Dispersion and space charge in circular accelerators

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Space charge workshop, Darmstadt, 5 Oct. 2017

Introduction

Previous work
- Generalized invariant 2D emittance with dispersion
- The well-known transverse envelope equation and instability
- Generalized envelope equations with dispersion

This work
- Stability analysis including dispersion → "dispersion mode"
- Envelope instability induced by space charge dispersion and the focusing structure

Relevant for: Bunch compression at FAIR, Recirculating ERLs

Next step: Measurement of the dispersion mode

Previous work: Invariant emittance with dispersion

- Generalized invariant emittance

Ref.[1]: S. Y. Lee and H. Okamoto, PRL. 80, 5133 (1998).
Ref.[2]: M. Venturini and M. Reiser, PRL. 81, 96 (1998).

Hamiltonian with dispersion and space charge:

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{\kappa_{x0}(s)}{2}x^2 + \frac{\kappa_{y0}(s)}{2}y^2 + \frac{m^2c^4}{E_0^2}\delta^2 \boxed{-\frac{x}{\rho(s)}\delta} + V_{\rm sc}(x, y, s)$$

Dispersion term

With coordinate transformation:

$$\begin{cases} x = \bar{x} + \delta D_x, \;\; p_x = \bar{p}_x + \delta D'_x, \\ y = \bar{y}, \;\; p_y = \bar{p}_y \end{cases}$$

$$\bar{H} = \frac{1}{2}(\bar{p}_x^2 + \bar{p}_y^2) + \frac{\kappa_{x0}(s)}{2}\bar{x}^2 + \frac{\kappa_{y0}(s)}{2}\bar{y}^2 + \frac{m^2c^4}{E_0^2}\delta^2 + V_{\rm sc}$$

Generalized emittance: $\varepsilon_{dx} = \sqrt{\langle\bar{x}^2\rangle\langle\bar{x}'^2\rangle - \langle\bar{x}\bar{x}'\rangle^2}$

With dispersion sc is no longer linear ! So this a simplification. However, also for nonl. sc a matched solution might exist.

Previous work: envelope instability

- The well-known envelope equations:

Ref.: J. Struckmeier and M. Reiser, Part. Accel. 14, 227 (1984)

$$\frac{\mathrm{d}^2\sigma_x}{\mathrm{d}s^2} + \left[\kappa_{x0}(s) - \frac{K_{\mathrm{sc}}}{2\sigma_x(\sigma_x+\sigma_y)}\right]\sigma_x - \frac{\varepsilon_x^2}{\sigma_x^3} = 0,$$

$$\frac{\mathrm{d}^2\sigma_y}{\mathrm{d}s^2} + \left[\kappa_{y0}(s) - \frac{K_{\mathrm{sc}}}{2\sigma_y(\sigma_x+\sigma_y)}\right]\sigma_y - \frac{\varepsilon_y^2}{\sigma_y^3} = 0$$

$\sigma_{x,y}$ rms beam size

K_{sc} perveance

- Envelope oscillation are characterized by
  - breathing mode ϕ_1
  - quadrupole mode ϕ_2

- Envelope instability or 90^0 stopband for periodic focusing

Criteria for instability: $k_{0,x,y} > 90^\circ$

Recent ref.[1] I. Hofmann and O. Boine-Frankenheim, Phys. Rev. Lett. 115, 204802 (2015).
[2] L. Groening, et.al. , Phys. Rev. Lett. 102, 234801 (2009)
[3] S. M. Lund and B. Bukh, Phys. Rev. ST Accel. Beams 7, 024801 (2004)

Previous work: Generalized envelope equations

- Based on the generalized emittance,

$$\sigma_x'' + \left[\kappa_{x0}(s) - \frac{K_{\mathrm{sc}}}{2\sigma_x(\sigma_x+\sigma_y)}\right]\sigma_x - \frac{\varepsilon_x^2}{\sigma_x^3} = 0$$

$$\sigma_y'' + \left[\kappa_{y0}(s) - \frac{K_{\mathrm{sc}}}{2\sigma_y(\sigma_x+\sigma_y)}\right]\sigma_y - \frac{\varepsilon_y^2}{\sigma_y^3} = 0$$

$$\Rightarrow \begin{cases} \sigma_x'' + \left[\kappa_{x0}(s) - \dfrac{K_{\mathrm{sc}}}{2X(X+Y)}\right]\sigma_x - \dfrac{\varepsilon_{dx}^2}{\sigma_x^3} = 0 \\ \sigma_y'' + \left[\kappa_{y0}(s) - \dfrac{K_{\mathrm{sc}}}{2Y(X+Y)}\right]\sigma_y - \dfrac{\varepsilon_{dy}^2}{\sigma_y^3} = 0 \end{cases}$$

with

Total beam size $X = \sqrt{\sigma_x^2 + \sigma_\delta^2 D_x^2}$ and $Y = \sigma_y$

Betatron beam size *Dispersion beam size*

- Space-charge-modified dispersion:

$$D_x'' + \left[\kappa_{x0}(s) - \frac{K_{\mathrm{sc}}}{2X(X+Y)}\right]D_x = \frac{1}{\rho(s)}$$

Ref. [1] S. Cousineau, *et.al*, PRSTAB 6, 034205 (2003).

[2] J. A. Holmes, *et.al*, PRSTAB 2, 114202 (1999).

Envelope oscillations with dispersion

- We performed perturbations on matched solution:

$$\begin{cases} \sigma_x'' + \left[\kappa_{x0}(s) - \frac{K_{\mathrm{sc}}}{2X(X+Y)}\right]\sigma_x - \frac{\varepsilon_{dx}^2}{\sigma_x^3} = 0 \\ \sigma_y'' + \left[\kappa_{y0}(s) - \frac{K_{\mathrm{sc}}}{2Y(X+Y)}\right]\sigma_y - \frac{\varepsilon_{dy}^2}{\sigma_y^3} = 0 \\ D_x'' + \left[\kappa_{x0}(s) - \frac{K_{\mathrm{sc}}}{2X(X+Y)}\right]D_x = \frac{1}{\rho(s)} \end{cases} \Leftarrow \begin{cases} \sigma_x = \sigma_{x0} + \xi, \\ \sigma_y = \sigma_{y0} + \zeta \\ D_x = D_{x0} + d_x \\ X = X_0 + \xi\sigma_{x0}/X_0 + \sigma_\delta^2 D_{x0} d_x / X_0 \end{cases}$$

Matched solution (σ_{x0}, σ_{y0}, D_{x0}, X_0) — *perturbations* (ξ, ζ, d_x, $\xi\sigma_{x0}/X_0 + \sigma_\delta^2 D_{x0} d_x / X_0$)

Then we obtain oscillation equations:

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\begin{pmatrix}\xi \\ \zeta \\ d_x\end{pmatrix} = \begin{pmatrix} -a_0 & -a_1 & -a_2 \\ -a_1 & -a_3 & -a_4 \\ -\frac{a_2}{\sigma_\delta^2} & -\frac{a_4}{\sigma_\delta^2} & -a_5 \end{pmatrix}\begin{pmatrix}\xi \\ \zeta \\ d_x\end{pmatrix}$$

Three equations are coupled because of space charge;

The coefficients a_0 to a_5 are functions of matched beam size $\sigma_{x0,y0}, X_0$ (see Appendix).

Three fundamental modes can be solved; ϕ_1, ϕ_2, ϕ_3

Dispersion mode

- Fundamental modes from oscillation eqs:
  1. Breathing mode ϕ_1
  2. Quadrupole mode ϕ_2
  3. *Dispersion mode*. ϕ_3

- Limit of zero space charge:

$$\phi_1 = \phi_2 = 2k_0, \quad \phi_3 = k_0$$

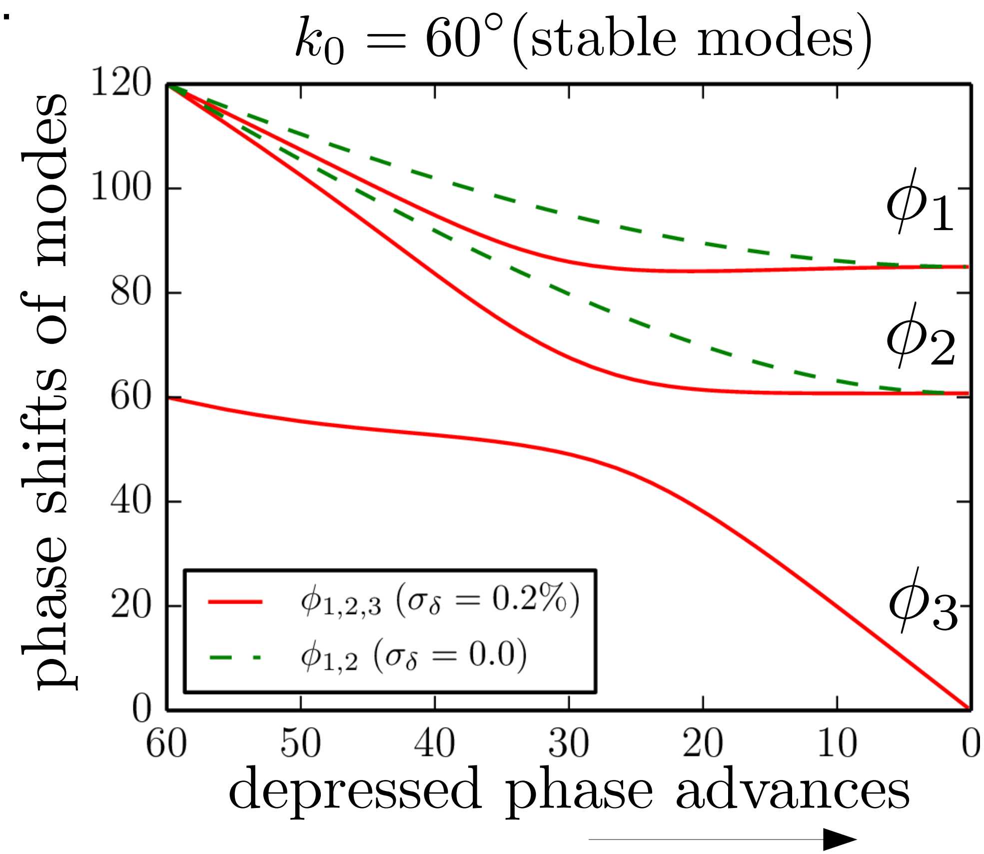
(k_0 is the phase advance)

- Limit of strong beam current:

$$\phi_1 = k_0, \quad \phi_2 = \sqrt{2}k_0, \quad \phi_3 = 0$$

(1) In both limits, ϕ_1 and ϕ_2 has identical form as in the case without dispersion.

(2) Without s.c., dispersion mode behaves like single particle



Dashed line: no dispersion
Solid line: with dispersion

Ref: M. Ikegami, et.al., PRSTAB 2, 124201 (1999)

Dispersion-induced instability

- The dispersion mode could lead to beam instability.

Stability analysis:

Vector of moments $\Sigma = (\sigma_x, \sigma'_x, \sigma_y, \sigma'_y, D_x, D'_x)^T$

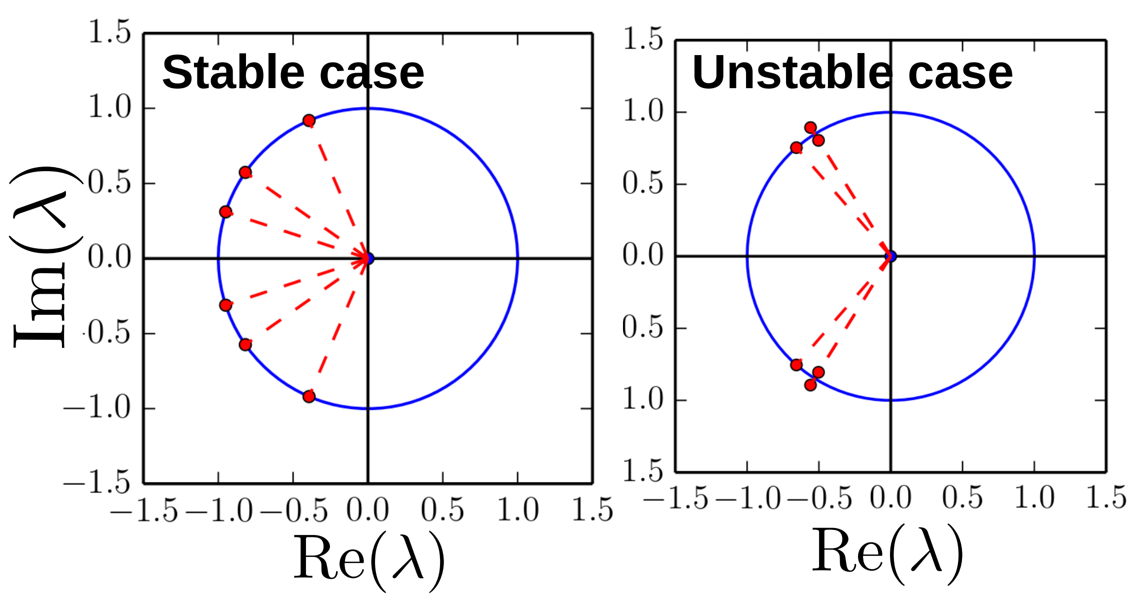
Equation of motion $\Sigma'_0 = G(\Sigma_0)$

Jacobian matrix $J_{k,l} = \dfrac{\partial G_k(\Sigma_l)}{\partial \Sigma_l}$

$$Z(s + nL_0) = \boxed{\lambda^n} Z(s)$$

$$Z(s) = J \qquad \lambda = |\lambda| e^{i\phi}$$

$|\lambda| > 1$ Indicates instability



Dispersion-induced instability

Ref. Y. S. Yuan, O. Boine-Frankenheim, G. Franchetti and I. Hofmann, PRL 118, 154801 (2017)

- An example of FODO with dipoles, and phase advance $k_{0,x} = 130^\circ$

  (Using ϕ_d to denote dispersion mode)

- Increasing beam current, beam experience three kinds of instabilities:

(1) Dispersion-induced: $\phi_d = \phi_1$

$\phi_3 + \phi_{1,2} = 360^\circ$

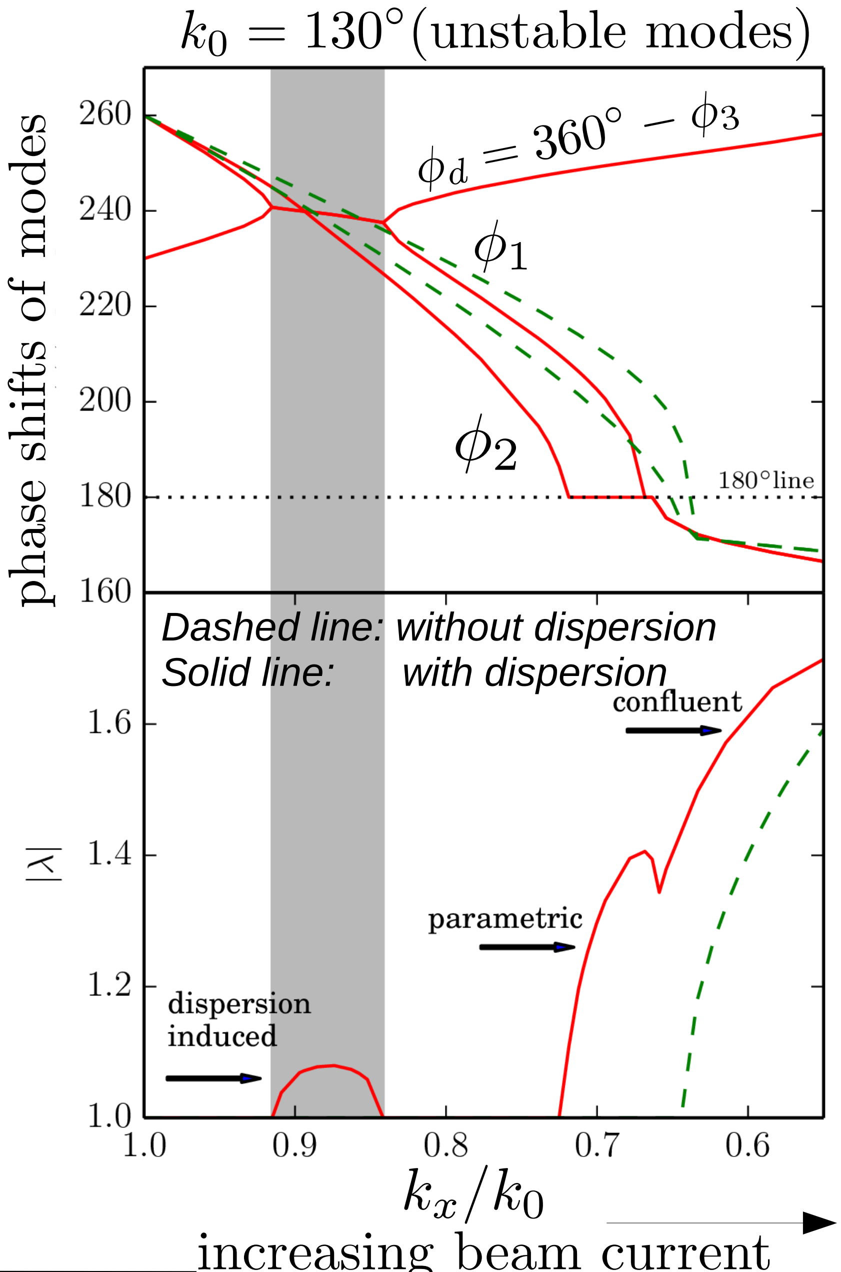
Criteria for lattice: $k_0 > 120^\circ$

(2) Parametric: $\phi_2 = 180^\circ$

(3) Confluent: $\phi_1 = \phi_2$

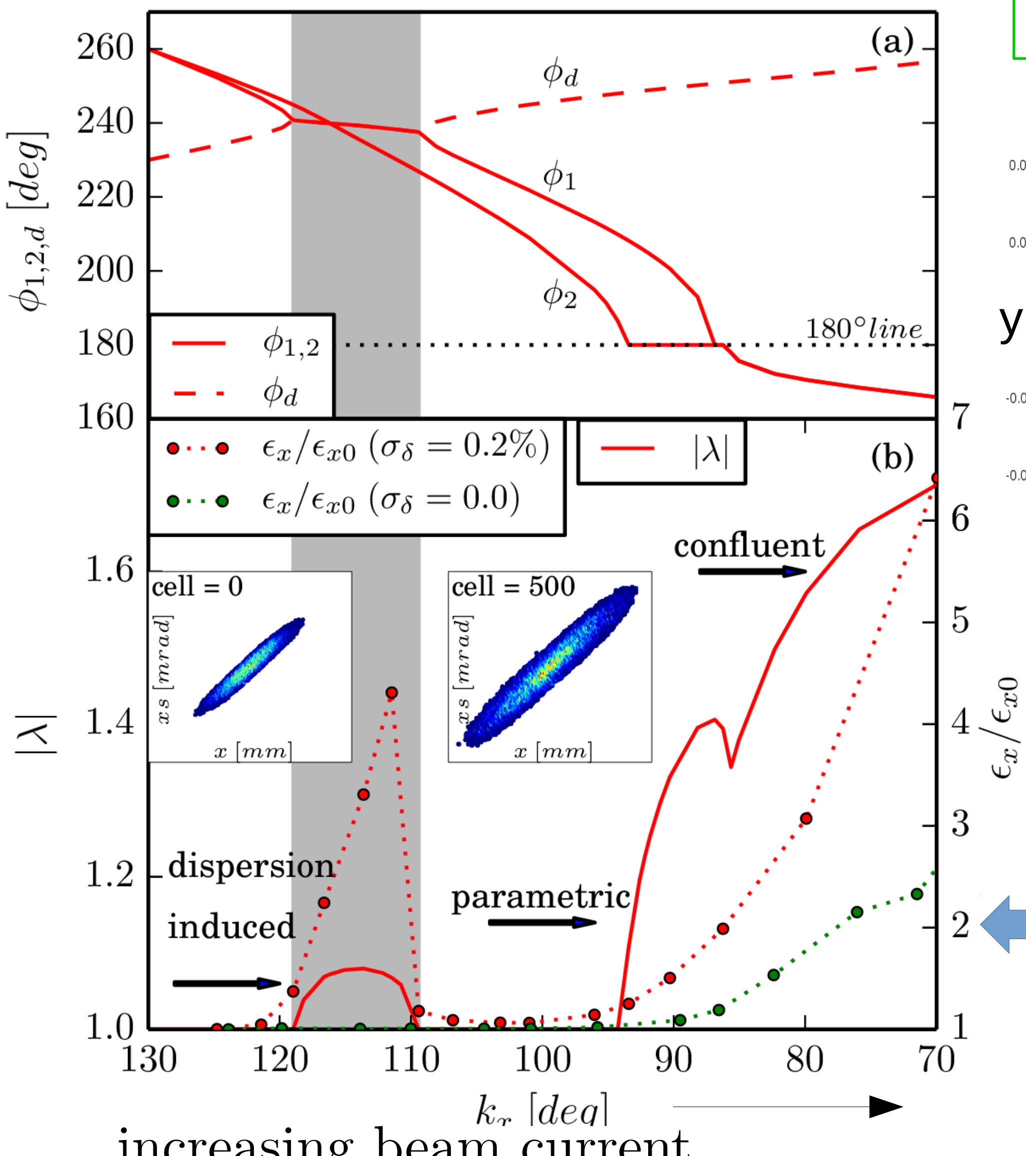
Envelope instability, modified by dispersion

Criteria for lattice: $k_0 > 90^\circ$

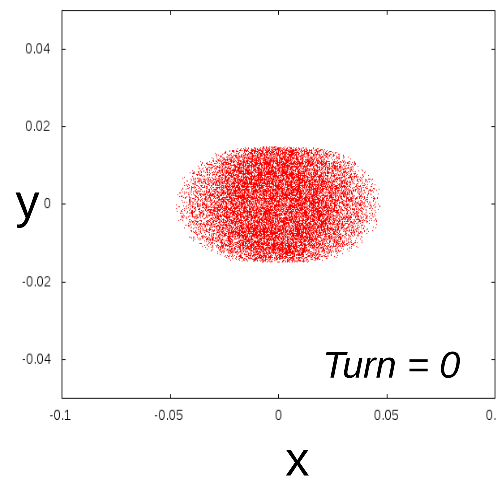


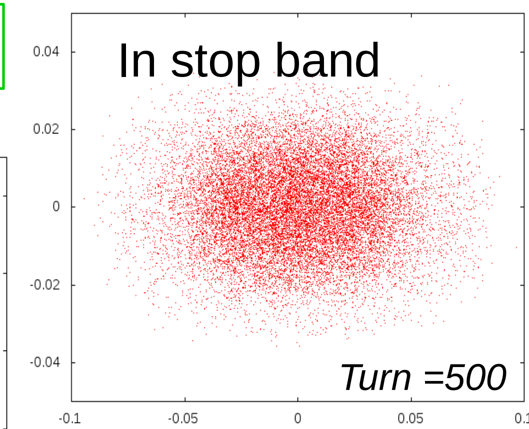
Dispersion-induced instability

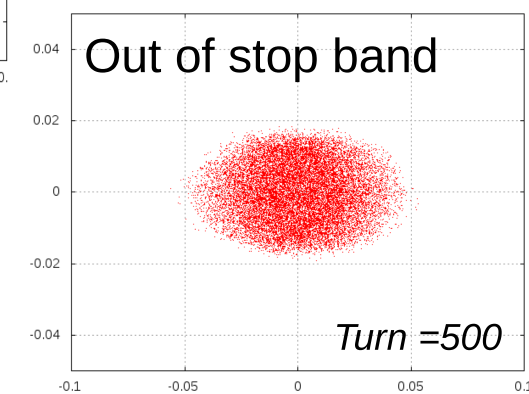
- PIC simulations



Particle distribution



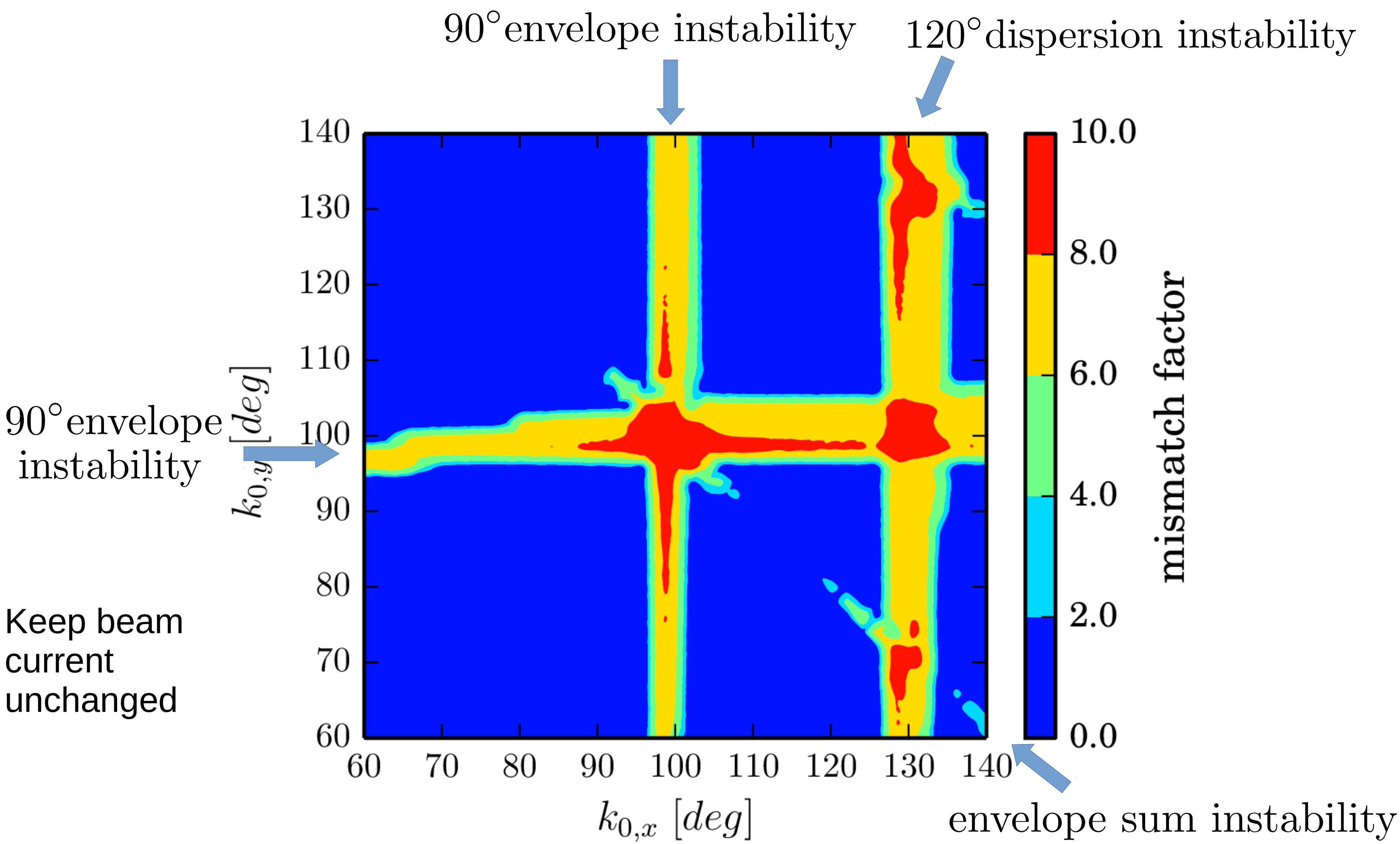




Dashed line: emittance growth from simulation
Solid line: growth factor from theory

Instability chart

- Scanning diagram of second order instability



Keep beam current unchanged

Ref.: O. Boine-Frankenheim, I. Hofmann, and J. Struckmeier, Phys. Plasmas 23, 090705 (2016)

Conclusion

- In the presence of space charge and dispersion, a dispersion mode exists
- The dispersion mode could becomes unstable in a lattice with $k_{0,x} > 120^\circ$
- The "120 dispersion instability" is an envelope instability, besides 90 instability

Outlook

- Dispersion-induced instability could be a limitation for high intensity circular accelerators

For example: Bunch compression in SIS100 at FAIR (phase advance > 120 deg)
Energy Recovery Linacs (ERL)

- The dispersion mode could be a method to characterize the s. c. modified dispersion
- Measurement of the dispersion mode could be achieved by using quadrupolar pickup signals

Thank you for your attention!

Appendix

- Coefficients in oscillation equations

$$a_0 = 4\kappa_x - \frac{2r+1}{r+1}\Delta\kappa_x \sin^2\theta_0$$

$$a_1 = \frac{r}{r+1}\Delta\kappa_x \sin\theta_0$$

$$a_2 = \frac{2r+1}{r+1}\sigma_\delta \Delta\kappa_x \sin\theta_0 \cos\theta_0$$

$$a_3 = 4\kappa_y + \frac{r+2}{r+1}\Delta\kappa_y$$

$$a_4 = \frac{r}{r+1}\sigma_\delta \Delta\kappa_x \cos\theta_0$$

$$a_5 = \kappa_x + \frac{2r+1}{r+1}\Delta\kappa_x \cos^2\theta_0$$

Here,

$$\sin\theta_0 = \sigma_{x0}/X_0$$

$$\cos\theta_0 = \sigma_\delta D_x/X_0$$

Appendix

- Dispersion mode can be observed in PIC simulations. (pyORBIT code)

(1) FFT are performed based on beam second moments in 1000 turns.

(2) Initial distribution are matched to lattice and space charge.