Astrophysics of color superconductivity in compact stars

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Three days of strong interactions

July 9, Wroclaw

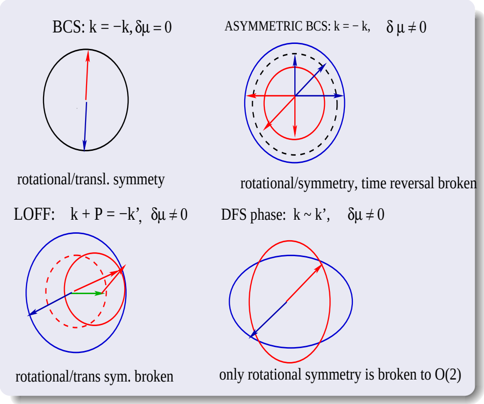
Searching for the ground state of cold, dense QCD

Key idea: Under realistic conditions of neutron stars phases with broken spatial symmetry will emerge. To understand their astrophysics we develop a simple example – the Fulde-Ferrell phase

Other related phases are:

- Various LO-type phases (crystalline structures); Studied in the GL theory, cold limit unknown
- Phase separation; proposed and studied in cold atoms
- Fermi-surface deformations (studied in QCD, but no comparison to other phases)

Fulde-Ferrell phase might be restrictive, but can help to understand the phases in a simple set-up



Model set-up

QCD Lagrangian:

$$\mathcal{L} = \overline{\psi}(i\gamma^{\mu}D_{\mu} + \hat{\mu}\gamma_0 - \hat{m})\psi - \frac{1}{4}G_a^{\mu\nu}G^a_{\mu\nu}, \quad G_a^{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu, \qquad (1)$$

ψ $4N_cN_f$-spinor, $D_\mu = \partial_\mu + igT_aA^a_\mu$ covariant derivative, A^a_μ gauge fields, $T^a = \lambda^a/2$ ($a = 1, \ldots, 8$) generators of $SU(3)_c$, and λ^a - Gell-Mann matrices. $G_a^{\mu\nu}$ the color field tensor. The partition function - integral over the quark fields $\psi(x)$

$$\mathcal{Z} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{S_0[\bar{\psi},\psi] + S_I[\bar{\psi},\psi]\right\}, \qquad (2)$$

$$S_0[\bar{\psi},\psi] = \int dx\, dy\, \bar{\psi}(x)\left[G_0^+\right]^{-1}(x,y)\psi(y), \qquad (3)$$

$$S_I[\bar{\psi},\psi] = \frac{g^2}{2}\int dx\, dy \sum_{a,b} \bar{\psi}(x)\Gamma^\mu_a\psi(x)D^{ab}_{\mu\nu}(x,y)\bar{\psi}(y)\Gamma^\nu_b\psi(y), \qquad (4)$$

Consider $N_f = 2$ and $N_c = 3$; define the basis

$$\bar{\psi} = (\bar{\psi}_r^u, \bar{\psi}_r^d, \bar{\psi}_g^u, \bar{\psi}_g^d, \bar{\psi}_b^u, \bar{\psi}_b^d). \tag{5}$$

Consider a local transformation on quark fields given by

$$\psi' \to \psi e^{-i\theta(x)}, \quad \bar{\psi}' \to \bar{\psi} e^{i\theta(x)}, \tag{6}$$

and special case $\theta(x) = Q_\mu x^\mu/2$, $Q = (0, \boldsymbol{Q})$. Quark Green's function becomes

$$\left[\tilde{G}_0^+\right]^{-1}(x,y) = [G_0^+]^{-1}(x,y) + [\gamma^\mu \partial_\mu \theta(y)]\delta(x-y). \tag{7}$$

Momentum space

$$[\tilde{G}_0^\pm]^{-1} = \gamma^\mu \left(\pm Q_\mu/2 + k_\mu\right) \pm \mu\gamma_0 - m. \tag{8}$$

Bosonize the action

$$\Delta^{+}(x,y) = g^2 \sum_{a,b} \bar{\Gamma}_a^{\mu} \langle \psi_C(x) \bar{\psi}(y) \rangle \Gamma_b^{\nu} \mathcal{D}_{\mu\nu}^{ab}(x,y), \tag{9}$$

Nambu-Gorkov spinor fields

$$\Psi = \begin{pmatrix} \psi \\ \psi_C \end{pmatrix}, \qquad \bar{\Psi} = \left(\bar{\psi}, \bar{\psi}_C\right), \tag{10}$$

Full propagator/self-energy

$$\mathcal{G} = \begin{pmatrix} \tilde{G}_0^{+} & F^{-} \\ F^{+} & \tilde{G}_0^{-} \end{pmatrix}, \qquad \Omega = \begin{pmatrix} \Sigma^{+} & \Delta^{-} \\ \Delta^{+} & \Sigma^{-} \end{pmatrix}. \tag{11}$$

Schwinger-Dyson equations

$$[G^{\pm}]^{-1} = \left[G_0^{\pm}\right]^{-1} + \Sigma^{\pm} - \Delta^{\mp} \left(\left[G_0^{\mp}\right]^{-1} + \Sigma^{\mp} \right)^{-1} \Delta^{\pm}, \tag{12}$$

$$F^{\pm} = - \left(\left[G_0^{\mp}\right]^{-1} + \Sigma^{\mp} \right)^{-1} \Delta^{\pm} G^{\pm}, \tag{13}$$

Partition function:

$$\mathcal{Z}_{\mathrm{MF}} = \left[\det_k(\beta\mathcal{G}^{-1})\right]^{1/2} \exp\left\{ \frac{g^2}{2\beta V} \int \frac{d^4k d^4p}{(2\pi)^8} \sum_{a,b} \mathrm{Tr}\left[\tilde{G}^-(k)\bar{\Gamma}_a^\mu \tilde{G}^+(p)\Gamma_b^\nu\right] D_{\mu\nu}^{ab}(k-p) \right\}, \quad (14)$$

Free quark propagator is diagonal in color and flavor space

$$G_0^\pm = \mathrm{diag}\left(G_{0\ r}^{\pm u}, G_{0\ r}^{\pm d}, G_{0\ g}^{\pm u}, G_{0\ g}^{\pm d}, G_{0\ b}^{\pm u}, G_{0\ b}^{\pm d}\right). \quad (15)$$

The gap functions, in the basis (5), are given by

$$\Delta^\pm = \begin{pmatrix} 0 & 0 & 0 & \Delta_1^\pm & 0 & 0 \\ 0 & 0 & \Delta_2^\pm & 0 & 0 & 0 \\ 0 & \Delta_2^\pm & 0 & 0 & 0 & 0 \\ \Delta_1^\pm & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (16)$$

$$\Delta_{1,2}^+(k) = \sum_e \eta_{1,2}^e(k)\Lambda^e(\boldsymbol{k}), \quad (17)$$

The full anomalous propagator

$$F^{\pm} = \begin{pmatrix} 0 & 0 & 0 & F^{\pm ud}_{rg} & 0 & 0 \\ 0 & 0 & F^{\pm du}_{rg} & 0 & 0 & 0 \\ 0 & F^{\pm ud}_{gr} & 0 & 0 & 0 & 0 \\ F^{\pm du}_{gr} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{18}$$

$$F^{\pm ud}_{rg} = -G^{\mp u}_{0\ r}\Delta^{\pm}_{1}G^{\pm d}_{0\ g}, \quad F^{\pm du}_{rg} = -G^{\mp d}_{0\ r}\Delta^{\pm}_{2}G^{\pm u}_{0\ g}, \qquad \text{etc} \tag{19}$$

The quasiparticle spectrum is determined from

$$\left(G^{-\,f}_{0\ i}\right)^{-1}\left(G^{+\,g}_{0\ j}\right)^{-1} - \sum_{e}|\eta^{efg}_{ij}|^2\Lambda^{e}(\boldsymbol{k}) = 0, \quad E^{\pm}_{e}(\eta^{e}_{1,2}) = E_{A,e} \pm \sqrt{E^2_{S,e} + |\eta^{e}_{1,2}|^2}, \tag{20}$$

$$E_{S,e}(|\boldsymbol{k}|, |\boldsymbol{Q}|, \theta, \bar{\mu})^2 = (|\boldsymbol{k}| - e\bar{\mu})^2\,, \quad E_{A,e}(|\boldsymbol{Q}|, \theta, \delta\mu) = \delta\mu + e|\boldsymbol{Q}|\cos\theta, \tag{21}$$

Integrate out gluons:

$$D_{\mu\nu}^{ab} = \delta^{ab}\frac{g_{\mu\nu}}{\Lambda^2}, \tag{22}$$

where Λ is a characteristic momentum scale. The partition function becomes

$$\ln\mathcal{Z}_{\rm MF} = \ln\mathcal{Z}_{\rm MF}^{\Delta} + \ln\mathcal{Z}_{\rm MF}^{0}, \tag{23}$$

the first term red-green quark condensate:

$$\begin{aligned}\ln\mathcal{Z}_{\rm MF}^{\Delta} \quad = \quad & \frac{3}{8}\frac{\Lambda^2}{g^2}\beta V\sum_n(|\eta_n^+|^2 + 3\sum_{n',n\neq n'}\eta_n^+\eta_{n'}^+) + \frac{1}{2}\sum_{e,n}\int\frac{d^3k}{(2\pi)^3}\Big\{\beta(E_e^+(\eta_n^e) - E_e^-(\eta_n^e)) \\ & +2\ln\left[f^{-1}\left(-E_e^+(\eta_n^e)\right)\right] + 2\ln\left[f^{-1}\left(-E_e^-(\eta_n^e)\right)\right]\Big\},\end{aligned} \tag{24}$$

Blue quarks:

$$\ln\mathcal{Z}_{\rm MF}^{0} = 2V\int\frac{d^3k}{(2\pi)^3}\sum_{e,f}\Big\{\ln\left[f^{-1}\left(-\xi_e^+(\boldsymbol{k},\mu_{b,f})\right)\right] + \ln\left[f^{-1}\left(-\xi_e^-(\boldsymbol{k},\mu_{b,f})\right)\right]\Big\}. \tag{25}$$

The thermodynamic potential is obtained form the log of the partition function as

$$\Omega_{\rm MF} = -\frac{1}{V\beta}\ln\,\mathcal{Z}_{\rm MF}. \tag{26}$$

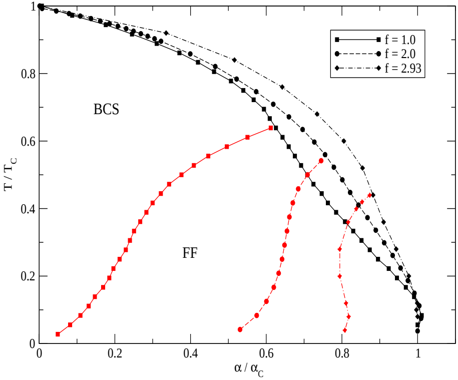
The stationary point(s) of the thermodynamic potential determine the equilibrium values of the order parameters

$$\frac{\partial\Omega_{\rm MF}}{\partial\eta_1^e} = 0, \quad \frac{\partial\Omega_{\rm MF}}{\partial\eta_2^e} = 0, \quad \frac{\partial\Omega_{\rm MF}}{\partial|\boldsymbol{Q}|} = 0. \tag{27}$$

The direction of the vector $\boldsymbol{Q}$ is chosen by the superconductor spontaneously. Densities of quarks

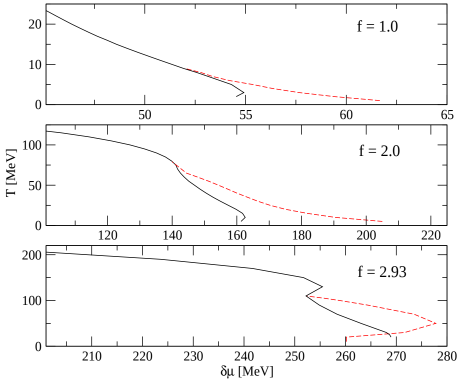
$$\frac{\partial\Omega_{\rm MF}}{\partial\mu_u} = n_u, \quad \frac{\partial\Omega_{\rm MF}}{\partial\mu_d} = n_d. \tag{28}$$

Phase-diagram: α-T plane



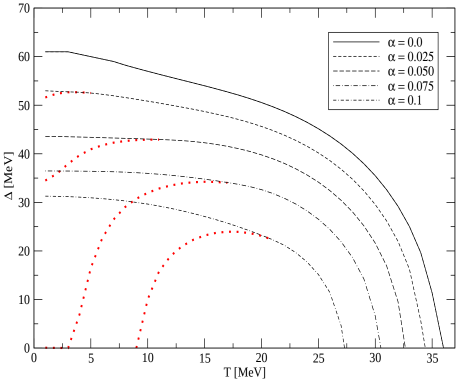
The phase diagram features a tricritical Lifshits point. The system belongs to the paramagnetic-ferromagnetic-helical universality class. From renormalization group $Q \sim (\alpha - \alpha_c)^{\beta}$, where $\beta = 1/2$.

Phase-diagram: $\delta\mu$-T plane



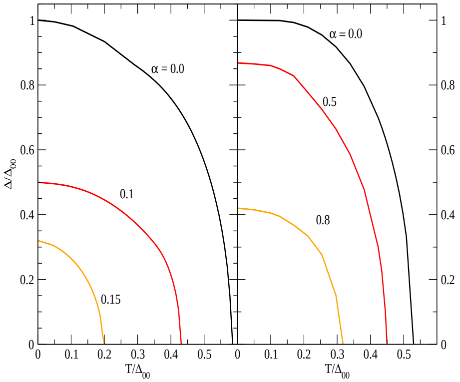
The transition from BCS to LOFF phase occurs at $\delta\mu_1 \sim 0.8$, from LOFF to the normal state at $\delta\mu_2 \sim 1$

Temperature dependence of the gap



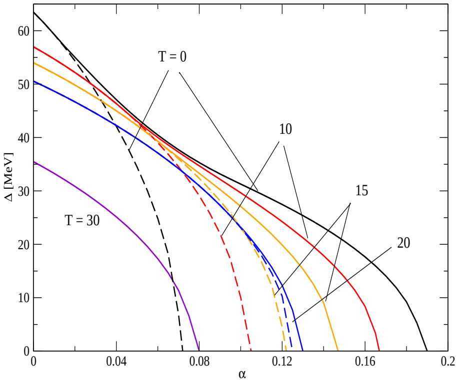
Anomalous behavior of the gap in the BCS phase (which leads to other anomalies in the thermodynamic quantities) is removed by the FF phase.

Weak vs strong coupling



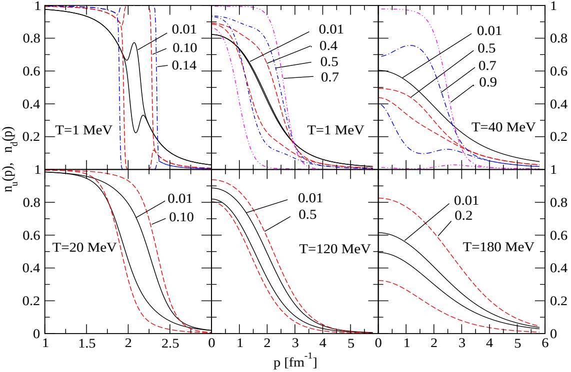
The temperature dependence in the strong coupling is self-similar to that of the weak coupling.

α-dependence of the gap



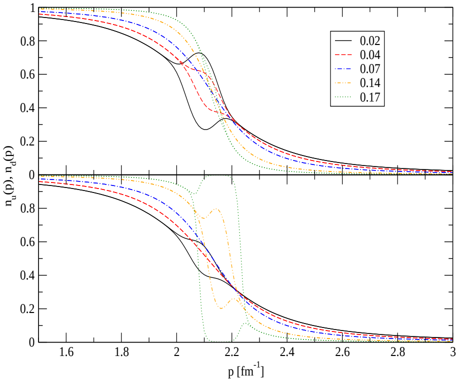
For large temperatures only BCS, at low temperatures large portions of α domain belongs to FF.

Occupation numbers: from weak to strong



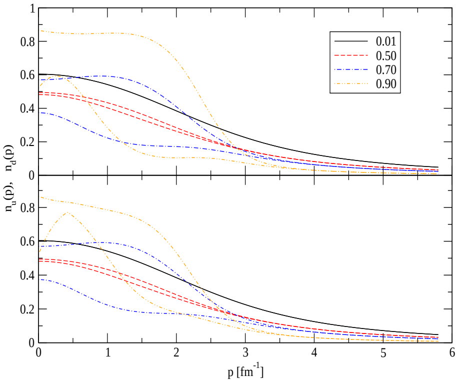
Upper panel: low temperatures, lower panel: high temperatures. Evolution from weak to strong coupling: topological change in the form of the Fermi sphere of minority across coupling (low temperatures).

Angle dependence of occupation numbers: weak coupling



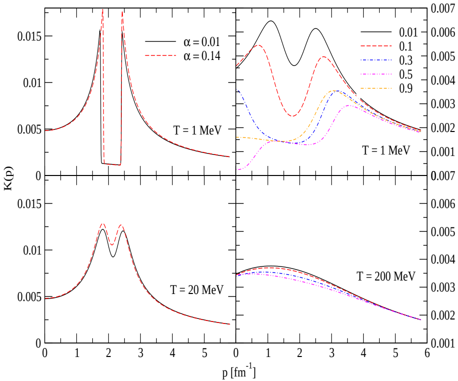
Occupation numbers at two angles $\theta = 45$ and $\theta = 90$.

Angle dependence of occupation numbers: strong coupling



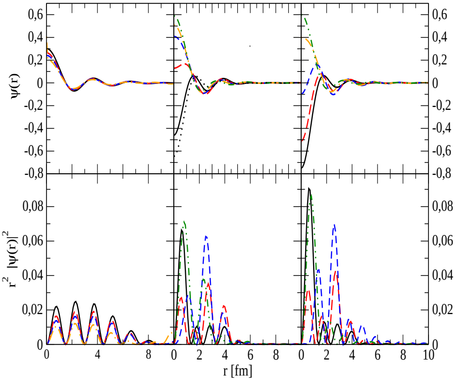
No blocking region; the differences between the directions are washed out.

Wave functions of Cooper pairs from weak to strong coupling



The depression around the Fermi sphere arises from the blocking region in the distribution of the minority component due to factor $1 - f_1 - f_2$, with $f_2 \simeq 0$ and $f_1 \simeq 1$.

Real space wave-function: from weak to strong coupling



Weak coupling: long range order, condensate over many periods of $2\pi k_F^{-1}$. Strong coupling: Localized structure at the origin: Bose condensate of molecules. New state of current carrying Bose condensate of quarks ?

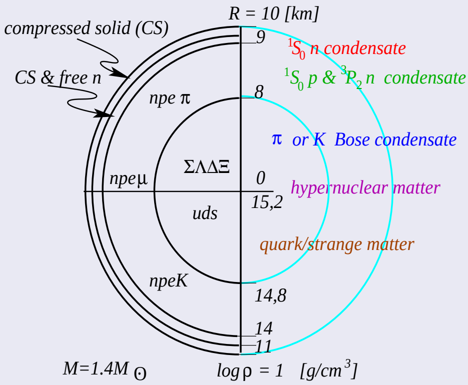
Learning about the interiors of neutrons stars

Key idea: Neutron (compact) star may feature solid phases; to lowest order quadrupole deformations generate gravitational radiation at twice the rotation frequency

Candidate phases:

- Solid crusts (nuclear matter)
- Solid neutron matter
  $n \simeq 3n_0$
- Solid unpaired quark matter
  $n \geq n_{\rm deconfinment}$
- Solid superconducting quark matter $n \geq n_{\rm deconfiment}$

Color superconducting phases can be solid with shear moduli by many orders exceeding that of the crusts.



A. S., Prog. Part. Nucl. Phys. **58**, 168 (2007)

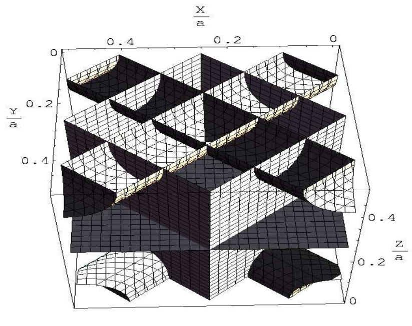
Shear modulus of CCS phase

Important: The superconducting phase has a nonzero shear modulus, i. e. it can support quadrupole and higher order deformations “mountains” d

The key quantities are:

- Breaking strain $10^{-5} \leq \sigma \leq 10^{-2}$
- Shear modulus

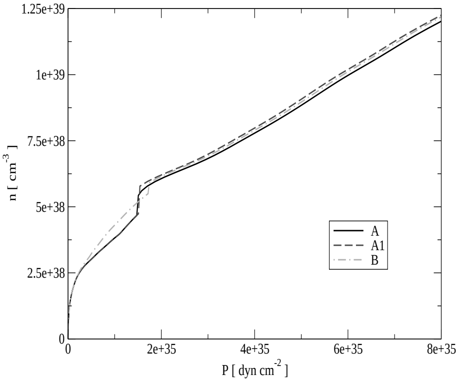
Color superconducting phases can be solid with shear moduli by many orders exceeding that of the crusts.



bcc superconducting lattice

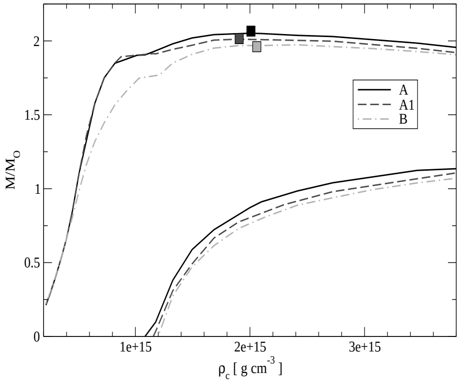
$$\mu = 2.47 \text{ MeV fm}^{-3} \left(\frac{\Delta}{10 \text{ MeV}}\right)^2 \left(\frac{\mu_q}{400 \text{ MeV}}\right)^2, \quad (29)$$

Equations of state



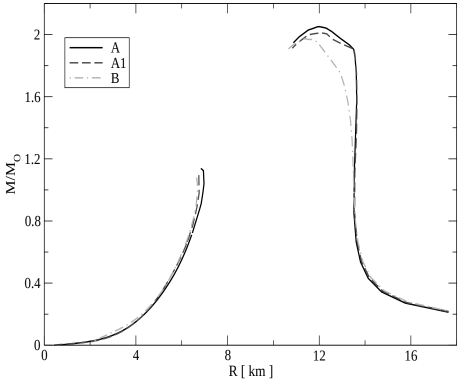
- The nuclear equation of state is taken from covariant BHF theory with two parameterizations (both stiff)
- The two quark equation of states differ by pressure normalization in the vacuum (slight vertical shift)

Stellar configurations



- Hybrid configurations can appear as "second family" of configurations; maximal masses are large $\sim 2M_\odot$
- Quark core masses are $\leq 0.8M_\odot$.

Mass-radius relationship



- Models satisfy constraints on $M - R$ relation
- Quark core radii below 7 km

Gravitation radiation

Given a deformation the characteristic strain amplitude:

$$h_0 = \frac{16\pi^2 G}{c^4} \frac{\epsilon I_{zz} \nu^2}{r}, \tag{30}$$

$\epsilon = (I_{xx} - I_{yy})/I_{zz}$ is the equatorial ellipticity. Strain amplitude can be expressed in terms of the $m = 2$ mass quadrupole moment as

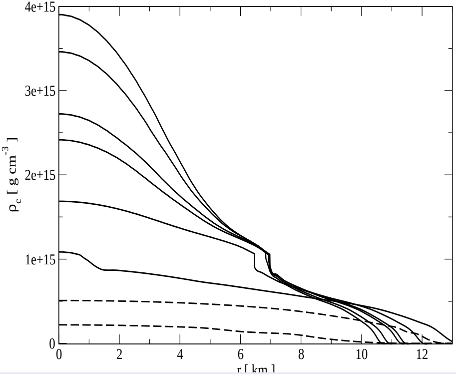
$$h_0 = \frac{16\pi^2 G}{c^4} \left(\frac{32\pi}{15}\right)^{1/2} \frac{Q_{22}\nu^2}{r}, \tag{31}$$

Quadrupole moment

$$Q_{22} = \int_0^{R_{\rm core}} \frac{drr^3}{g(r)} \left[\frac{3}{2}(4-U)t_{rr} + \frac{1}{3}(6-U)t_\Lambda + \sqrt{\frac{3}{2}}\left(8 - 3U + \frac{1}{3}U^2 - \frac{r}{3}\frac{dU}{dr}\right) t_{r\perp}\right], \tag{32}$$

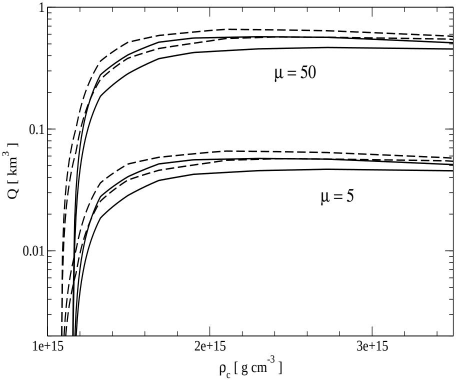
where $U = 2 + d\ln g(r)/d\ln r$ and t_{rr}, t_Λ and $t_{r\perp}$ are the coefficients of the expansion of the shear stress tensor in spherical harmonics.

Comparison to the previous work



- L.-M. Lin, Phys. Rev. D **76**, 081502(R) (2007, *incompressible models without nuclear crusts*
- Haskell et al, Phys. Rev. Lett. **99**, 231101 (2007), *incompressible quark matter plus* $n = 1$ *polytrope*
- B. Knippel, A. Sedrakian, Phys. Rev. D **79**, 083007 (2009), *microscopic equations of state*

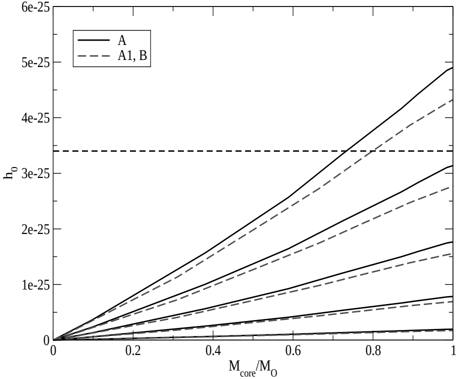
Quadrupole moments



- Quadrupole moments for fixed breaking strain 10^{-4}
- Dashed lines, same mass and radius, but incompressible model of core

The main difference is in the integral parameters of the configurations.

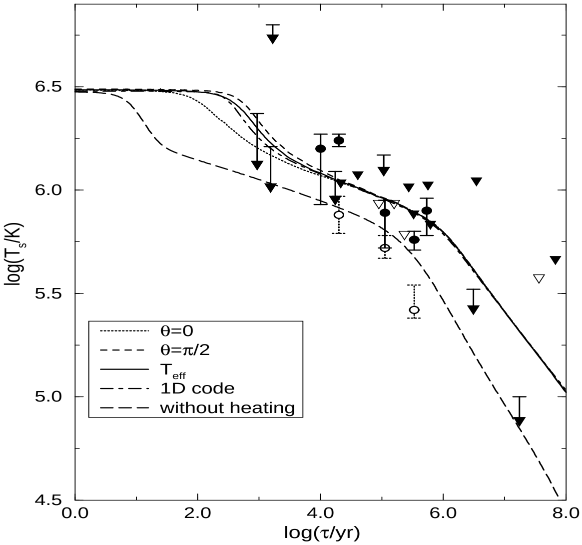
Strain amplitudes



GW strain amplitudes for breaking strain 10^{-4}, Gaps from 10 to 50 MeV.
Dashed line Crab pulsars' upper limit from S5 run

h_0 can pin down the product $\sigma\Delta^2$, currently $\bar{\sigma}_{\max}\Delta^2 \sim 0.25$ MeV^2 (under the assumptions of the present model).

Neutrinos in superconducting quark matter



Transport equations

- ν and $\bar{\nu}$ - Boltzmann equations with KB collision integrals

$$\left[\partial_t + \vec{\partial}_q\, \omega_\nu(\vec{q})\vec{\partial}_x\right] f_\nu(\vec{q}, x)$$
$$= \int_0^\infty \frac{dq_0}{2\pi} \mathrm{Tr}\left[\Omega^<(q,x) S_0^>(q,x) - \Omega^>(q,x) S_0^<(q,x)\right],$$

- ν-quasiparticle propagators:

$$S_0^<(q,x) = \frac{i\pi/q}{\omega_\nu(\vec{q})}\Big[\delta\left(q_0 - \omega_\nu(\vec{q})\right) f_\nu(q,x)$$
$$-\delta\left(q_0 + \omega_\nu(\vec{q})\right)\left(1 - f_{\bar{\nu}}(-q,x)\right)\Big]. \qquad (33)$$

- definition of the Poisson bracket

$$\{f,g\}_{P.B.} = \partial_\omega f\ \partial_t g - \partial_t f\ \partial_\omega g - \partial_{\vec{p}} f\ \partial_{\vec{r}} g + \partial_{\vec{r}} f\ \partial_{\vec{p}} g. \qquad (34)$$

Neutrinos in a color superconductors

energy loss per unit time and volume

$$\epsilon_{\nu\bar{\nu}} = \frac{d}{dt}\int \frac{d^3q}{(2\pi)^3}\left[f_\nu(\vec{q}) + f_{\bar{\nu}}(\vec{q})\right]\omega_\nu(\vec{q}) \qquad (35)$$

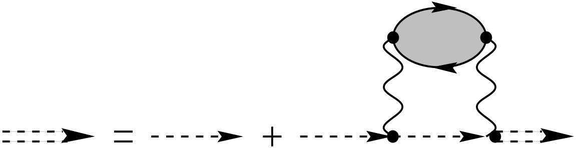
expressed through the collision integrals

$$\begin{aligned}\epsilon_{\nu\bar{\nu}} = & -2\left(\frac{G}{2\sqrt{2}}\right)^2 \sum_f \int \frac{d^3q_2}{(2\pi)^3 2\omega_\nu(\vec{q}_2)} \int \frac{d^3q_1}{(2\pi)^3 2\omega_\nu(\vec{q}_1)} \int \frac{d^4q}{(2\pi)^4} \\ & (2\pi)^4\delta^3(\vec{q}_1 + \vec{q}_2 - \vec{q})\delta(\omega_\nu(\vec{q}_1) + \omega_\nu(\vec{q}_2) - q_0)\ [\omega_\nu(\vec{q}_1) + \omega_\nu(\vec{q}_2)] \\ & g_B(q_0)\,[1 - f_\nu(\omega_\nu(\vec{q}_1))]\,[1 - f_{\bar{\nu}}(\omega_\nu(\vec{q}_2))]\,\Lambda^{\mu\lambda}(q_1, q_2)\mathrm{Im}\,\Pi^R_{\mu\lambda}(q).\end{aligned}$$

Self-energies

- ν and $\bar{\nu}$-self-energies (second order in weak force)

$$-i\Omega^{>,<}(q_1,x) = \int \frac{d^4q}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}(2\pi)^4\delta^4(q_1-q_2-q)$$
$$i\Gamma^{\mu}_{L\,q}\, iS_0^{<}(q_2,x) i\Gamma^{\dagger\,\lambda}_{L\,q}\, i\Pi^{>,<}_{\mu\lambda}(q,x), \qquad (36)$$

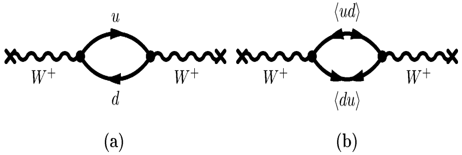
- 
- the problem is to compute the polarization tensor!

One loop results

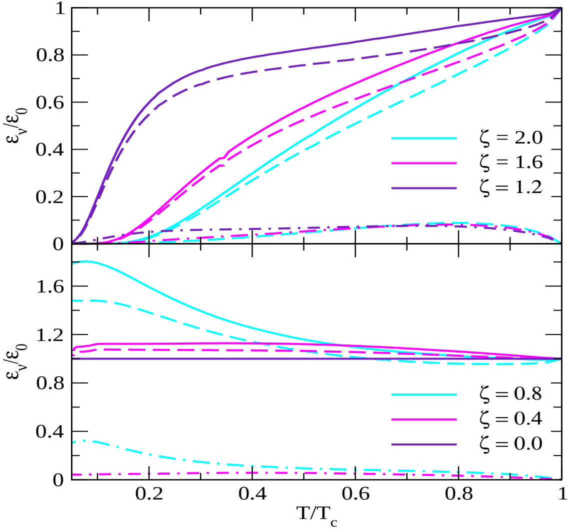
Polarization tensors

$$\Pi_{\mu\lambda}(q) = -i\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[(\Gamma_-)_\mu S(p)(\Gamma_+)_\lambda S(p+q)\right]$$

$$\Gamma_\pm(q) = \gamma_\mu(1-\gamma_5)\otimes\tau_\pm$$



$$S_{f=u,d} = i\delta_{ab}\frac{\Lambda^+(p)}{p_0^2-\epsilon_p^2}(\not p - \mu_f\gamma_0), \quad F(p) = -i\epsilon_{ab3}\epsilon_{fg}\Delta\frac{\Lambda^+(p)}{p_0^2-\epsilon_p^2}\gamma_5 C$$



$\zeta = \Delta/\delta\mu$, where $\delta\mu = \mu_d - \mu_u = \mu_e$.

form P. Jaikumar, C. D. Roberts, and A. S., Phys. Rev. C **73** (2006) 042801.

Summary, conclusions, and outlook

- The temperature-asymmtery phase diagram shows that phases with broken spatial symmetries are favored compared to the BCS state in weak coupling
- Constructed hybrid configurations of CCS featuring compact stars
- The sequence contains entirely heavy mass (2 solar) objects with core masses 0.8 solar mass and radii up to 7 km. We think this is model independent (largely).
- If the core is maximally strained then the h_0 is detectable for realistic values $\sigma \sim 10^{-4}$ and $\Delta \sim 40$ MeV.

Questions for the feature

- Is there quark matter in the CCS state in compact stars?
- Is it strained and to which extent?
- What are the dynamic avenues for obtaining stressed cores?
- Other signatures of quark superconducting phases:
  - Cooling behavior
  - Global oscillations modes
  - Dynamics of glitches and post-glitch relaxations

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Craig Roberts (Argonne)