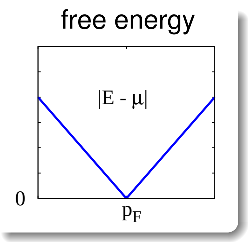


➜ imbalanced pairing unavoidable (Rajagopal & Schmitt, PRD (2006))

BCS pairing

- Cooper instability:

  Fermi gas



BCS pairing

- Cooper instability:

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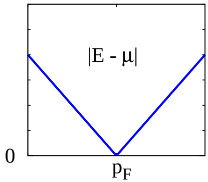
  ➜ condensation of Cooper pairs

free energy



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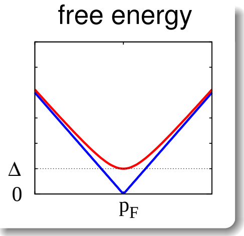

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  - close to the Fermi surface
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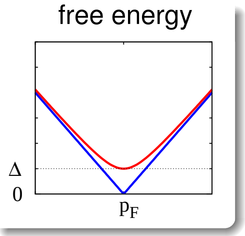

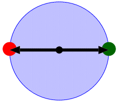

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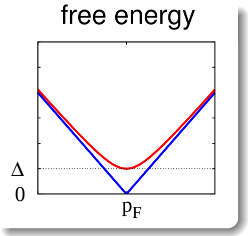
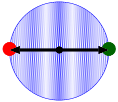
  ➜ works only if $p_F^a = p_F^b$





Imbalanced pairing

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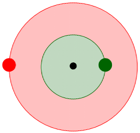

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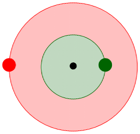

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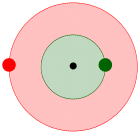

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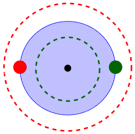

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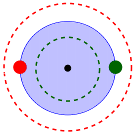

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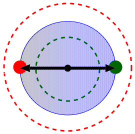

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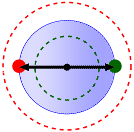
    (Chandrasekhar, Clogston (1962))



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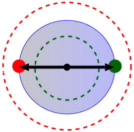

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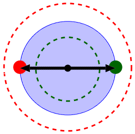

- option 2: phase separation (globally neutral mixed phases)
  - Coulomb and surface effects ? ➜ pattern sizes ?
- option 3: gapless phases (= Sarma phases = breached pairing)
  - chromomagnetic instabilities

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  pairs with nonzero total momentum

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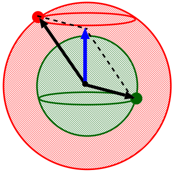
  - single plane wave
    $\langle q(\vec{x})q(\vec{x})\rangle \sim \Delta\, e^{2i\vec{q}\cdot\vec{x}}$ for fixed $\vec{q}$
  - disfavored by phase space



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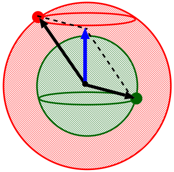

- LO (Larkin, Ovchinnikov, 1964):
  - multiple plane waves (e.g., $\cos(2\vec{q}\cdot\vec{x})$)
  - Ginzburg-Landau expansion

Inhomogeneous phases in color superconductivity

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  - single plane wave (Alford, Bowers, Rajagopal, 2001)

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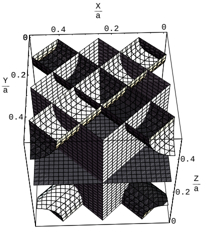
    (e.g., Bowers, Rajagopal, 2002; Casalbuoni et al., 2006 . . .)
    - different directions,

      but equal wave lengths
    - mostly Ginzburg-Landau



(Rajagopal, Sharma, 2006)

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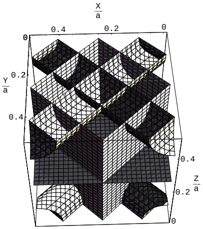

(Rajagopal, Sharma, 2006)

- aim of our work:
    - NJL model for inhomogeneous pairing beyond Ginzburg-Landau
    - superimpose different wave lengths

Model

- Model Lagrangian with NJL-type qq interaction:

$$\mathcal{L} = \bar{q}\,(i\partial\!\!\!/ + \hat{\mu}\gamma^0)\,q + \mathcal{L}_{int}$$

$$\mathcal{L}_{int} = H \sum_{A,A'=2,5,7} (\bar{q}\,i\gamma_5\tau_A\lambda_{A'}\,q_c)(\bar{q}_c\,i\gamma_5\tau_A\lambda_{A'}\,q)\,, \qquad q_c = C\bar{q}^T$$

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- bosonize: $\varphi_{AA'}(x) \;=\; -2H\,\bar{q}_c(x)\,\gamma_5\tau_A\lambda_{A'}\,q(x)$

$$\Rightarrow \quad \mathcal{L}_{int} = \frac{1}{2}\sum_{A,A'}\Big\{\, (\bar{q}\,\gamma_5\tau_A\lambda_{A'}\,q_c)\,\varphi_{AA'} \,+\, h.c. \,-\, \frac{1}{2H}\,\varphi^\dagger_{AA'}\,\varphi_{AA'}\Big\}$$

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- mean-field approximation:

$$\langle\,\varphi_{AA'}(x)\rangle = \Delta_A(x)\,\delta_{AA'}\,, \qquad \langle\,\varphi^\dagger_{AA'}(x)\rangle = \Delta^*_A(x)\,\delta_{AA'}$$

  - $\Delta_A(x)$ *classical* fields
  - retain space-time dependence!

Mean-field model

- Lagrangian:

$$\mathcal{L}_{MF}(x) = \bar{\Psi}(x)\, S^{-1}(x)\, \Psi(x) \; - \; \frac{1}{4H} \sum_A |\Delta_A(x)|^2$$

  - Nambu-Gor'kov bispinors: $\Psi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} q(x) \\ q_c(x) \end{pmatrix}$
  - inverse dressed propagator:

$$S^{-1}(x) = \begin{pmatrix} i\partial\!\!\!/ + \hat{\mu}\gamma^0 & \hat{\Delta}(x)\,\gamma_5 \\ -\hat{\Delta}^*(x)\,\gamma_5 & i\partial\!\!\!/ - \hat{\mu}\gamma^0 \end{pmatrix}, \quad \hat{\Delta}(x) := \sum_A \Delta_A(x)\,\tau_A\lambda_A$$

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- Thermodynamic potential:

$$\Omega_{MF}(T,\mu) = -\frac{1}{2}\frac{T}{V}\mathrm{Tr}\ln\frac{S^{-1}}{T} + \frac{T}{V}\sum_A \int_{[0,\frac{1}{T}]\otimes V} d^4x\, \frac{|\Delta_A(x)|^2}{4H}$$

  - $\mathrm{Tr}\ln S^{-1}$ nontrivial because of x-dependent gap functions!

Crystalline ansatz

- gap functions: time-independent, periodic in space
  $\hat{\Delta}(x) \equiv \hat{\Delta}(\vec{x}) = \hat{\Delta}(\vec{x} + \vec{a}_i)\,, \quad i = 1, 2, 3$

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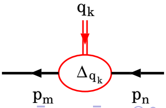

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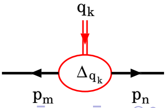

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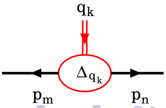
  - condensates couple different momenta!
  - diagonal in energy ➜ Matsubara sum as usual



Hamiltonian

- The inverse propagator can be put into the form

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- thermodynamic potential:

$$\Omega_{MF} = -\frac{1}{4V} \sum_{\lambda} \left[E_\lambda + 2T \ln\left(1 + 2e^{-E_\lambda/T}\right) \right] + \sum_{A} \sum_{q_k} \frac{|\Delta_{A,q_k}|^2}{4H}$$

  - E_λ: eigenvalues of $\mathcal{H}$

Thermodynamic potential

- remaining problem: diagonalize $\mathcal{H}$

$$\mathcal{H}_{\vec{p}_m,\vec{p}_n} = \begin{pmatrix} (\gamma^0 \vec{\not{p}}_n - \hat{\mu})\,\delta_{\vec{p}_m,\vec{p}_n} & -\sum_{\vec{q}_k} \hat{\Delta}_{q_k} \gamma^0 \gamma_5\, \delta_{\vec{q}_k,\vec{p}_m-\vec{p}_n} \\ \sum_{\vec{q}_k} \hat{\Delta}^*_{q_k} \gamma_0 \gamma_5\, \delta_{\vec{q}_k,\vec{p}_n-\vec{p}_m} & (\gamma^0 \vec{\not{p}}_n + \hat{\mu})\,\delta_{\vec{p}_m,\vec{p}_n} \end{pmatrix}$$

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➜ We finally obtain:

$$\Omega_{MF} = -\frac{1}{4}\int_{B.Z.} \frac{d^3k}{(2\pi)^3} \sum_\lambda \left[E_\lambda(\vec{k}) + 2T\,\ln\left(1 + 2e^{-E_\lambda(\vec{k})/T}\right)\right] + \sum_A \sum_{q_k} \frac{|\Delta_{A,q_k}|^2}{4H}$$

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- remaining diagonalization problem:

$$(\mathcal{H}_{\Delta,\delta\mu})_{\vec{p}_m,\vec{p}_n} = \begin{pmatrix} (p_m - \bar{\mu} - \delta\mu)\,\delta_{\vec{p}_m,\vec{p}_n} & \Delta_{p_m - p_n} \\ \Delta^*_{p_n - p_m} & -(p_m - \bar{\mu} + \delta\mu)\,\delta_{\vec{p}_m,\vec{p}_n} \end{pmatrix}$$

Regularization

- unregularized expression for Ω_{MF} divergent
  - ➜ needs to be regularized
- 3-momentum cutoff ?
- inhom. phases: $\mathcal{H}$ depends on two momenta!
  - cut off both of them, e.g., $|\vec{p}_m|, |\vec{p}_n| \leq \Lambda$?
  - ➜ strong regularization artifacts:
    - large $|\vec{q}|$ suppressed, e.g., $|\vec{q}| < 2\Lambda$
    - violates "model independent" low-energy results

➜ Pauli-Villars-like scheme:

$$F(E_\lambda) \to \sum_{j=0}^{2} F(E_{\lambda,j}) \,, \qquad E_{\lambda,j}(\vec{k}) = \sqrt{E_\lambda^2(\vec{k}) + j\Lambda^2}$$

Numerical results

- external parameters: $T, \bar{\mu}, \delta\mu$
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- model parameters: $H,\ \Lambda$
  - $\Lambda = 400$ MeV
  - $H \ \leftrightarrow\ \Delta_{BCS} = 80$ MeV
- crystal structure:
  - general problem (too) difficult

  ➜ consider one-dimensional modulations (in 3+1 D):

$$\Delta(z) = \sum_k \Delta_k\, e^{2ikqz}$$

  - further restriction: $\Delta(z) = \text{real} \quad \Leftrightarrow \quad \Delta_k = \Delta^*_{-k}$

step 1: minimize Ω at fixed q

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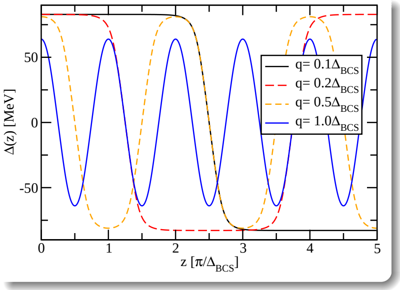
- example: $\delta\mu = 0.7\ \Delta_{BCS}$



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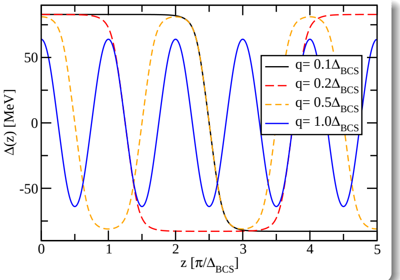

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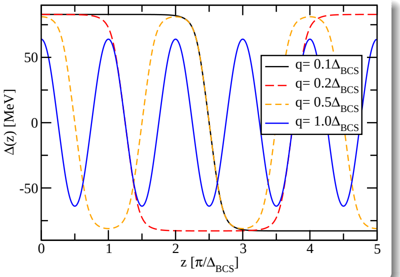

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  - shape of transition region almost independent of q
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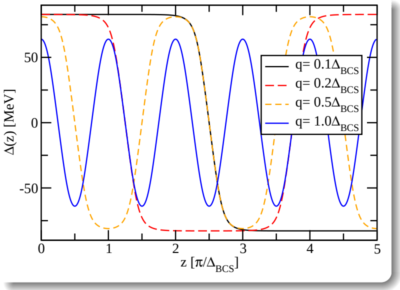

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- parametrization of the gap function:

$$\Delta(z) = A\,\mathrm{sn}\big(\kappa(z - z_0); \nu\big)$$

(works extremely well)

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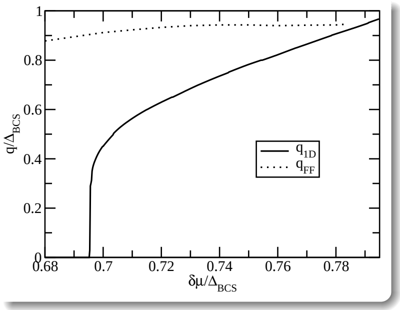

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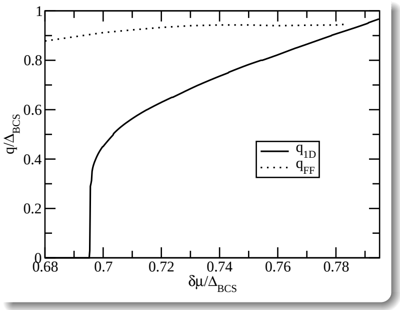

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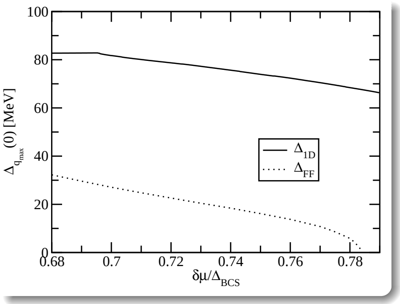

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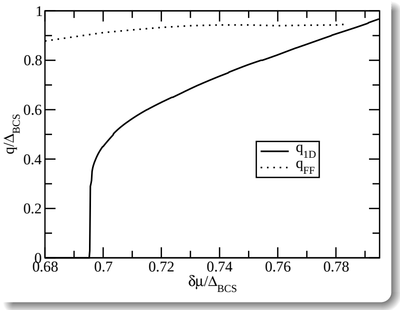

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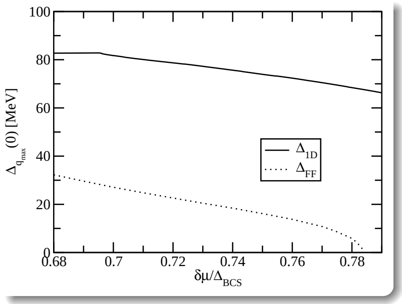

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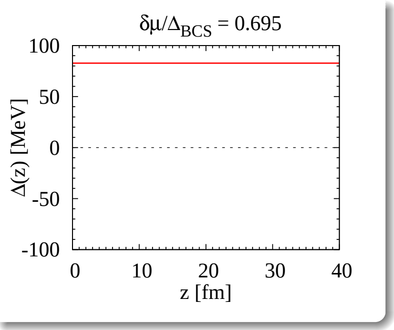
- $\Delta_{inhom.} \gg \Delta_{FF}$

Gap functions

- putting everything together:



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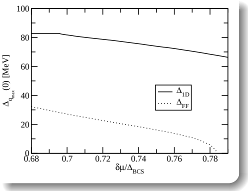

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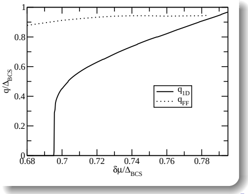

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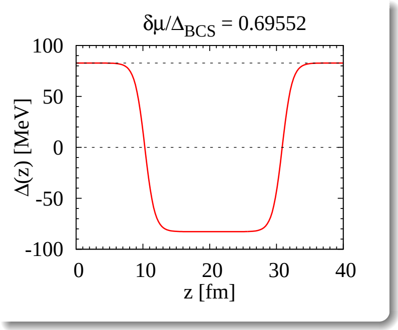

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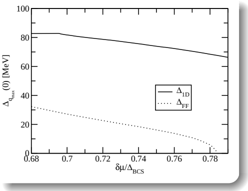

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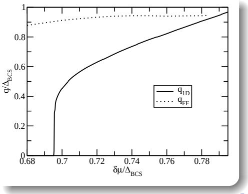

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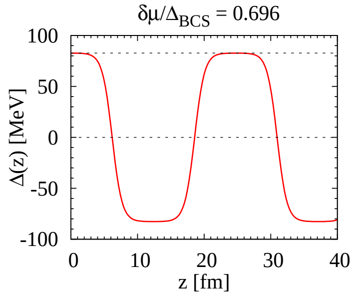

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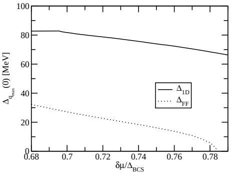

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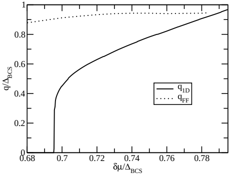

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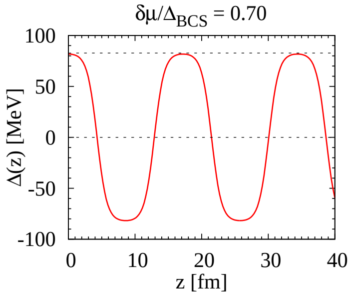

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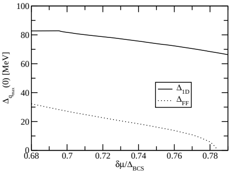

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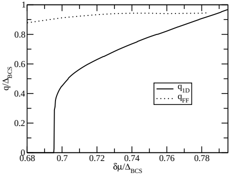

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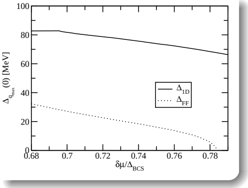

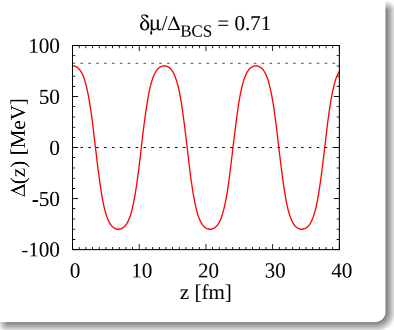

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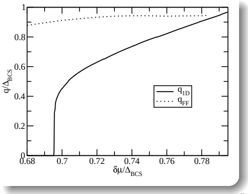

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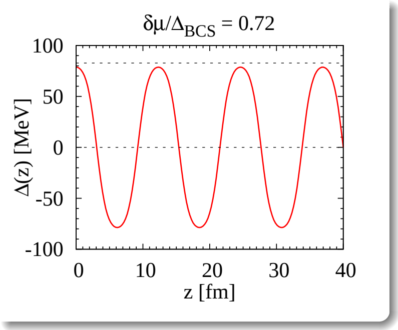

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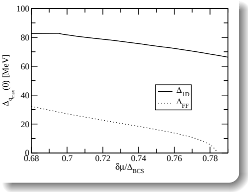

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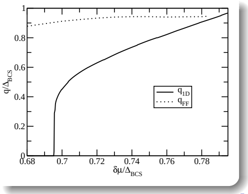

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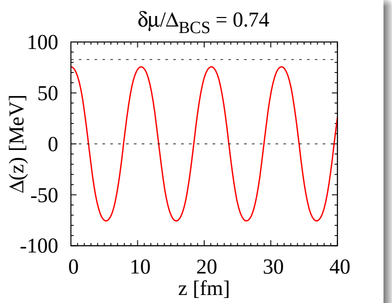

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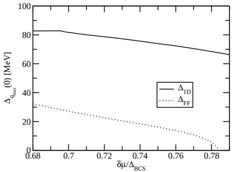

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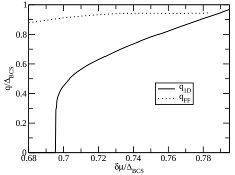

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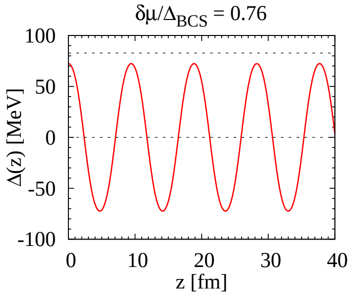

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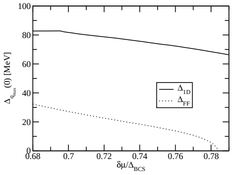

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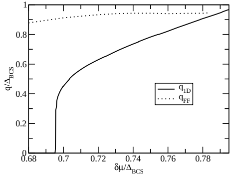

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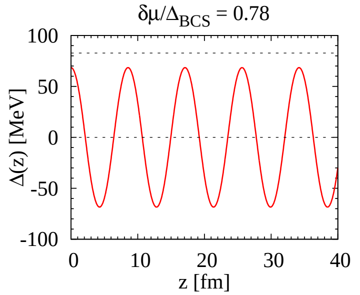

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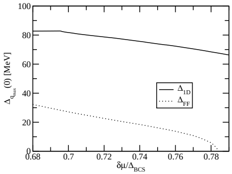

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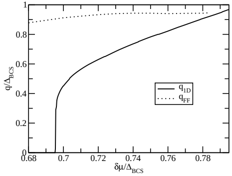

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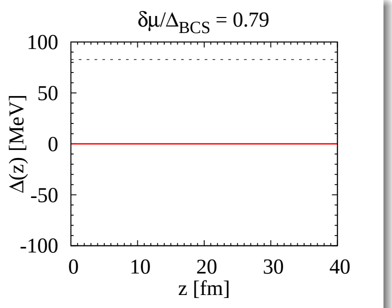

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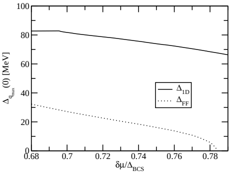

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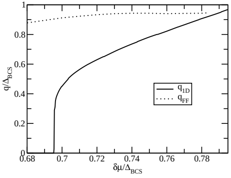
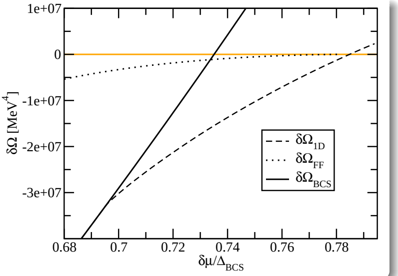


General one-dimensional solutions

- free-energy gain:



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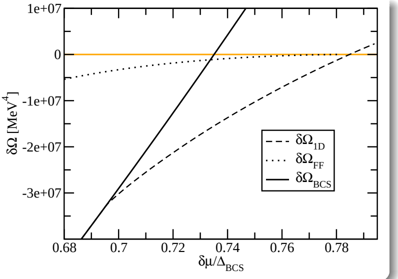

- LO window

  $\sim 2 \times$ FF window

Quasiparticle spectra

- anisotropic dispersion relations: $E_\lambda(\vec{k}) = E_\lambda(\vec{k}_\perp, k_z)$

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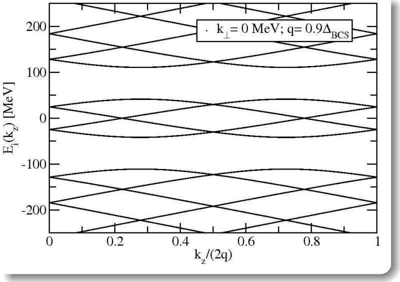
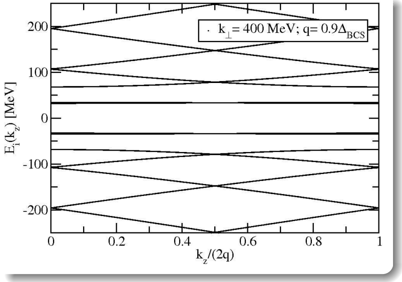
- typical examples at fixed $k_\perp$:

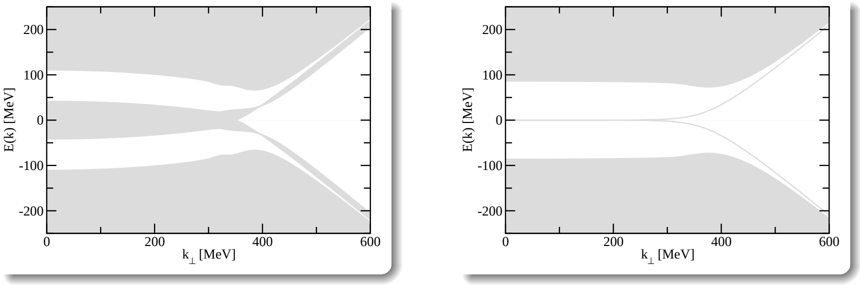




Band structure

- superposition of the eigenvalue spectra of all k_z:

sinusoidal ($q = 0.9\ \Delta_{BCS}$) soliton lattice ($q = 0.2\ \Delta_{BCS}$)



- "almost gapped" regions between low- and high-lying modes
- low-lying modes related to solitons: $q \to 0$ ➜ $E \to 0$

Outlook I

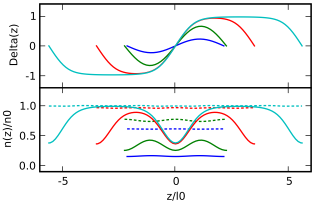
- to be done:
  - globally neutral two-flavor quark matter in beta equilibrium
  - density profiles

    (example:

    A. Bulgac, M.M. Forbes, PRL (2008),

    Density Functional Theory calculation for

    polarized Fermi systems at unitarity)



  - three flavors (including masses)
  - finite temperature, phase diagram
  - two- and three-dimensional crystals

Chiral phase transition (D. Nickel, arXiv:0902.1778, arXiv:0906.5295)

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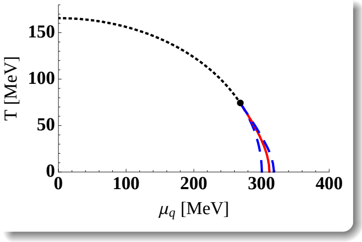

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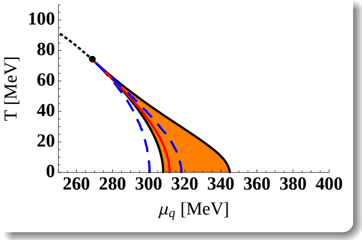

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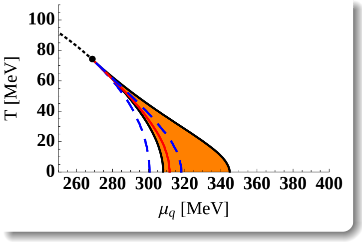

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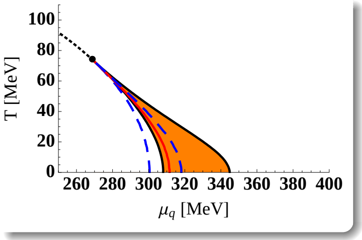

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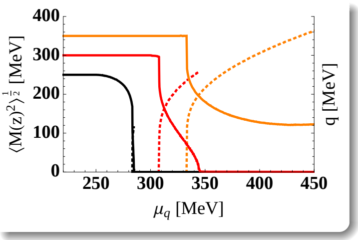

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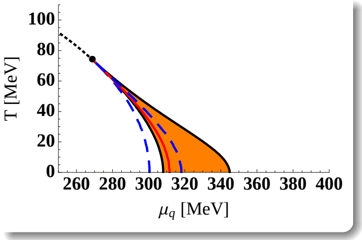

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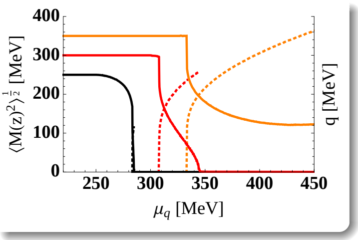

- 1st-order line covered by the inhomogeneous phase
- all phase boundaries 2nd order

Outlook II

- to be done:
  - include Polyakov loop
  - three flavors
  - two- and three-dimensional crystals
  - combine with color superconductivity