Three-Loop HTL Free Energy for QED

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Quark-Gluon Plasma meets Cold Atoms II

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FIAS

Frankfurt Institute for Advanced Studies

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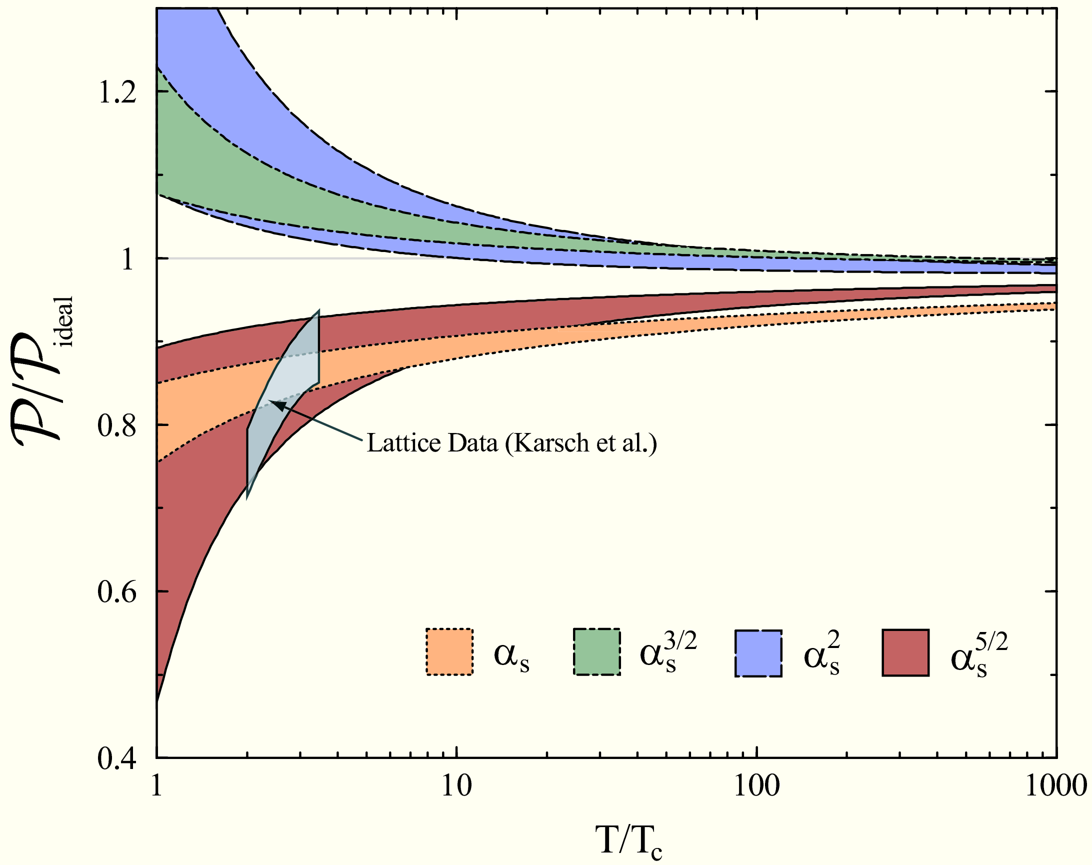
Introduction



Perturbative QCD free energy with $N_c = 3$ and $N_f = 2$ vs temperature. ($\pi T \le \mu \le 4\pi T$)
4-d lattice results from Karsch et al, 03.

(Here $\alpha_s = g_s^2/4\pi$)

- The weak-coupling expansion of the QCD free energy, $\mathcal{F}$, has been calculated to order $\alpha_s^3 \log \alpha_s$. [1,2,3,4]
- At temperatures expected at RHIC energies, $T \sim 0.3$ GeV, the running coupling constant $\alpha_s(2\pi T)$ is approximately 1/3, or $g_s \sim 2$.
- The successive terms contributing to $\mathcal{F}$ can strictly only form a decreasing series if $\alpha_s \lesssim 1/20$ which corresponds to $T \sim 10^5$ GeV.

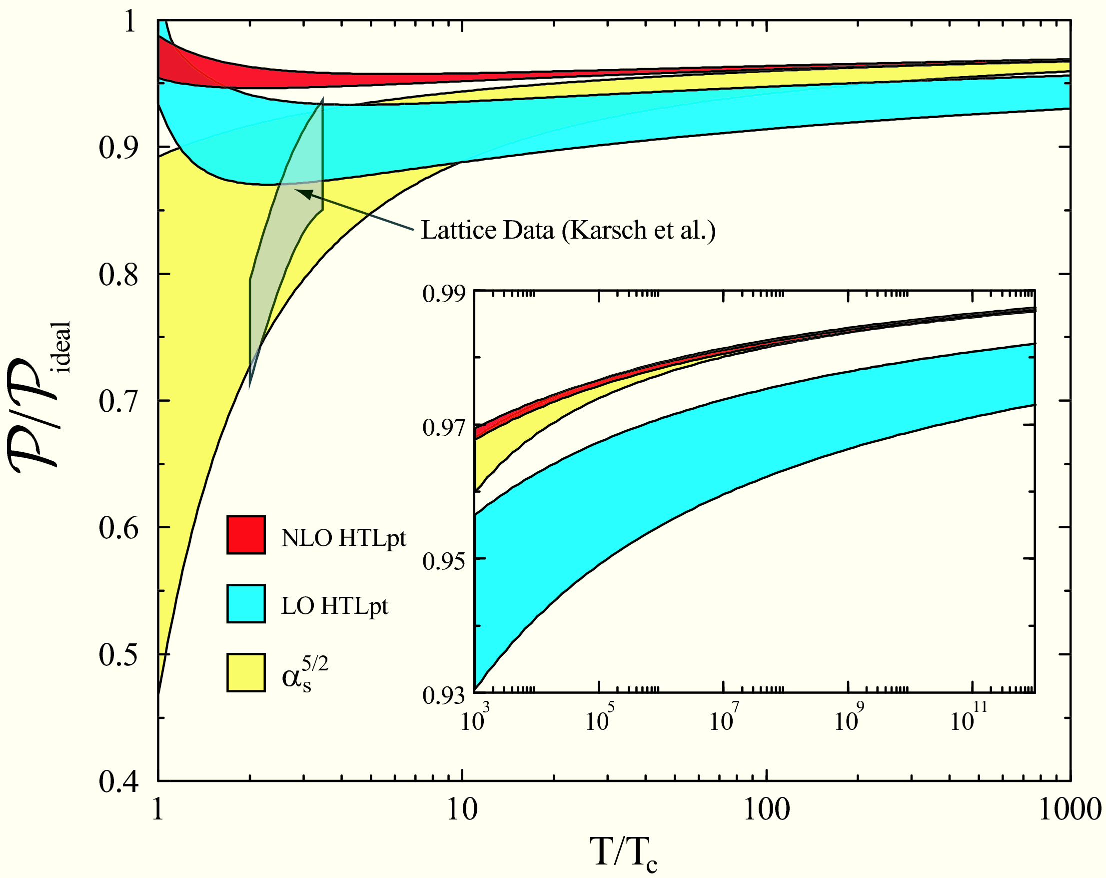
[1] Arnold and Zhai, 94/95.

[2] Kastening and Zhai, 95.

[3] Braaten and Nieto, 96.

[4] Kajantie, Laine, Rummukainen and Schröder, 02.

Introduction



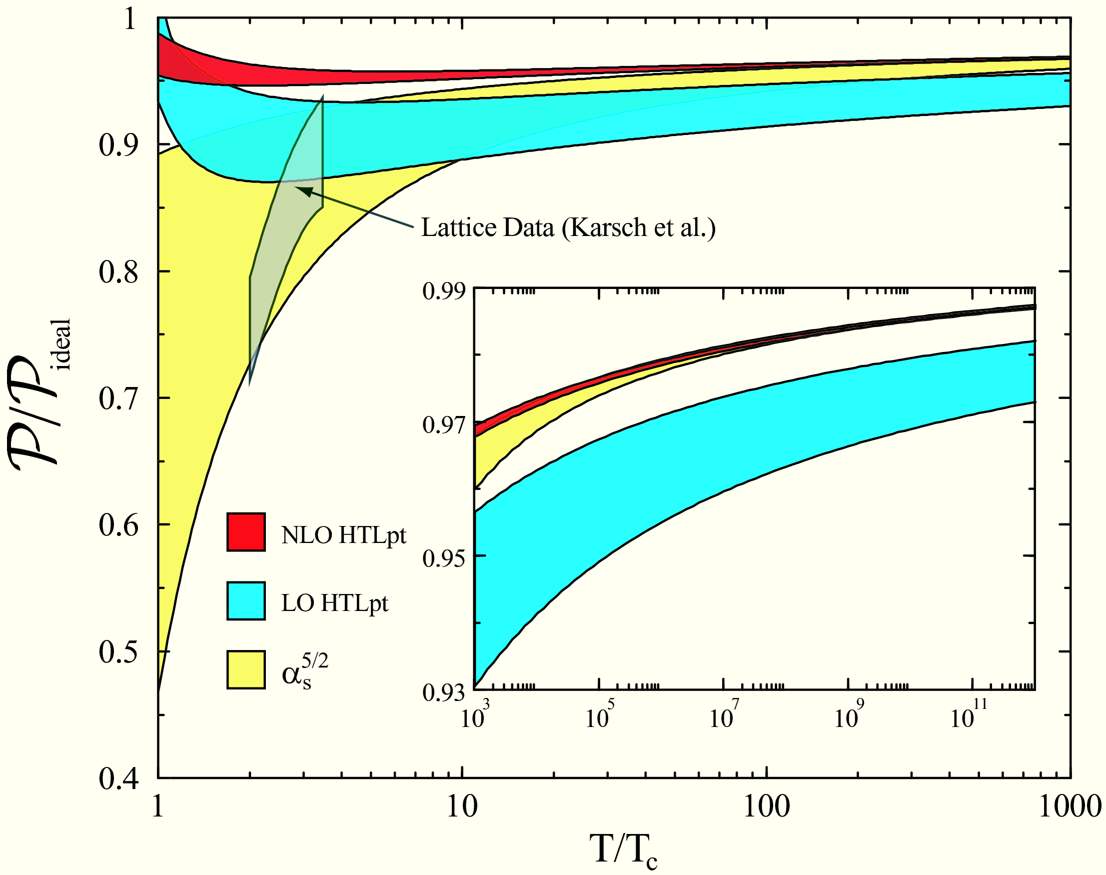
LO and NLO HTLpt free energy of QCD with $N_c = 3$ and $N_f = 2$ together with the perturbative prediction accurate to g^5.

- Hard-thermal-loop (HTL) perturbation theory [4,5] is a systematic, self-consistent and gauge-invariant reorganization of thermal quantum fields.
- Hard-thermal-loop perturbation theory is formulated in Minkowski space, therefore it is in principle possible to carry out real time calculations.
- Interested in $T > 2 - 3\,T_c$.

[4] Andersen, Braaten, Strickland, 99/99/99.

[5] Andersen, Braaten, Petitgirard, Strickland, 02; Andersen, Petitgirard, Strickland, 03.

But there is still work to do!



LO and NLO HTLpt free energy of QCD with $N_c = 3$ and $N_f = 2$ together with the perturbative prediction accurate to g^5.

- g^4 and g^5 terms can't be fully fixed at NLO. Some of them enter at NNLO. The result has the right magnitude, but the wrong sign.
- Running coupling effect doesn't enter at NLO. At this order, running coupling needs to be put by hand. Coupling constant renormalization enters at NNLO as well.

NNLO is needed!

Anharmonic Oscillator

- Consider the perturbation series for the ground state energy, E, of a simple anharmonic oscillator with potential

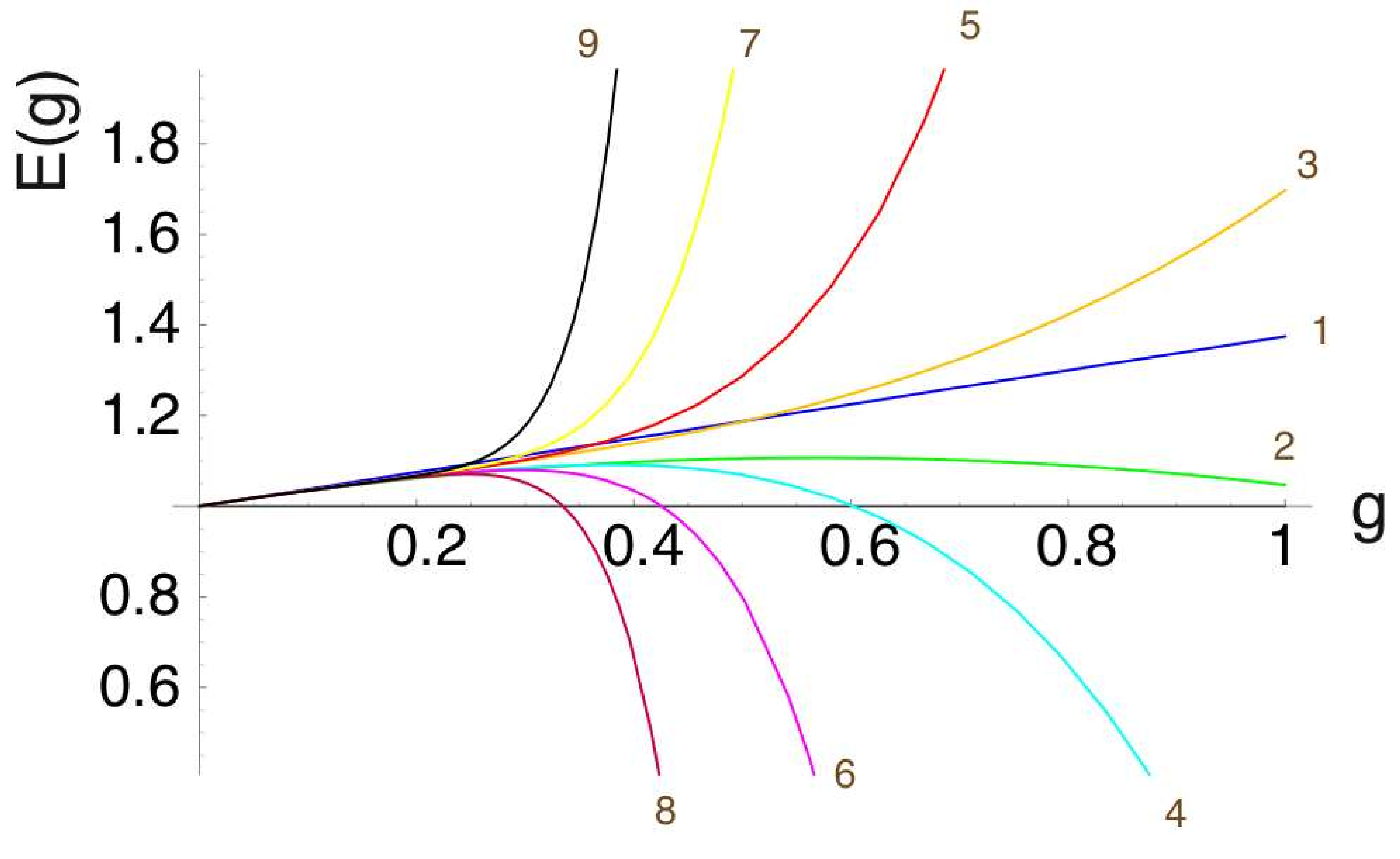
$$V(x) = \frac{1}{2}\omega^2 x^2 + \frac{g}{4}x^4 \qquad (\omega^2, g > 0)$$

- Weak-coupling expansion of the ground state energy $E(g)$ is known to all orders (Bender and Wu 69/73)

$$E(g) = \omega \sum_{n=0}^{\infty} c_n^{\mathrm{BW}} \left(\frac{g}{4\omega^3}\right)^n, \quad c_n^{\mathrm{BW}} = \left\{\frac{1}{2}, \frac{3}{4}, -\frac{21}{8}, \frac{333}{16}, -\frac{30885}{16}, \ldots\right\}$$

- $\lim_{n\to\infty} c_n^{\mathrm{BW}} = (-1)^{n+1}\sqrt{\frac{6}{\pi^3}}3^n(n-\frac{1}{2})!$
- Because of the factorial growth, the expansion is an asymptotic series with zero radius of convergence!

Anharmonic Oscillator



Variational Perturbation Theory (Janke and Kleinert 95/97)

- Split the harmonic term into two pieces and treat the second as part of the interaction

$$\omega^2 \to \Omega^2 + \left(\omega^2 - \Omega^2\right) \quad \Longrightarrow \quad E_N(g,r) = \Omega \sum_{n=0}^{N} c_n(r) \left(\frac{g}{4\Omega^3}\right)^n$$

where $r \equiv \frac{2}{g}\left(\omega^2 - \Omega^2\right)$

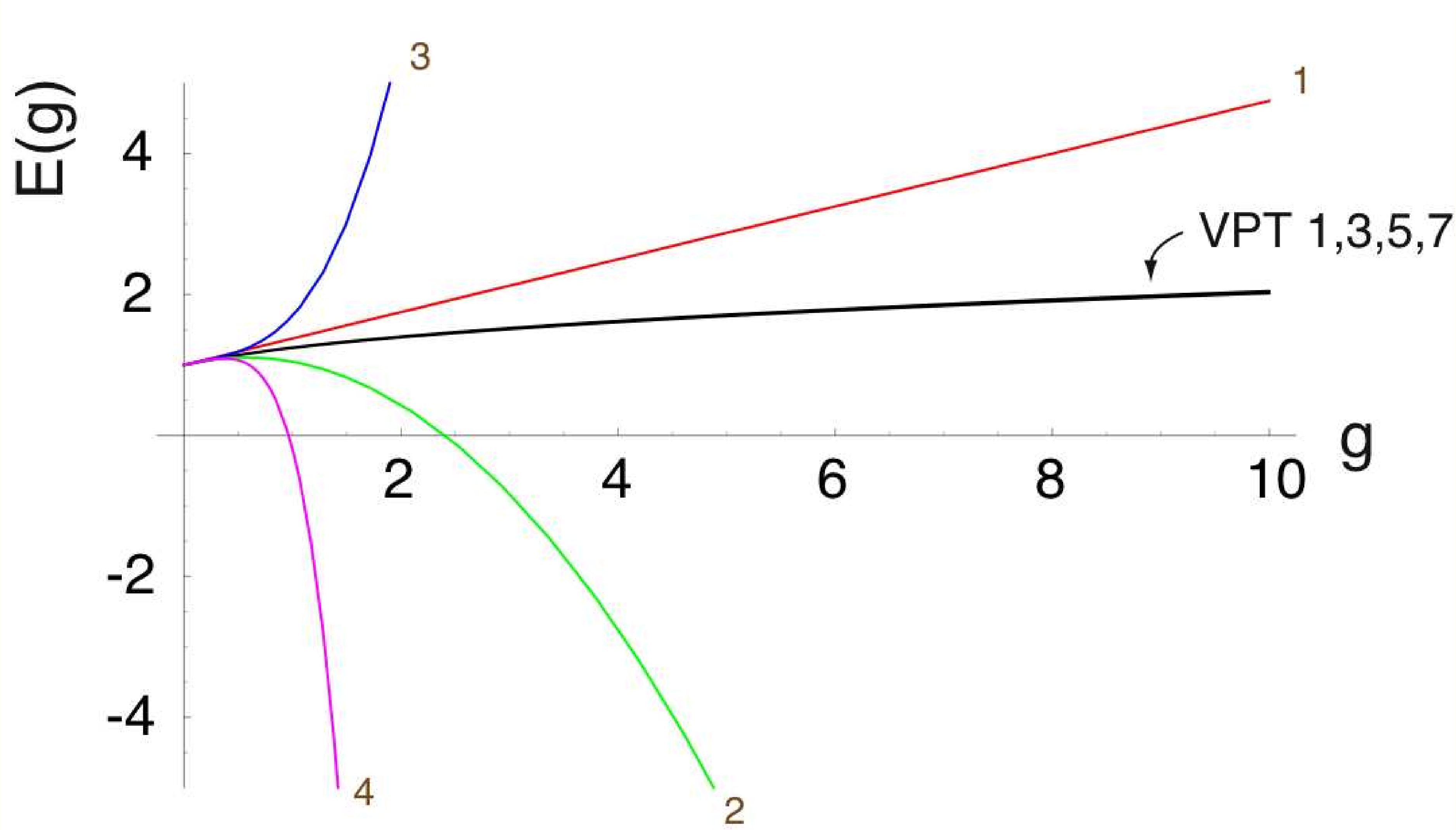
- The new coefficients c_n can be obtained by

$$c_n(r) = \sum_{j=0}^{n} c_j^{\mathrm{BW}} \binom{(1-3j)/2}{n-j} (2r\Omega)^{n-j}$$

- Fix Ω_N by requiring that at each order N

$$\left.\frac{\partial E_N}{\partial \Omega}\right|_{\Omega=\Omega_N} = 0$$

Variational Perturbation Theory



Hard-Thermal-Loop Perturbation Theory (HTLpt)

- Hard-thermal-loop perturbation theory is a reorganization of the perturbative series for QCD which is similar in spirit to variational perturbation theory

$$\mathcal{L}_{\rm HTLpt} = \left(\mathcal{L}_{\rm QCD} + \mathcal{L}_{\rm HTL}\right)\Big|_{g \to \sqrt{\delta} g} + \Delta\mathcal{L}_{\rm HTL}(g, m_D^2(1-\delta))$$

The HTL "improvement" term is

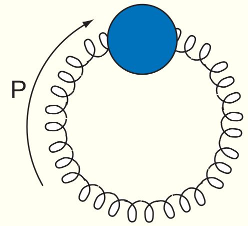
$$\mathcal{L}_{\rm HTL} = -\frac{1}{2}(1-\delta) m_D^2 \,{\rm Tr}\left(G_{\mu\alpha} \left\langle \frac{y^\alpha y^\beta}{(y \cdot D)^2} \right\rangle_y G^\mu{}_\beta \right)$$

where $\langle \cdots \rangle_y$ indicates angle average

HTLpt: 1-loop free energy for pure glue

- Separation into hard and soft contributions ($d = 3 - 2\epsilon$)

$$\mathcal{F}_g = -\frac{1}{2}\sum\!\!\!\!\!\!\!\int_P \{(d-1)\log[-\Delta_T(P)] \;+\; \log\Delta_L(P)\}$$



  - Hard momenta $(\omega, \mathbf{p} \sim T)$

$$\mathcal{F}_g^{(h)} = \frac{d-1}{2}\sum\!\!\!\!\!\!\!\int_P \log(P^2) + \frac{1}{2}m_D^2\sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2} - \frac{1}{4(d-1)}m_D^4\sum\!\!\!\!\!\!\!\int_P \left[\frac{1}{(P^2)^2} - 2\,\frac{1}{p^2P^2} - 2d\,\frac{1}{p^4}\mathcal{T}_P + 2\,\frac{1}{p^2P^2}\mathcal{T}_P + d\,\frac{1}{p^4}(\mathcal{T}_P)^2\right] + \mathcal{O}(m_D^6)$$

  - Soft momenta $(\omega, \mathbf{p} \sim gT)$

$$\mathcal{F}_g^{(s)} = \frac{1}{2}T\int_{\mathbf{p}} \log(p^2 + m_D^2)$$

HTLpt: 1-loop free energy for pure glue

- LO thermodynamical potential

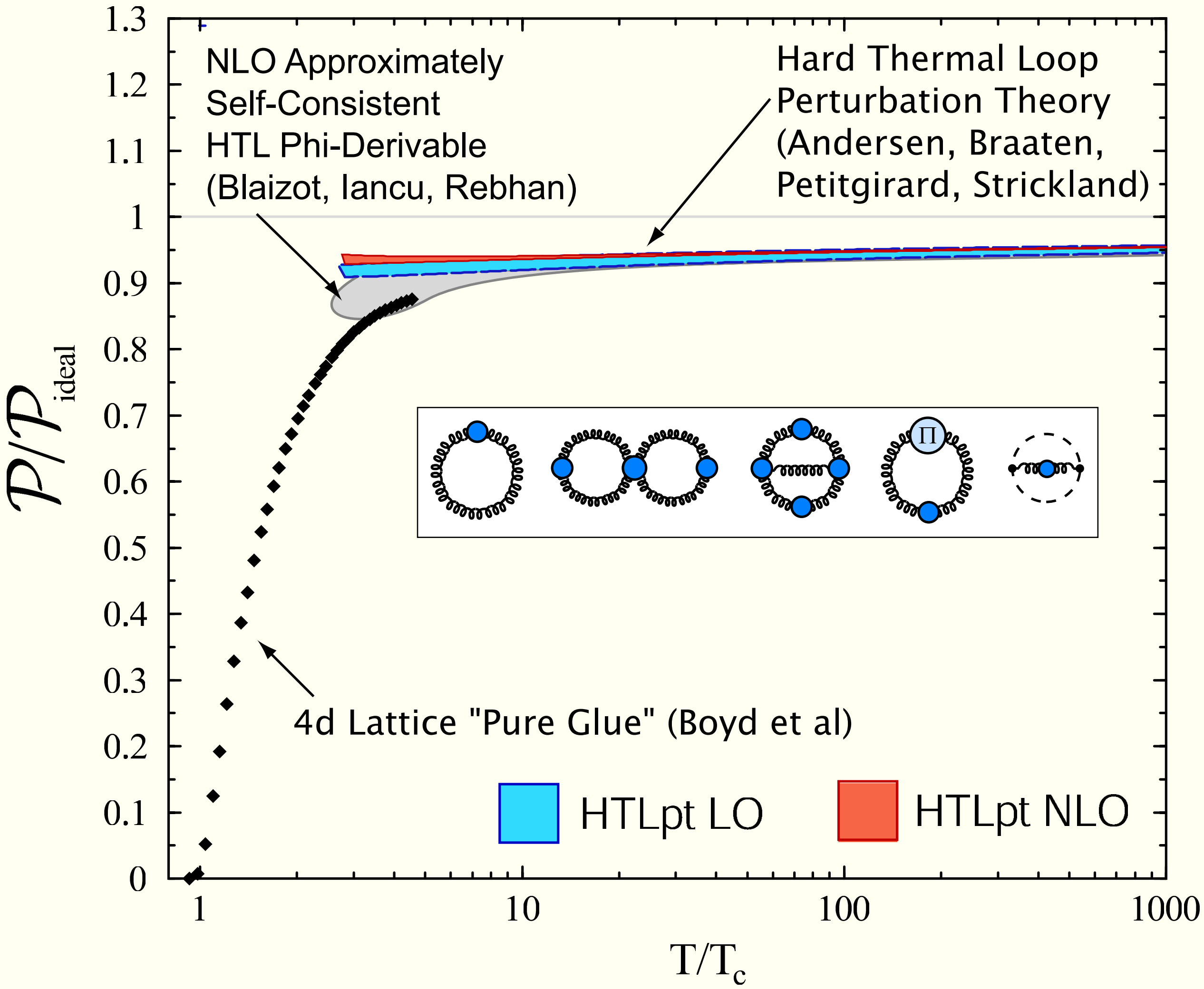
$$\frac{\Omega_{\rm LO}}{\mathcal{F}_{\rm ideal}} = 1 - \frac{15}{2}\hat{m}_D^2 + 30\hat{m}_D^3 + \frac{45}{4}\left(\log\frac{\hat{\mu}}{2} - \frac{7}{2} + \gamma + \frac{\pi^2}{3}\right)\hat{m}_D^4 + \mathcal{O}(\hat{m}_D^6),$$

where $\hat{m}_D = \frac{m_D}{2\pi T}$ and $\hat{\mu} = \frac{\mu}{2\pi T}$.

- The gap equation is not well-defined at LO (α_s does not appear above). However, we can get LO free energy by setting

$$m_D = gT.$$

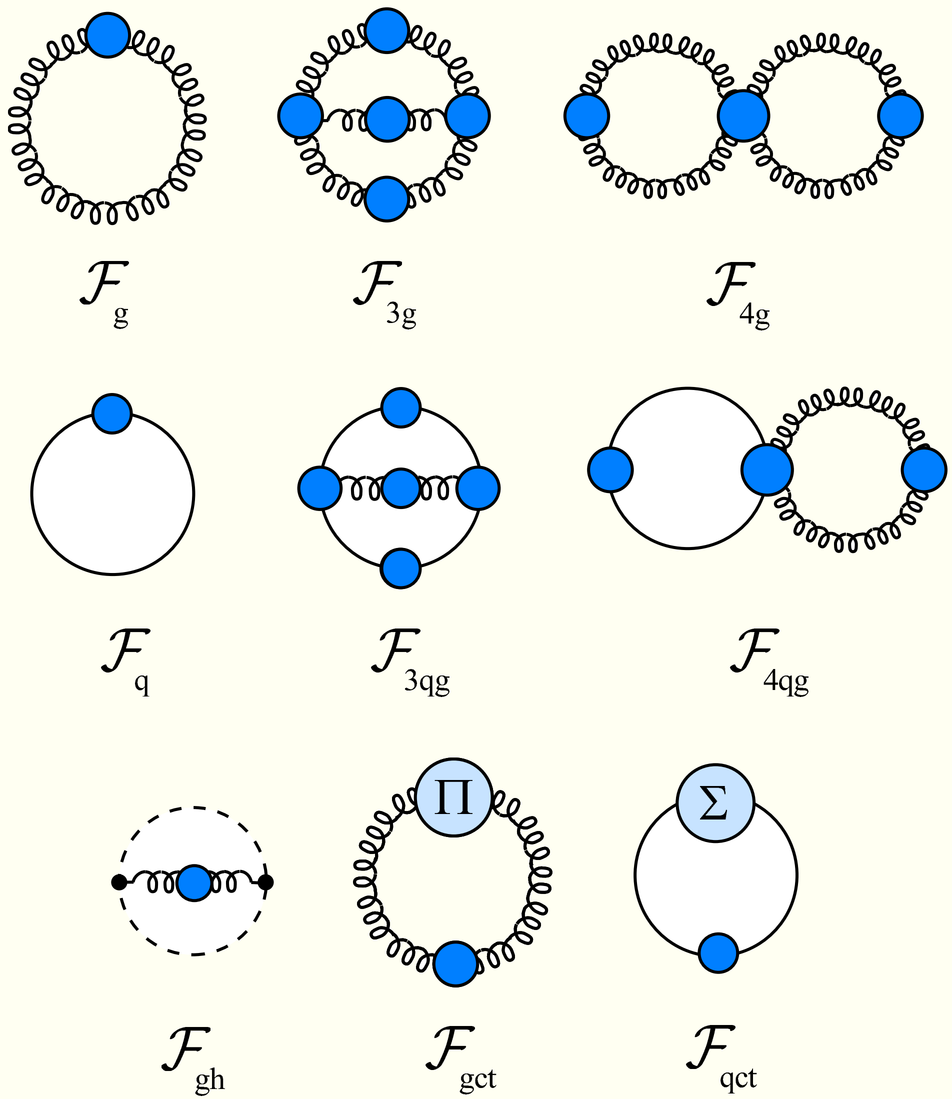
HTLpt: 1- and 2-loop free energy for pure glue



LO and NLO HTLpt free energy of pure glue vs temperature

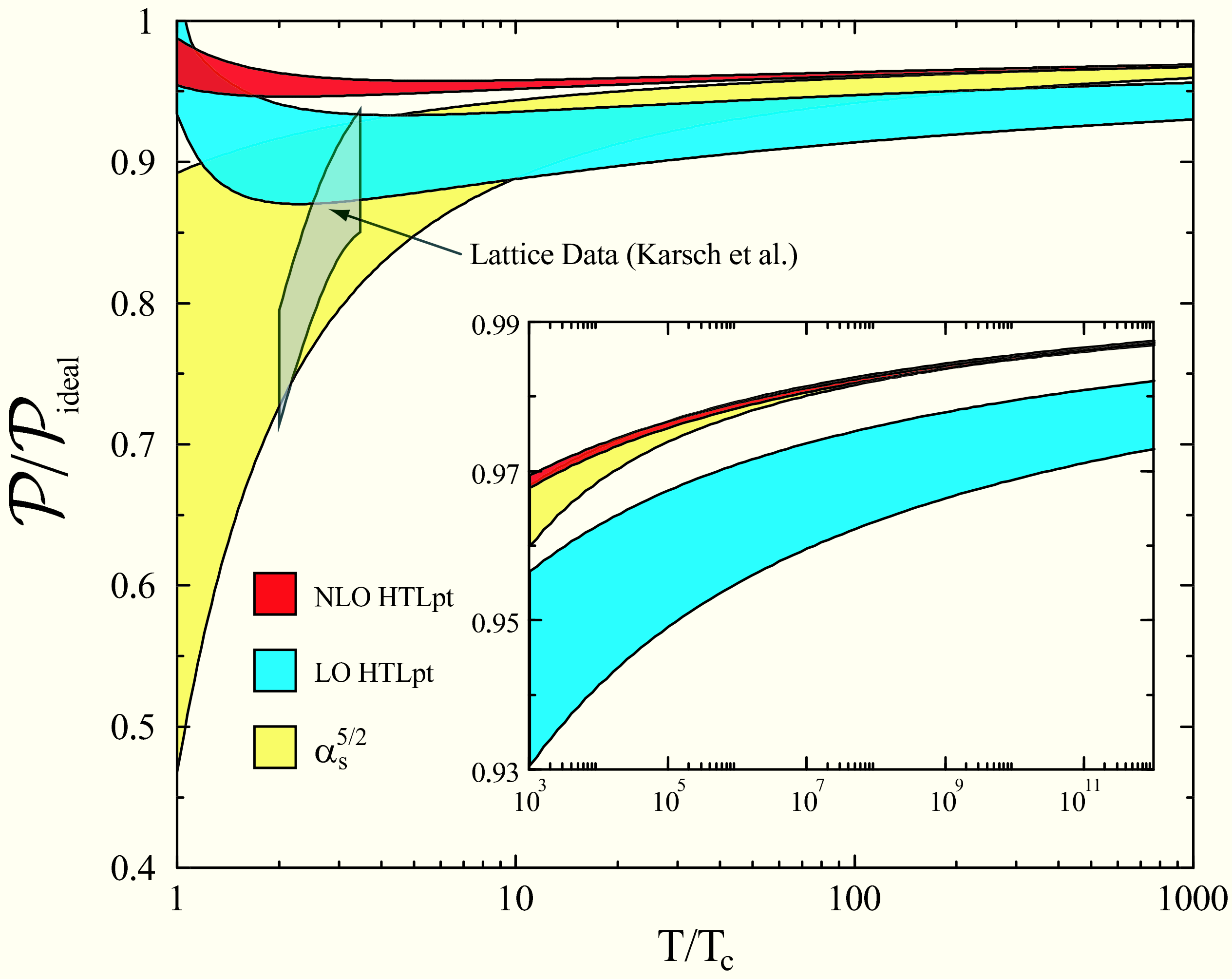
Andersen, Braaten, Petitgirard, Strickland, 02.

HTLpt: 1- and 2-loop diagrams for QCD



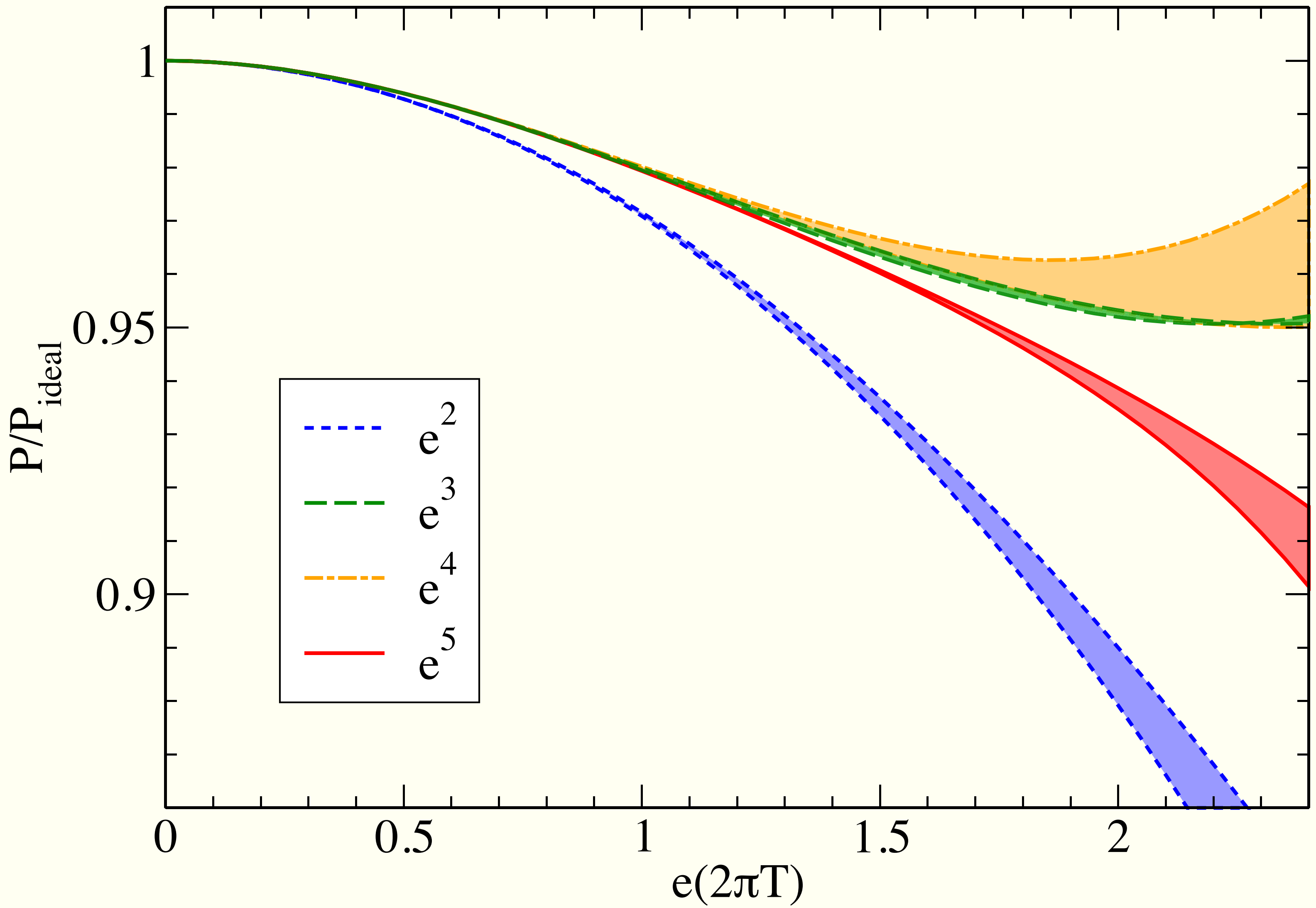
1- and 2-loop QCD diagrams contributing to HTLpt

HTLpt: 1- and 2-loop free energy for QCD



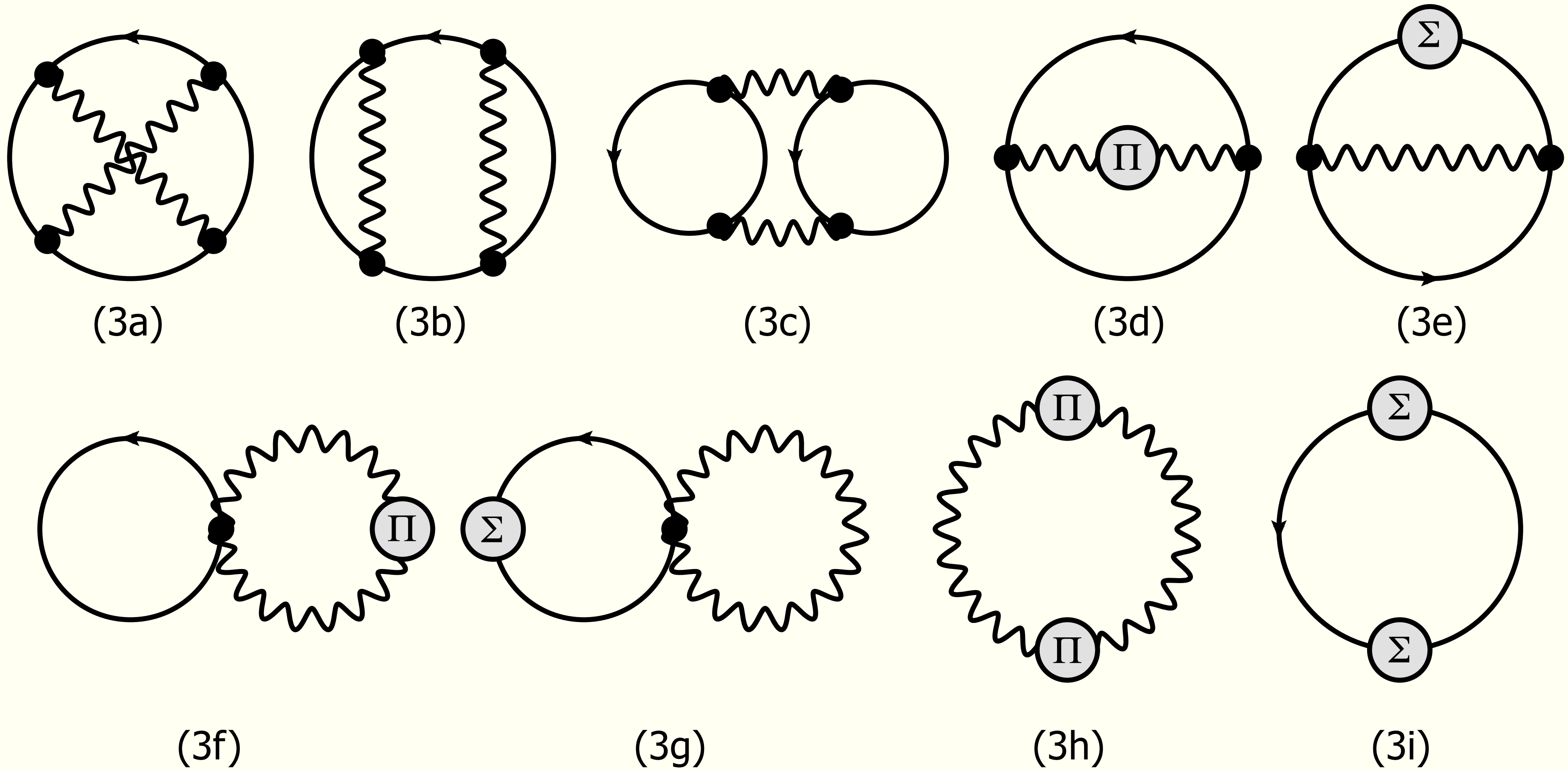
LO and NLO HTLpt free energy of QCD vs temperature
Andersen, Petitgirard, Strickland, 03.

HTLpt: naive pert. expansion of QED free energy



Perturbative QED free energy (Kastening and Zhai, 95)

HTLpt: 3-loop diagrams for QED



3-loop QED diagrams contributing to HTLpt

HTLpt: Counterterms

- The counterterms we need in the 3rd loop renormalization are

$$\begin{aligned}
\Delta\mathcal{E}_0 &= \frac{1}{128\pi^2\epsilon} m_D^4 \\
\Delta m_D^2 &= -N_f \frac{\alpha}{3\pi\epsilon} m_D^2 \\
\Delta m_f^2 &= \frac{3\alpha}{4\pi\epsilon} m_f^2 \\
\Delta\alpha &= N_f \frac{\alpha^2}{3\pi\epsilon} \quad \text{(same as zero T!)}
\end{aligned}$$

HTLpt: 3-loop thermodynamic potential for QED

- The NNLO thermodynamic potential reads

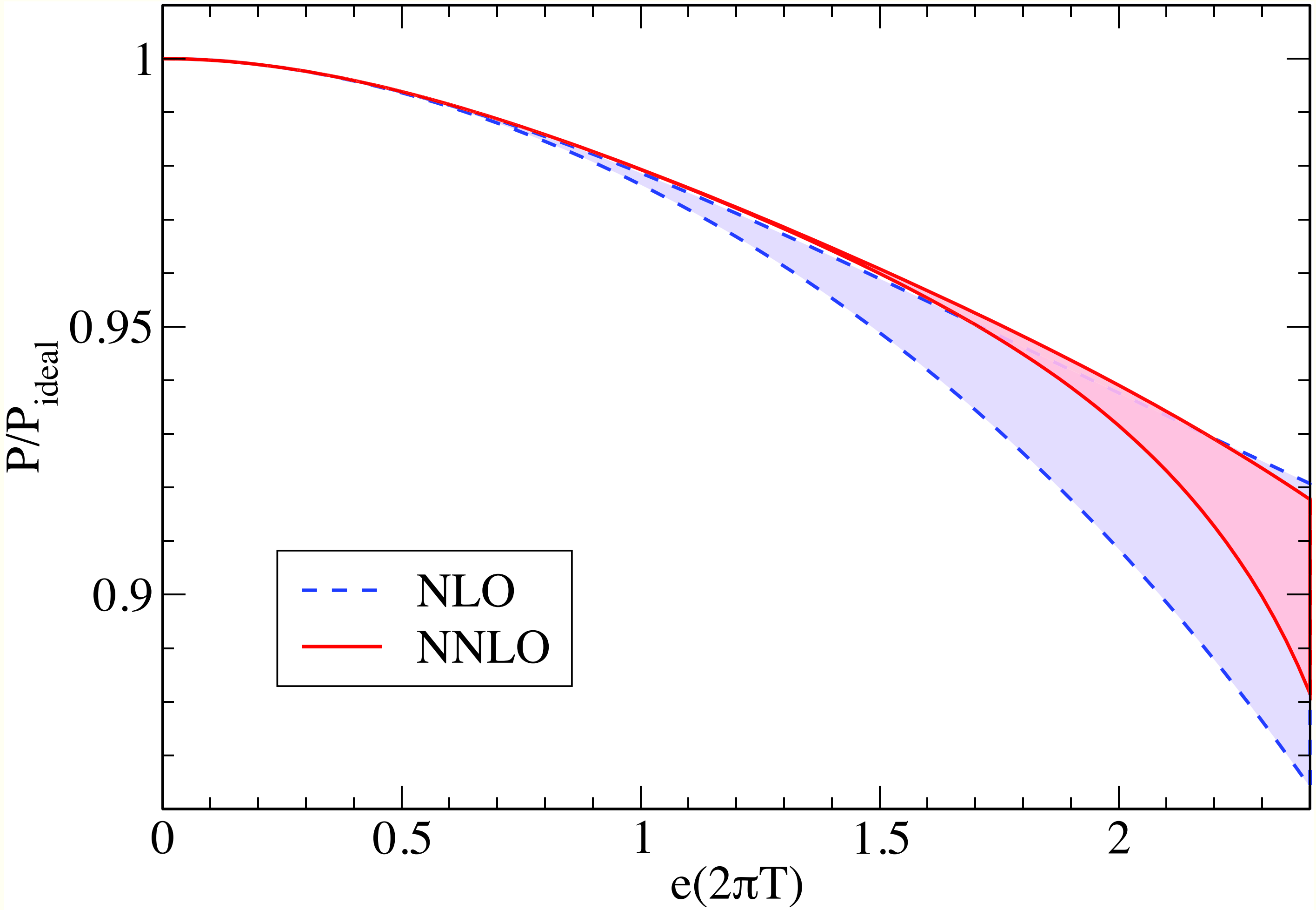
$$
\begin{aligned}
\Omega_{\rm NNLO} \quad = \quad & -\frac{\pi^2 T^4}{45}\Big\{1+\frac{7}{4}N_f-\frac{15}{4}\hat{m}_D^3 \\
& +N_f\frac{\alpha}{\pi}\Big[-\frac{25}{8}+\frac{15}{2}\hat{m}_D+15\Big(\log\frac{\hat{\mu}}{2}-\frac{1}{2}+\gamma+2\log 2\Big)\hat{m}_D^3-90\hat{m}_D\hat{m}_f^2\Big] \\
& +N_f\Big(\frac{\alpha}{\pi}\Big)^2\Big[\frac{15}{64}(35-32\log 2)-\frac{45}{2}\hat{m}_D\Big] \\
& +N_f^2\Big(\frac{\alpha}{\pi}\Big)^2\Big[\frac{25}{12}\Big(\log\frac{\hat{\mu}}{2}+\frac{1}{20}+\frac{3}{5}\gamma-\frac{66}{25}\log 2+\frac{4}{5}\frac{\zeta'(-1)}{\zeta(-1)}-\frac{2}{5}\frac{\zeta'(-3)}{\zeta(-3)}\Big) \\
& \qquad +\frac{5}{4}\frac{1}{\hat{m}_D}-15\Big(\log\frac{\hat{\mu}}{2}-\frac{1}{2}+\gamma+2\log 2\Big)\hat{m}_D\Big]\Big\}
\end{aligned}
$$

PURELY ANALYTIC!

- To eliminate the m_D and m_f dependence, the gap equations are imposed

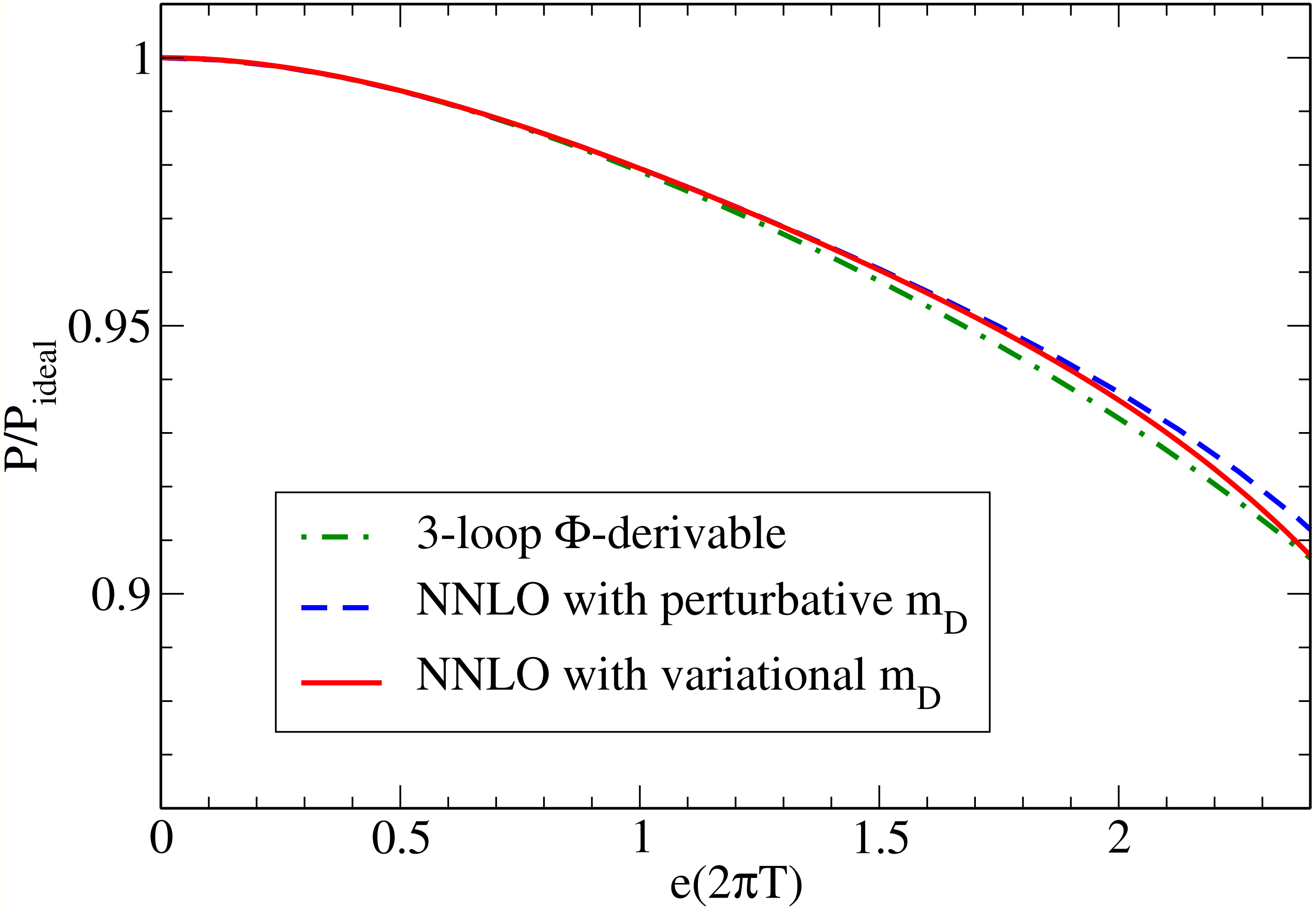
$$
\begin{aligned}
\frac{\partial}{\partial m_D}\Omega(T,\alpha,m_D,m_f,\delta=1) &= 0 \\
\frac{\partial}{\partial m_f}\Omega(T,\alpha,m_D,m_f,\delta=1) &= 0
\end{aligned}
$$

HTLpt: 2- and 3-loop free energy for QED



NLO and NNLO HTLpt predictions for QED free energy

HTLpt: comparison of different schemes



Comparison of three different predictions for QED free energy at $\mu = 2\pi T$

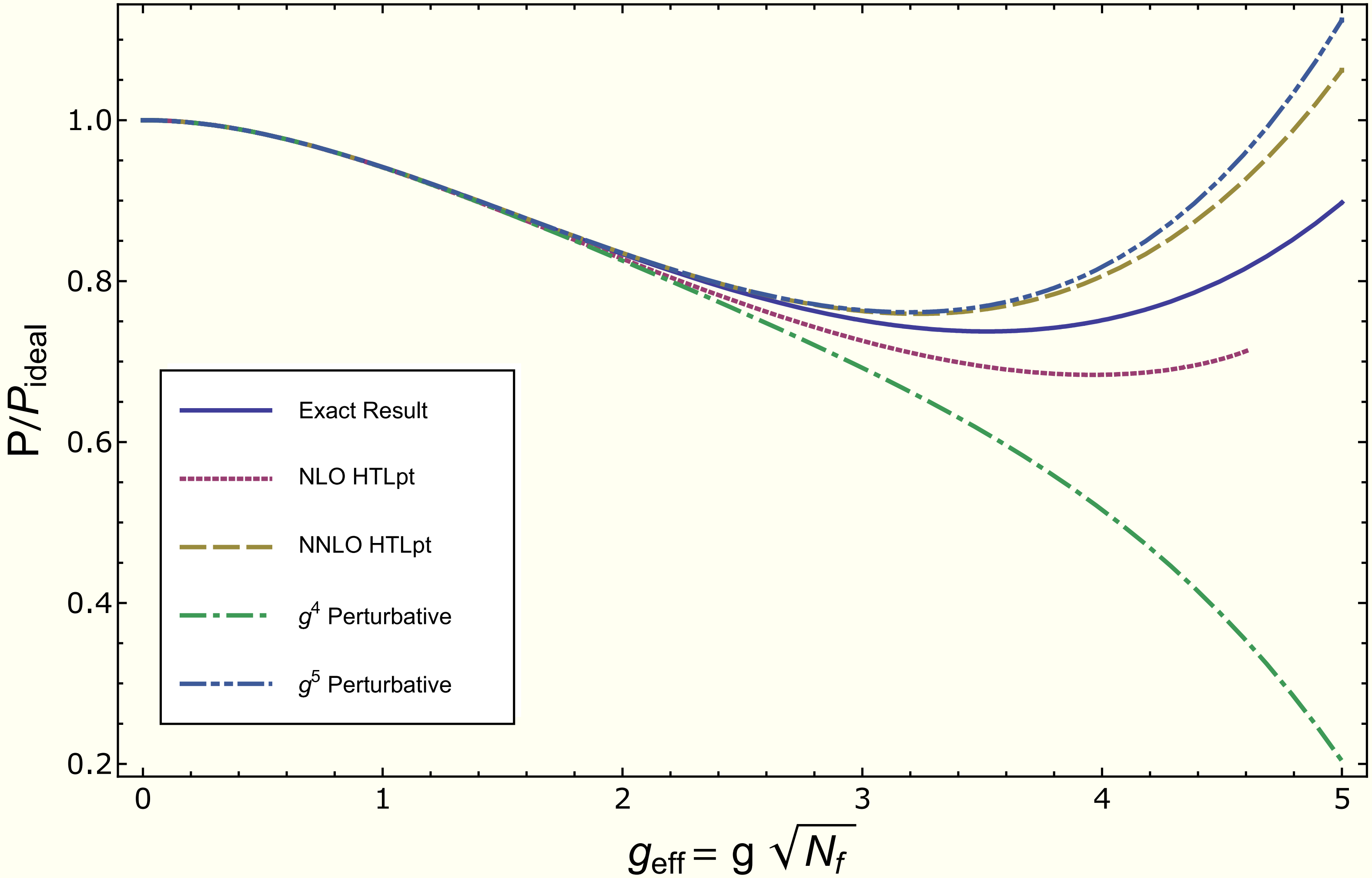
3-loop Φ-derivable result is taken from Andersen and Strickland, 05

Conclusions and Outlook

- The problem of bad convergence of weak-coupling expansion at finite temperature is generic.
- It does not just happen in gauge theories, but also in scalar theories, and even in quantum mechanics.
- Generalized from the idea of variational perturbation theory, hard-thermal-loop perturbation theory, which is formulated in Minkowski space, can improve the convergence of perturbative calculations in a gauge-invariant manner.
- By pushing forward, hopefully, hard-thermal-loop perturbation theory can provide a generic way towards a convergent gauge theory at high temperature, $T > 2 - 3\,T_c$.
- Once the NNLO QCD thermodynamics is obtained, running coupling fixed self-consistently, and m_D fixed via NNLO gap equation, we can begin to calculate dynamic quatities.

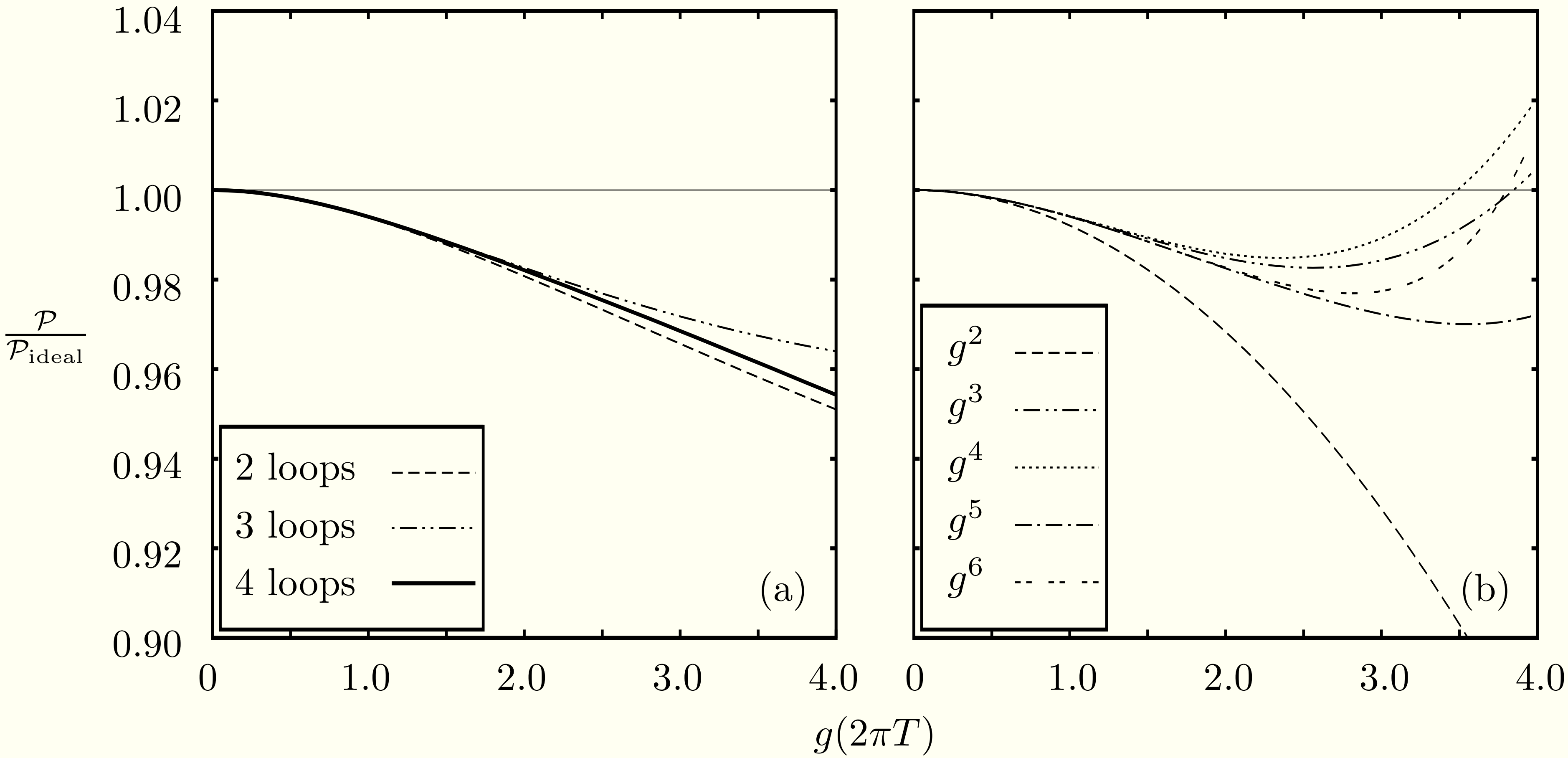
Back-up

Large-N Limit



QED free energy in the large N_f limit. The exact result is taken from Ipp, Moore and Rebhan 03.

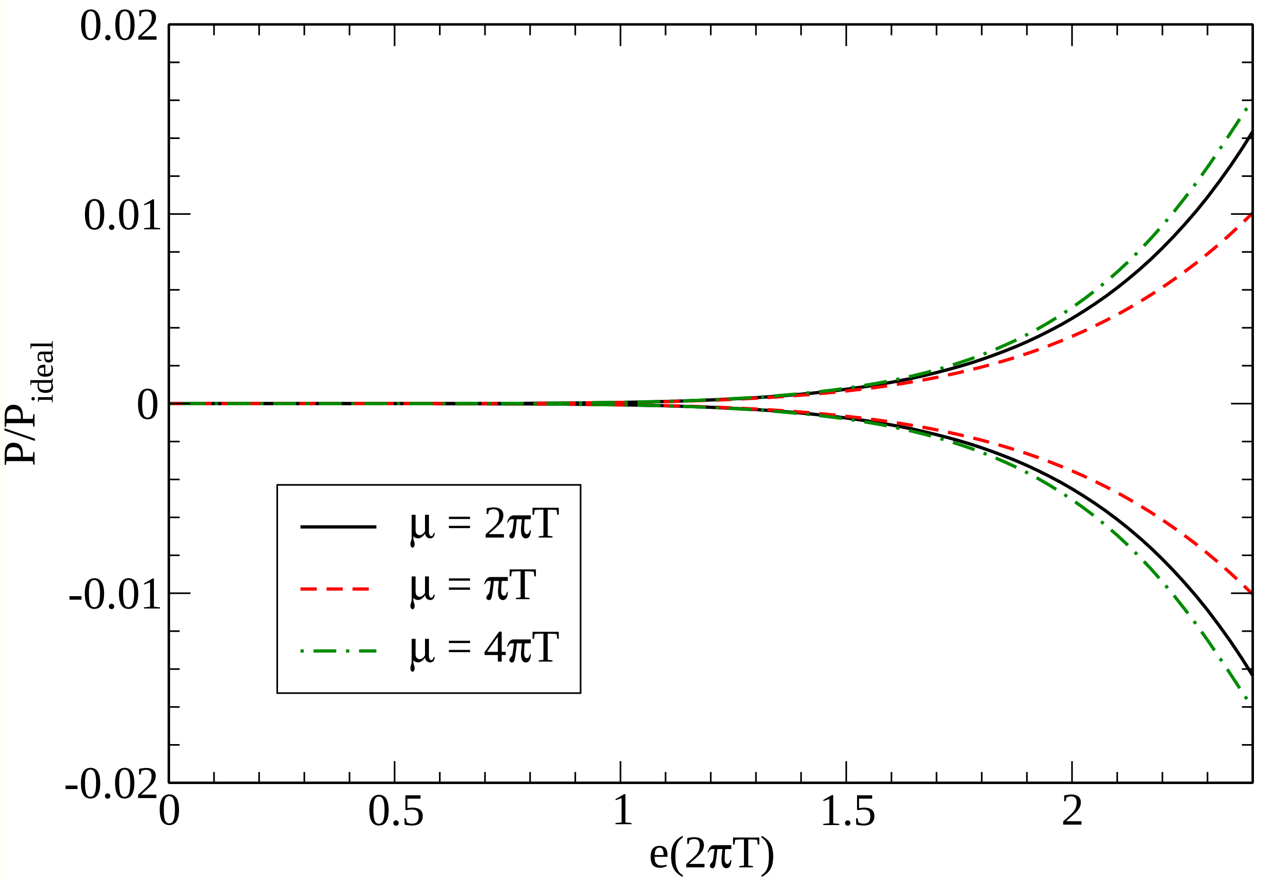
Screened Perturbation Theory



4-loop SPT pressure vs weak-couping pressure

Andersen, Braaten and Strickland, 00. Andersen and Strickland, 01.
Andersen and Kyllingstad, 08.

HTLpt: 3-loop free energy for QED



The imaginary part of NNLO HTLpt predictions for QED free energy

Weinberg and Wu, 87