

Space Charge from an Action Principle

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Relativistic Charged Particle– Electromagnetic Field Lagrangian

$$\begin{aligned}\mathcal{L} = \int d\mathbf{v}_0 d\mathbf{x}_0 \left\{ -mc^2 \sqrt{1 - \frac{1}{c^2} \left(\frac{\partial \mathbf{x}}{\partial t} \right)^2} + \right. \\ \left. -q\phi(\mathbf{x}) + \frac{q}{c} \frac{\partial \mathbf{x}}{\partial t} \cdot \mathbf{A} \right\} f(\mathbf{v}_0, \mathbf{x}_0) + \\ \frac{1}{8\pi} \int d\mathbf{x} \left(-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \right)^2 - (\nabla \times \mathbf{A})^2\end{aligned}$$

B. Shadwick, A. Stamm, & E. Evstatiev, “Variational formulation of macro-particle plasma simulation algorithms”, Phys. Plasmas **21**, 5 (2014).

F. Low, Proc. R. Soc. London Ser. A. Math.
Phys. Sci., **248** (1253) pp. 282–287 (1958).

(1) Change the independent variable

$$S = \int \mathcal{L} d\tau = \int \underbrace{\left(\frac{d\tau}{ds} \mathcal{L} \right)}_{\mathcal{L}} ds$$

Meaning

Now I'm tracking time-of-flight to a given point in the accelerators

$$\mathcal{L} = \int d\mathbf{q}_0 d\mathbf{q}'_0 \left\{ -mc^2 \sqrt{(\tau')^2 - (\mathbf{x}'_{\perp})^2} - 1 - \frac{q}{c} \frac{d\tau}{ds} \phi + \frac{q}{c} \mathbf{x}'_{\perp} \cdot \mathbf{A}_{\perp} + \frac{q}{c} A_s \right\} f$$

$$+ \frac{1}{8\pi} \int d\mathbf{x}_{\perp} d\tau \left(-\frac{\partial \mathbf{A}}{\partial \tau} - \nabla \phi \right)^2 - (\nabla \times \mathbf{A})^2$$

Approximation

Particles do not “turn around”; can't use this for a gun code

(2) Split off external & self-consistent fields

$$\mathbf{A} = \begin{pmatrix} A_x^{(ext.)} \\ A_y^{(ext.)} \\ A_s + \mathcal{A} \end{pmatrix}$$

Meaning

I only want to solve my self-consistent fields. I already know my magnets.

$$\begin{aligned} \mathcal{L} = & \int d\mathbf{q}_0 d\mathbf{q}'_0 \left\{ -mc^2 \sqrt{(\tau')^2 - (\mathbf{x}'_{\perp})^2} - 1 \right. \\ & \left. - \frac{q}{c} \frac{d\tau}{ds} \phi + \frac{q}{c} \mathbf{x}'_{\perp} \cdot \mathbf{A}_0 + \frac{q}{c} (A_s + \mathcal{A}) \right\} f \\ & + \frac{1}{8\pi} \int d\mathbf{x}_{\perp} d\tau \left(-\hat{e}_{\mathbf{z}} \frac{\partial \mathcal{A}}{\partial \tau} - \nabla \phi \right)^2 - (\nabla \times \hat{e}_{\mathbf{z}} \mathcal{A})^2 \end{aligned}$$

Approximation

No transverse self-consistent vector potential precludes synchrotron radiation, no electrostatic elements.

(3) Switch to deviation coordinates

$$z = s - \beta_0 \tau$$

Meaning

My beam is moving with the same velocity, which will make it easy to justify future approximations

$$\begin{aligned} \mathcal{L} = & \int d\mathbf{q}_0 d\mathbf{q}'_0 \left\{ -mc^2 \sqrt{\frac{1}{\beta_0^2} (1 - z')^2 - (\mathbf{x}'_{\perp})^2} - 1 \right. \\ & \left. - \frac{q}{c} \frac{1}{\beta_0} (1 - z') \phi + \frac{q}{c} \mathbf{x}'_{\perp} \cdot \mathbf{A}_0 + \frac{q}{c} (A_s + \mathcal{A}) \right\} f \\ & + \frac{1}{8\pi} \int d\mathbf{x}_{\perp} d\tau \left(-\hat{e}_{\mathbf{z}} \frac{\partial \mathcal{A}}{\partial \tau} - \nabla \phi \right)^2 - (\nabla \times \hat{e}_{\mathbf{z}} \mathcal{A})^2 \end{aligned}$$

Approximation

No transverse self-consistent vector potential precludes synchrotron radiation, no electrostatic elements.

(4) Assume that z' is “small”

$$z = s - \beta_0 \tau$$

Meaning

All the particles really
are moving at the same
speed

$$\begin{aligned} \mathcal{L} = & \int d\mathbf{q}_0 d\mathbf{q}'_0 \left\{ -mc^2 \sqrt{\frac{1}{\beta_0^2} (1 - z')^2 - (\mathbf{x}'_{\perp})^2} - 1 \right. \\ & \left. - \frac{q}{c} \frac{1}{\beta_0} \phi + \frac{q}{c} \mathbf{x}'_{\perp} \cdot \mathbf{A}_0 + \frac{q}{c} (A_s + \mathcal{A}) \right\} f \\ & + \frac{1}{8\pi} \int d\mathbf{x}_{\perp} d\tau \left(-\hat{e}_{\mathbf{z}} \frac{\partial \mathcal{A}}{\partial \tau} - \nabla \phi \right)^2 - (\nabla \times \hat{e}_{\mathbf{z}} \mathcal{A})^2 \end{aligned}$$

Approximation

No pesky self-
consistent vector-
potential-like terms
with the electrostatic
potential going forward

(5) Apply a quasi-static approximation to the fields

$$\{\mathcal{A}, \phi\} = \{\mathcal{A}, \phi\}(x, y, z = s - \beta_0 \tau)$$

Meaning

All the fields are co-moving with the sources

$$\begin{aligned} \mathcal{L} = \int d\mathbf{q}_0 d\mathbf{q}'_0 \bigg\{ & -mc^2 \sqrt{\frac{1}{\beta_0^2} (1 - z')^2 - (\mathbf{x}'_{\perp})^2} - 1 \\ & - \frac{q}{c} \frac{1}{\beta_0} \phi + \frac{q}{c} \mathbf{x}'_{\perp} \cdot \mathbf{A}_0 + \frac{q}{c} (A_s + \mathcal{A}) \bigg\} f \\ & + \frac{1}{8\pi} \int d\mathbf{x}_{\perp} d\tau (\beta_0 \partial_z \mathcal{A} - \partial_z \phi)^2 \\ & + (\partial_x \phi)^2 + (\partial_y \phi)^2 - (\partial_x \mathcal{A})^2 - (\partial_y \mathcal{A})^2 \end{aligned}$$

Approximation

REALLY can't have radiation now

May I have a Hamiltonian yet?

No, because of the pesky scalar potentials

$$\mathcal{P} = \frac{\delta \mathcal{L}}{\delta \partial_s \mathcal{A}} = 0$$

$$\Upsilon = \frac{\delta \mathcal{L}}{\delta \partial_s \phi} = 0$$

Degenerate Lagrangian
i.e. the self-consistent
field dynamics are
determined by a
constraint equation, not
an equation of motion.

(6) Insert a “gauge fixing term” for the scalar potential

$$\partial_i \frac{\delta \mathcal{L}}{\delta \partial_i \phi} - \frac{\delta \mathcal{L}}{\delta \phi} = 0$$

Meaning

Need to eliminate the scalar potential using what amounts to a constraint

$$\nabla_{\perp}^2 \phi - \partial_z (\beta_0 \partial_z \mathcal{A} - \partial_z \phi) + 4\pi \frac{q}{c} \frac{1}{\beta_0} \int d\mathbf{q}' f = 0$$

$$-\nabla_{\perp}^2 \mathcal{A} + \beta_0 \partial_z (\beta_0 \partial_z \mathcal{A} - \partial_z \phi) - 4\pi \frac{q}{c} \int d\mathbf{q}' f = 0$$

Approximation

Nope, this is exact

(6 cont'd.) One particular combination of the scalar potentials is relevant

$$\psi = \frac{1}{\beta_0} \phi - \mathcal{A}$$

$$-\nabla_{\perp}^2 \psi - (1 - \beta_0^2) \partial_z^2 \psi - 4\pi \frac{q}{c} \left(1 - \frac{1}{\beta_0^2} \right) \int d\mathbf{q}' f = 0$$

$$\nabla_{\perp}^2 \psi + \frac{1}{\gamma_0^2} \partial_z^2 \psi = 4\pi \frac{q}{c} \left(\frac{1}{\beta_0^2 \gamma_0^2} \right) \int d\mathbf{q}' f$$

(6) Now I can have a Hamiltonian

$$\mathcal{H} = p_z + \sqrt{\beta_0^2 p_z^2 + \left(\mathbf{p}_\perp - \frac{q}{c} \mathbf{A}_0\right)^2 + m^2 c^2} - \frac{q}{c} A_s + \frac{q}{c} \psi$$

with the auxiliary condition on the scalar potential

$$\nabla_\perp^2 \psi + \frac{1}{\gamma_0^2} \partial_z^2 \psi = 4\pi \frac{q}{c} \left(\frac{1}{\beta_0^2 \gamma_0^2} \right) \int d\mathbf{q}' f$$

Thank you!

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