The equation of state from lattice QCD

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for the
RBC-Bielefeld – and
hotQCD –
– Collaborations

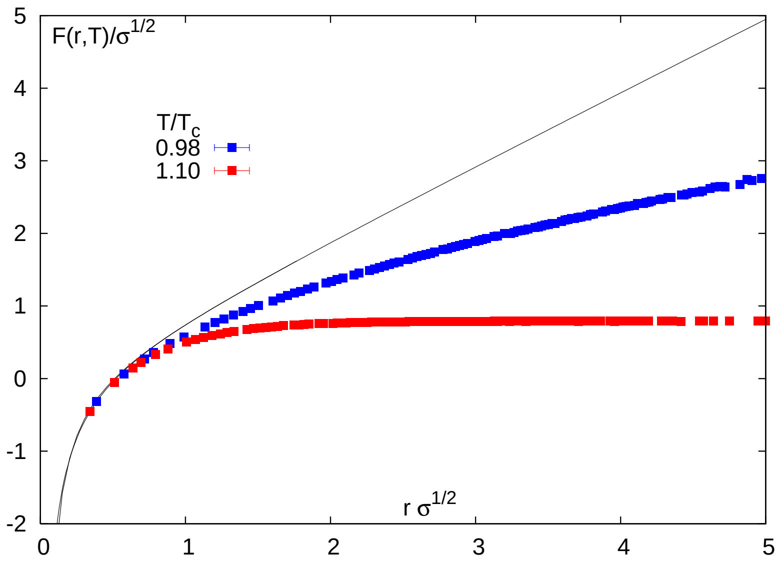
at high temperature and/or density

QCD undergoes a transition from the hadronic phase to the quark-gluon plasma

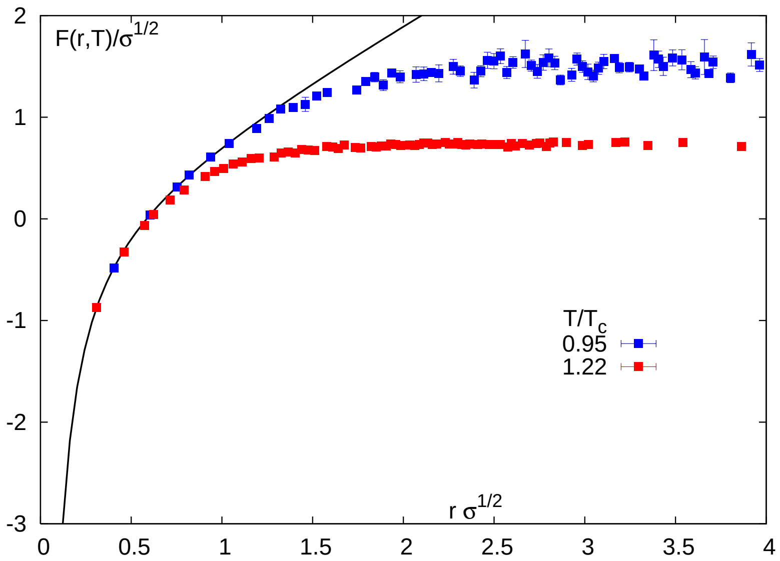
confinement - deconfinement

heavy (static) quark potential

quenched:

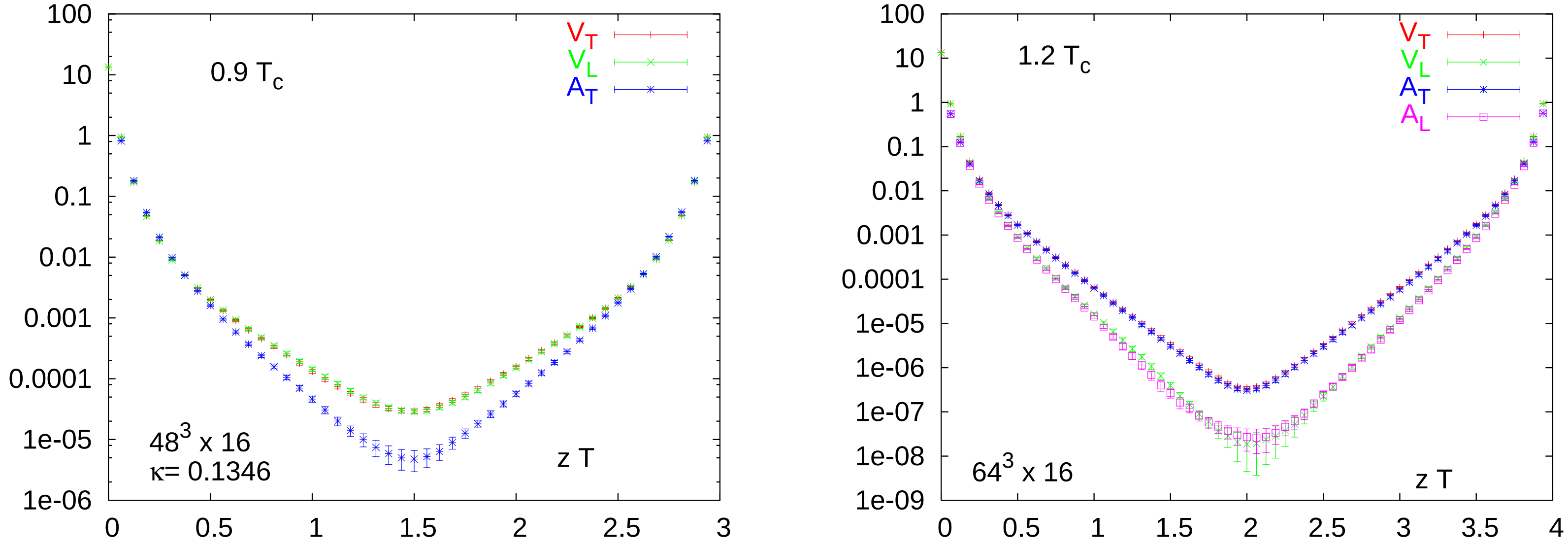


full QCD:



↑ string breaking

Chiral symmetry restoration $SU_V(N_F) \to SU_L(N_F) \times SU_R(N_F)$



$T < T_c$: $V_T = V_L \neq A_T = A_L$ $T > T_c$: $V_T = A_T \neq V_L = A_L$

- at $T > T_c$, chiral symmetry restoration: $V = A$
- at $T \neq 0$, for spatial correlations: rotational $SO(3) \to SO(2) \times Z(2)$

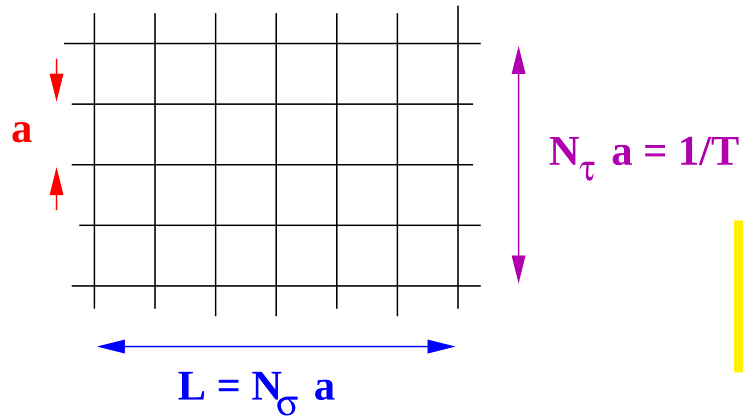
$$\Rightarrow \quad V_T \neq V_L, \quad A_T \neq A_L \quad \text{possible}$$

some technical remarks unavoidable: must control systematic errors !

numerical treatment of QCD ⇒ discretize (Euclidean) space-time

⇒ **lattice**

$N_\sigma^3 \times N_\tau$



$$Z(T,V) = \int \prod_{i=1}^{N_\tau N_\sigma^3} d\phi(x_i) \exp\{-S[\phi(x_i)]\}$$

finite yet high-dimensional path integral

→ **Monte Carlo**

- thermodynamic limit, IR - cut-off effects $\qquad LT = \frac{N_\sigma}{N_\tau} \rightarrow \infty$
- continuum limit, UV - cut-off effects $\qquad aT = \frac{1}{N_\tau} \rightarrow 0$ $\qquad$ → improved actions
- physical quark masses $\qquad m \rightarrow m_{\mathrm{phys}} \simeq 0$

  numerical effort $\sim (1/m)^p$

Choice of (improved) fermion actions

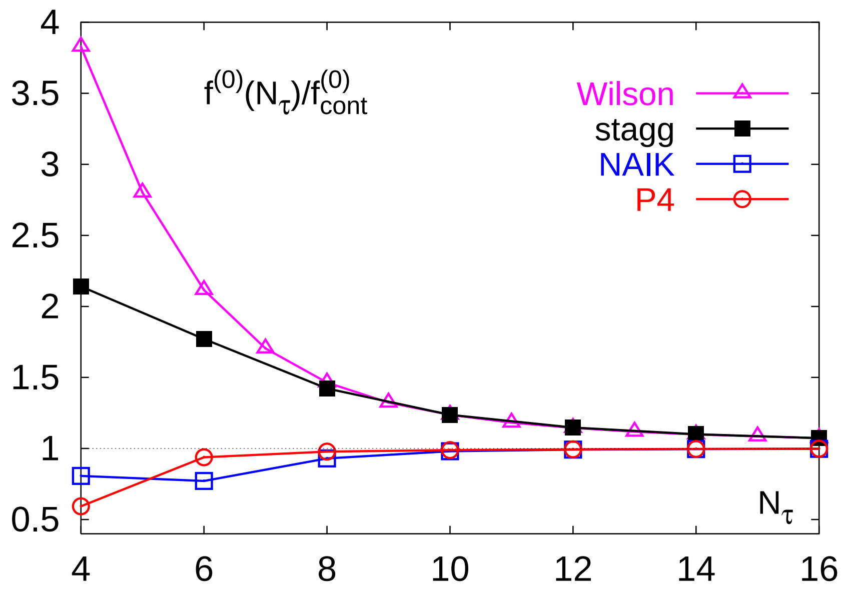
- free energy density, for instance (see later): $f/T^4 \sim N_\tau^4 \times \text{signal}$

$\Rightarrow$ signal $\sim 1/N_\tau^4$

$\Rightarrow$ keep N_τ small

$\Rightarrow$ coarse lattices $a = 1/N_\tau T$

$\Rightarrow$ improved actions



⋆ in the following: `p4` (to improve thermodynamics) and `fat3` (to improve flavor symmetry) `p4fat3`
or `Naik` (to improve thermodynamics) and `fat7` (to improve flavor symmetry)
and perturbative $\mathcal{O}(a^2 g^2)$ improvement = `asqtad`

Simulation parameters

- $N_F = 2+1$: two degenerate u/d quarks + strange quark
- RHMC algorithm, exact to machine precision
  - polynomial approximation: 16/10 for light/strange quarks in molecular dynamics
    20/16 for light/strange quarks in heatbath/ Metropolis
  - Hasenbusch trick: light/strange 1:5
  - Sexton/Weingarten in ratio 1:15
- lattice sizes $16^3 \times 4$, $24^3 \times 6$, $32^3 \times 8$ $(T > 0)$
  $16^3 \times 32, 24^3 \times 32, 32^3 \times 32, 24^2 \times 32 \times 48$ ($T = 0$, for scales and normalization)
- statistics $\mathcal{O}(10k - 60k)$ for $N_\tau = 4$, each $(\beta, \hat{m}_q, \hat{m}_s)$
  $\mathcal{O}(8k - 30k)$ for $N_\tau = 6$, each $(\beta, \hat{m}_q, \hat{m}_s)$
  $\mathcal{O}(10k - 40k)$ for $N_\tau = 8$, each $(\beta, \hat{m}_q, \hat{m}_s)$
  $\mathcal{O}(\geq 5k)$ for $T = 0$, each $(\beta, \hat{m}_q, \hat{m}_s)$
- "line of constant physics" i.e. constant physical m_K, m_π

I Equation of State

start from energy-momentum tensor $\frac{\Theta^{\mu}_{\mu}(T)}{T^4} = \frac{\epsilon - 3p}{T^4} = T\frac{d}{dT}(p/T^4)$

where $p = \frac{T}{V}\ln Z(T,V) - \lim_{T\to 0}\frac{T}{V}\ln Z(T,V)$ taking care of UV divergencies

thus $$\frac{p(T)}{T^4} - \frac{p(T_0)}{T_0^4} = \int_{T_0}^{T} dT' \frac{1}{T'^5}\Theta^{\mu}_{\mu}(T')$$

now $Z(T,V;g,m_l,m_s) = Z(N_\tau, N_\sigma, a; \beta = 6/g^2, \hat{m}_l, \hat{m}_s) \to Z(N_\tau, N_\sigma, a; \beta, m_\pi, m_K)$

and tune bare lattice parameters $\hat{m}_l, \hat{m}_s$ with β such that $m_{\pi,K} = \text{const} \Rightarrow \hat{m}_{l,s}(\beta), a(\beta)$

$$\Rightarrow \quad \frac{\Theta^{\mu}_{\mu}(T)}{T^4} = -R_\beta(\beta) N_\tau^4 \left(\left\langle \frac{d\bar{S}}{d\beta} \right\rangle_T - \left\langle \frac{d\bar{S}}{d\beta} \right\rangle_{T=0} \right)$$

with $$R_\beta(\beta). = T\frac{d\beta}{dT} = -a\frac{d\beta}{da}$$

furthermore, will need $(\hat{m}_s(\beta) = \hat{m}_l(\beta) \times h(\beta))$

$$R_m(\beta). = \frac{1}{\hat{m}_l(\beta)}\frac{d\hat{m}_l(\beta)}{d\beta} \qquad\qquad R_h(\beta). = \frac{1}{h(\beta)}\frac{dh(\beta)}{dh}$$

action $S = \beta\, S_G \;+\; 2\,\hat{m}_l(\beta)\; \bar{\psi}_l \psi_l \;+\; \hat{m}_s(\beta)\; \bar{\psi}_s \psi_s \;+\; \beta$ independent

such that Θ^{μ}_{μ} consists of three pieces (`asqtad` slightly more complicated)

$$\frac{\Theta^{\mu\mu}_G(T)}{T^4} = R_\beta\; N_\tau^4\; \Delta\langle \bar{S}_G \rangle$$

where $\Delta\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_{T=0} - \langle \mathcal{O} \rangle_T$

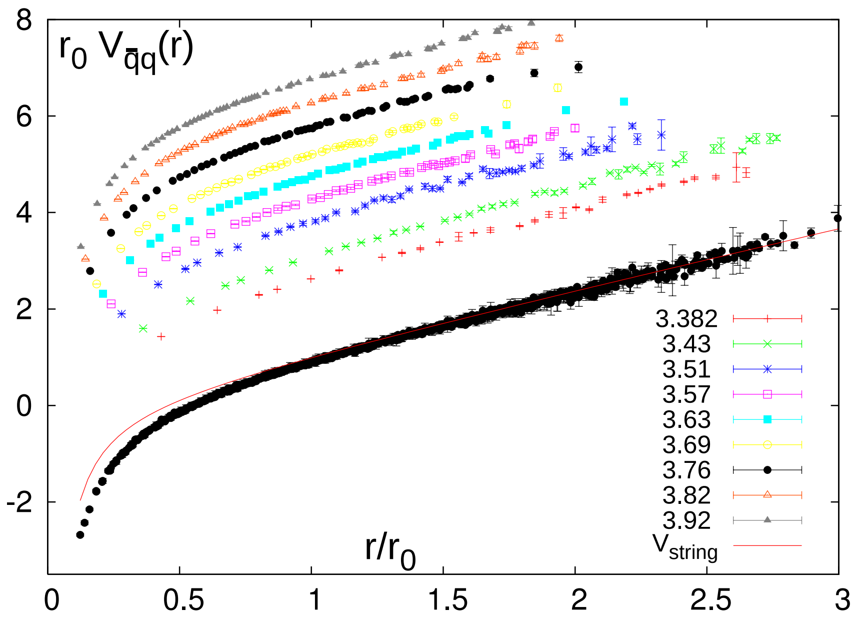
$$\frac{\Theta^{\mu\mu}_F(T)}{T^4} = -R_\beta\, R_m\; N_\tau^4\; \{\, 2\,\hat{m}_l\, \Delta\langle \bar{\psi}\psi \rangle_l \;+\; \hat{m}_s\, \Delta\langle \bar{\psi}\psi \rangle_s \,\}$$

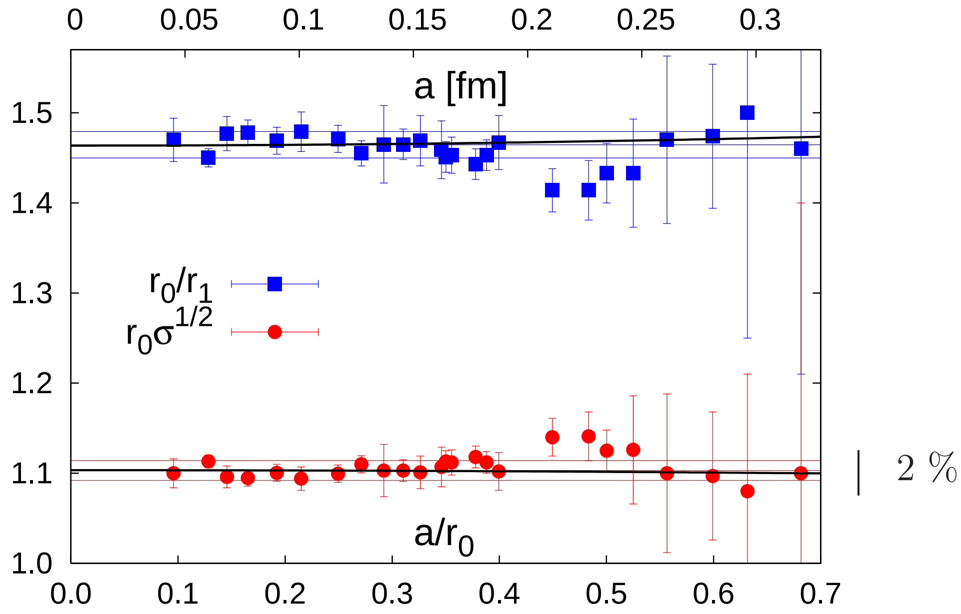
$$\frac{\Theta^{\mu\mu}_h(T)}{T^4} = -R_\beta\, R_h\; N_\tau^4\; \hat{m}_s\, \Delta\langle \bar{\psi}\psi \rangle_s$$

need: non-perturbative β functions $R_\beta(\beta), R_m(\beta), R_h(\beta)$

"action differences" $\Delta \bar{S}_G, \Delta\langle \bar{\psi}\psi \rangle_{l,s}$

⋆ T = 0 scale taken from $\Upsilon\, 2S - 1S$ splitting [A. Gray et al.] via the heavy quark potential $V(r)$





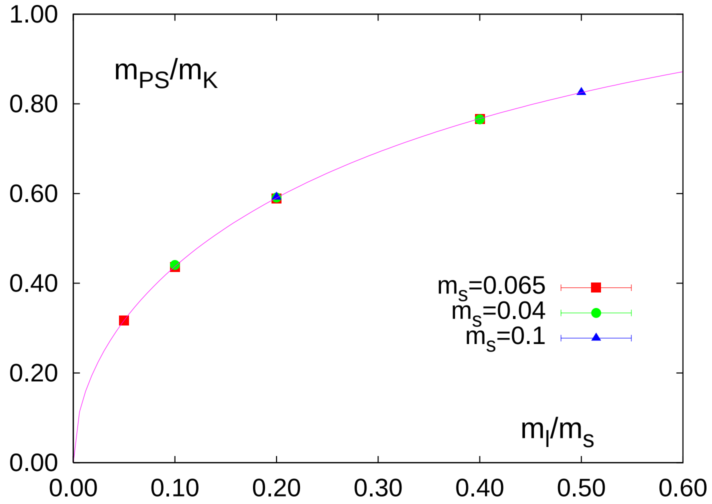
potential well decribed by

$$V(r) = c_0 - \frac{\alpha}{r_{imp}} + \sigma r_{imp} \qquad \text{with} \qquad \frac{a}{r_{imp}} = 4\pi \int_{-\pi}^{\pi} \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\vec{r}} \left[4 \sum_i^3 \sin^2(ak_i/2) + \frac{1}{3}\sin^4(ak_i/2)\right]^{-1}$$

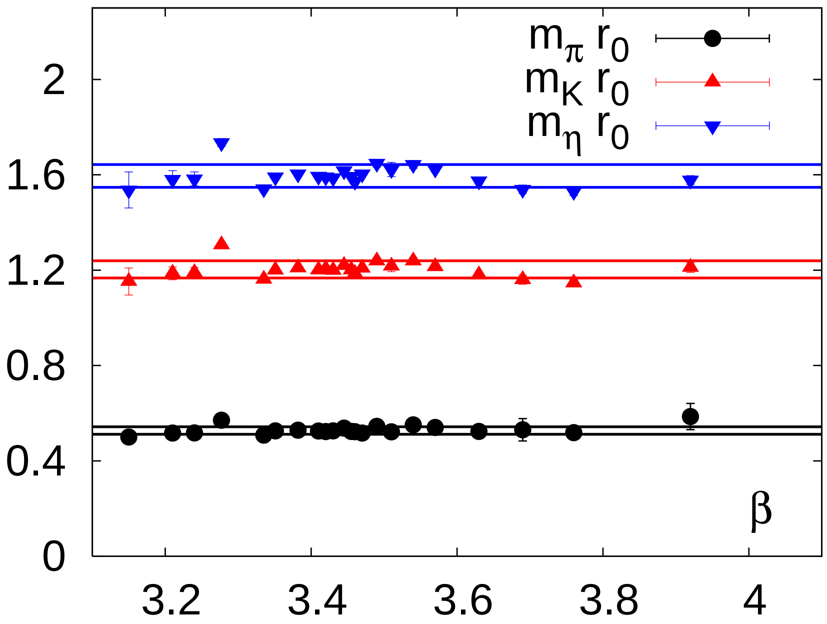
r_0/a from $r^2 \frac{dV(r)}{dr}\Big|_{r=r_0} = 1.65$ together with $r_0 = 0.469(7)$ fm [A. Gray et al.] $\Rightarrow$ $a(\beta)$

$m_{\pi,K} = \text{const}$: **Line of Constant Physics (LoCP)**

- to sufficient precision,
  m_π/m_K depends on $h = \hat{m}_s/\hat{m}_l$ only
  $\Rightarrow$ fix $h = 10$
  $\Rightarrow$ $R_h(\beta) = 0$



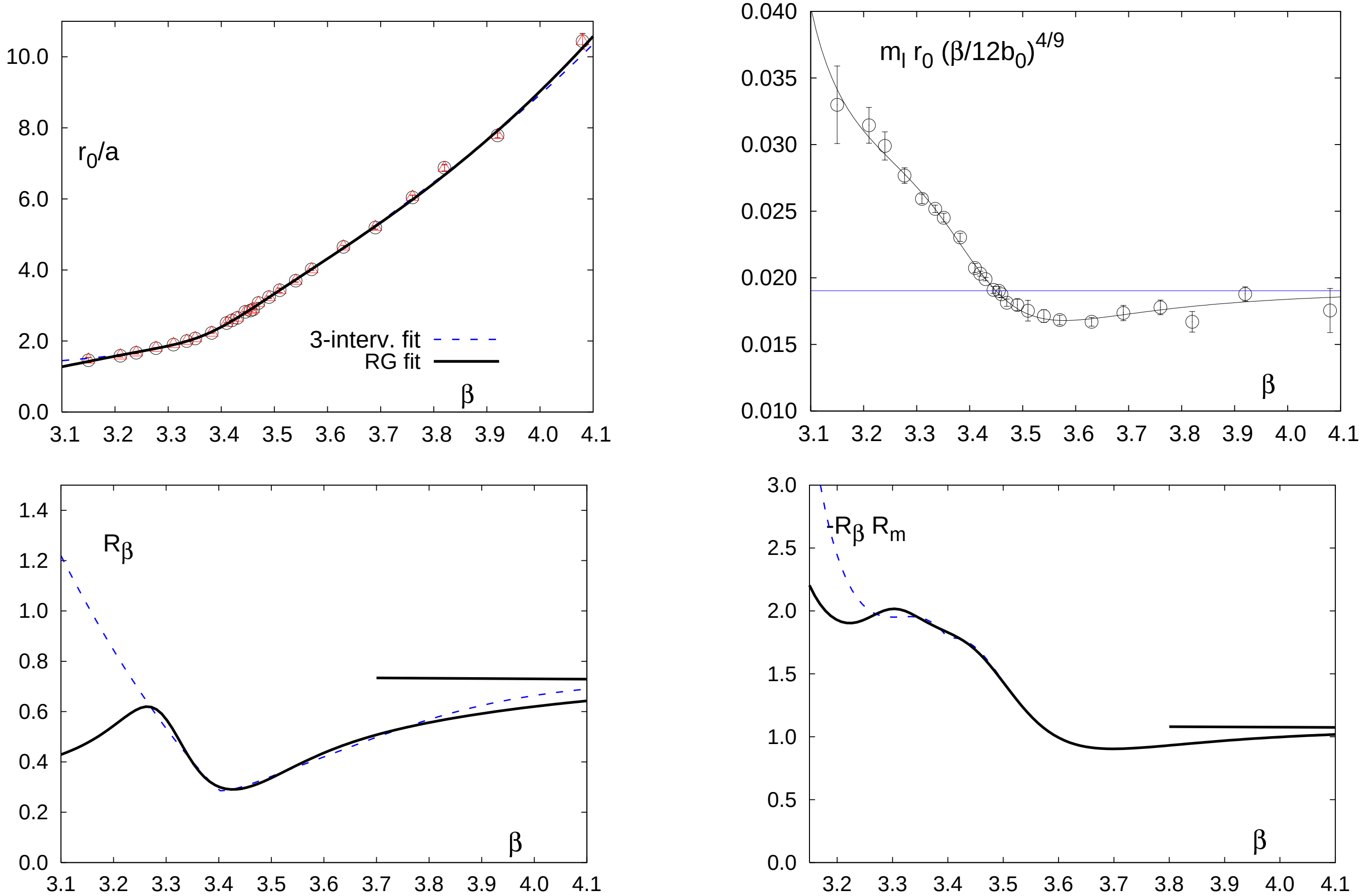
- fine tune $\hat{m}_s(\beta)$



3 %

$m_K \simeq m_K^{\rm phys}$

$m_\pi \simeq 220\,\text{MeV}$



Allton inspired parametrization with rational fct. in $\hat{a}(\beta) = R_\beta^{(2-loop)}(\beta)/R_\beta^{(2-loop)}(\beta=3.4)$

$$\frac{a}{r_0} = a_r R_\beta^{(2-loop)} \frac{1 + b_r \hat{a}^2 + c_r \hat{a}^4 + d_r \hat{a}^6}{1 + e_r \hat{a}^2 + f_r \hat{a}^4} \quad \Rightarrow \quad R_\beta = \frac{r_0}{a}\left(\frac{dr_0/a}{d\beta}\right)^{-1}$$

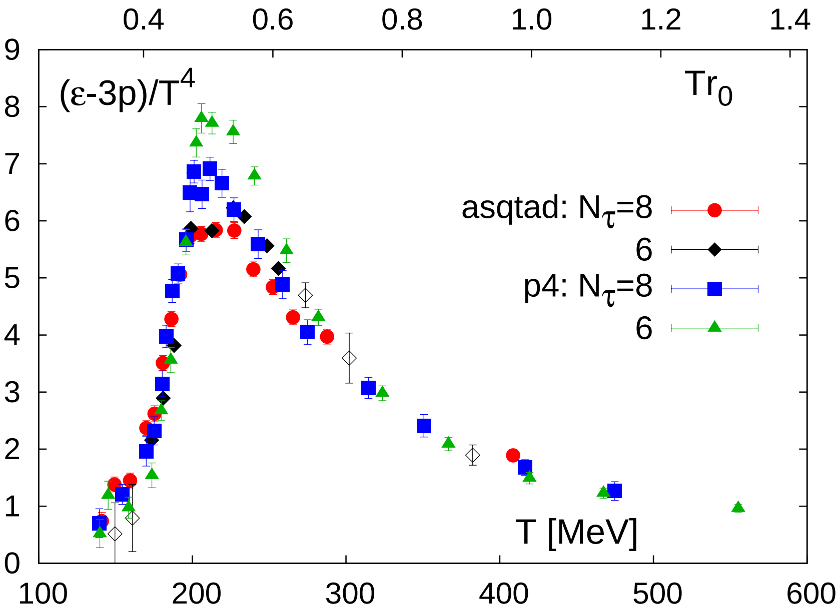
$$\hat{m}_l = a_m R_\beta^{(2-loop)} \left(\frac{12 b_0}{\beta}\right)^{4/9} \frac{1 + b_m \hat{a}^2 + c_m \hat{a}^4 + d_m \hat{a}^6}{1 + e_m \hat{a}^2 + f_m \hat{a}^4 + g_m \hat{a}^6} \quad \Rightarrow \quad R_m$$

results:

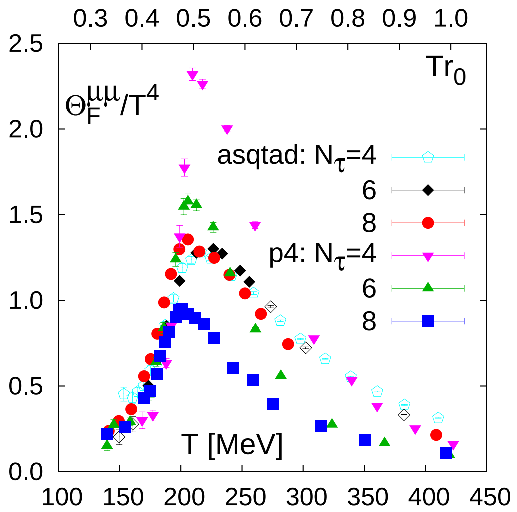
$\Theta^{\mu}_{\mu}(T)/T^4$ (the central quantity)

⋆ small discretization effects

⋆ marvellous agreement between `p4fat3` and `asqtad` actions



⋆ remaining discretization differences in the peak region mainly due to fermionic part Θ_F



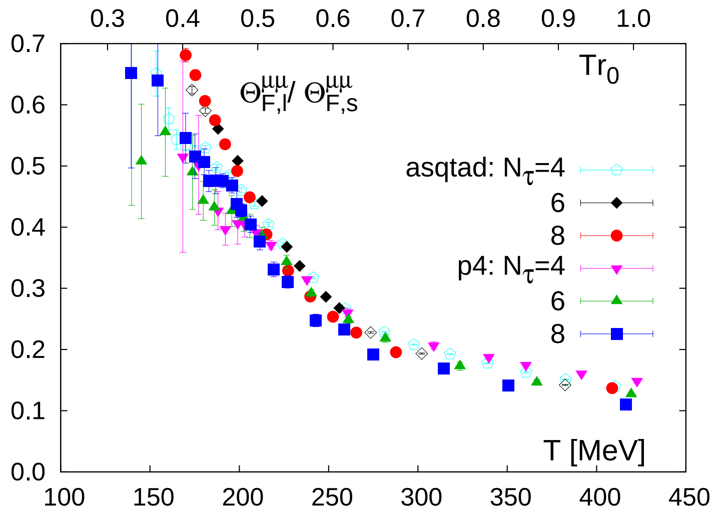
there: 15 % Θ_F contribution

$\Rightarrow$ 5 % overall

affected by R_m at low β

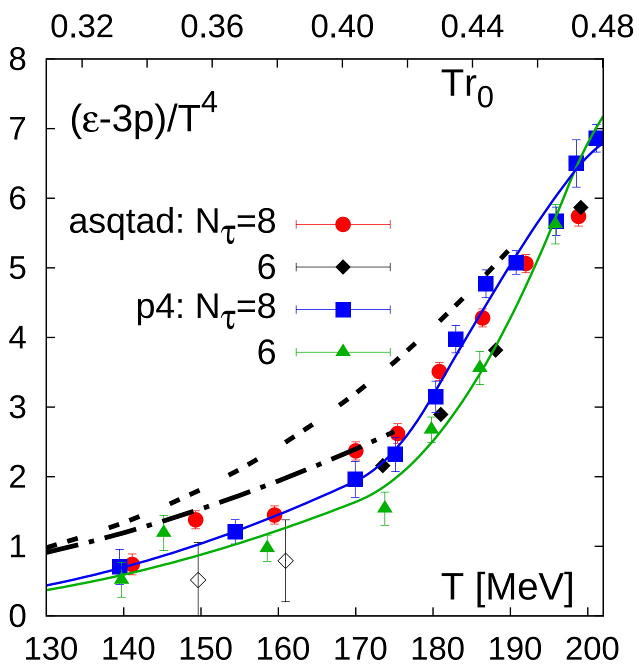
$\rightsquigarrow$ drops out in $\Theta_{F,l}/\Theta_{F,s}$

remaining differences due to slight mismatch in m_s^{phys}



results

low T region:



model to compare with:

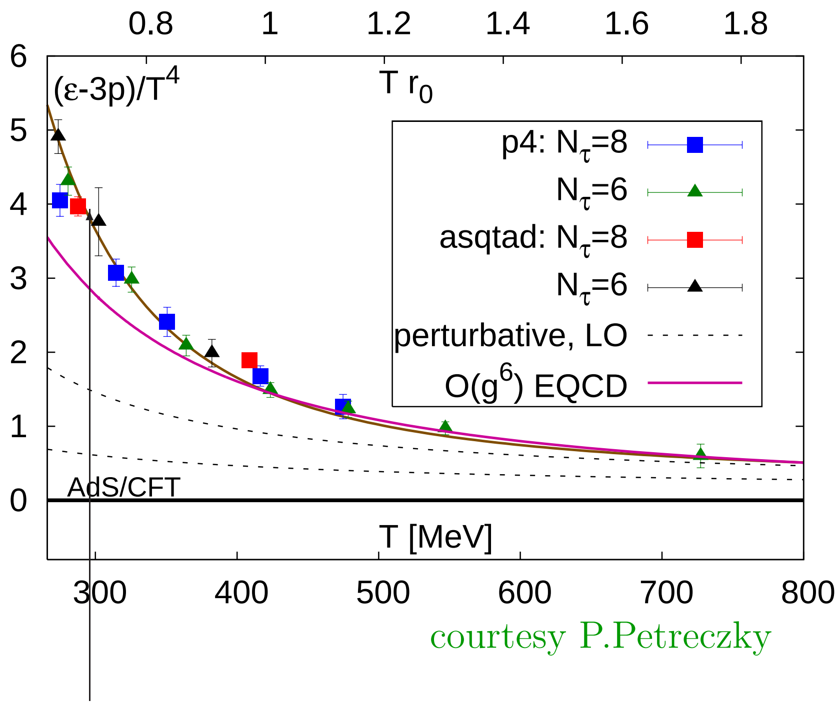
hadron resonance gas (HRG)

$$\left(\frac{\epsilon - 3p}{T^4}\right)_{lowT} = \sum_{m_i \le m_{max}} \frac{d_i}{2\pi^2} \sum_{k=1}^{\infty} (\pm)^{k+1} \frac{1}{k} \left(\frac{m_i}{T}\right)^3 K_1(km_i/T)$$

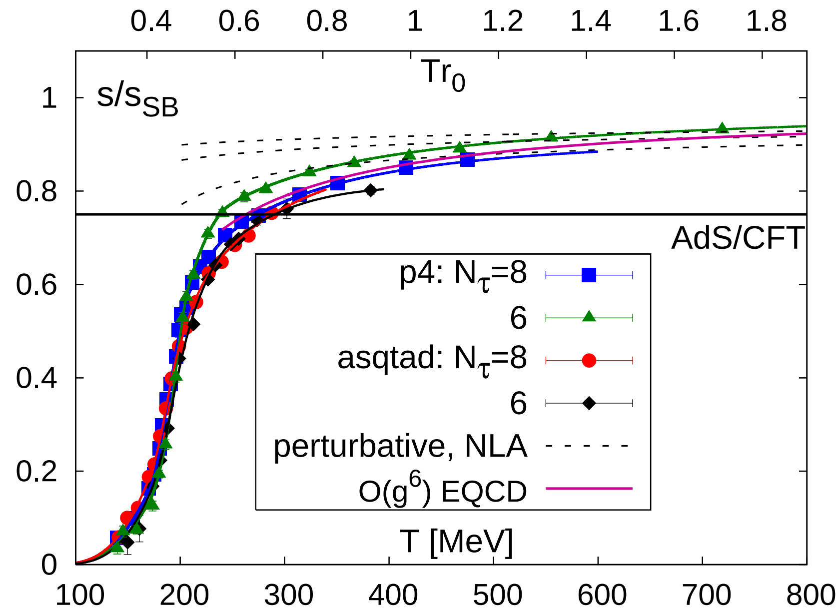
putting HRG to scrutiny

- ⋆ reducing discretization effects will lower the crossover temperature by a few MeV
- ⋆ smaller/physical quark masses will raise $\epsilon - 3p$ somewhat, but
- ⋆ the number of resonances is more important than the number of light pions

results: high T



courtesy P.Petreczky



$$s/T^3 = (\epsilon + p)/T^4$$

$$\left(\frac{\epsilon - 3p}{T^4}\right)_{highT} = \frac{3}{4} b_0 g^4(T) + \frac{d_2}{T^2} + \frac{d_4}{T^4}$$

EQCD: 3d effective theory, pert. matching and non-perturbative input

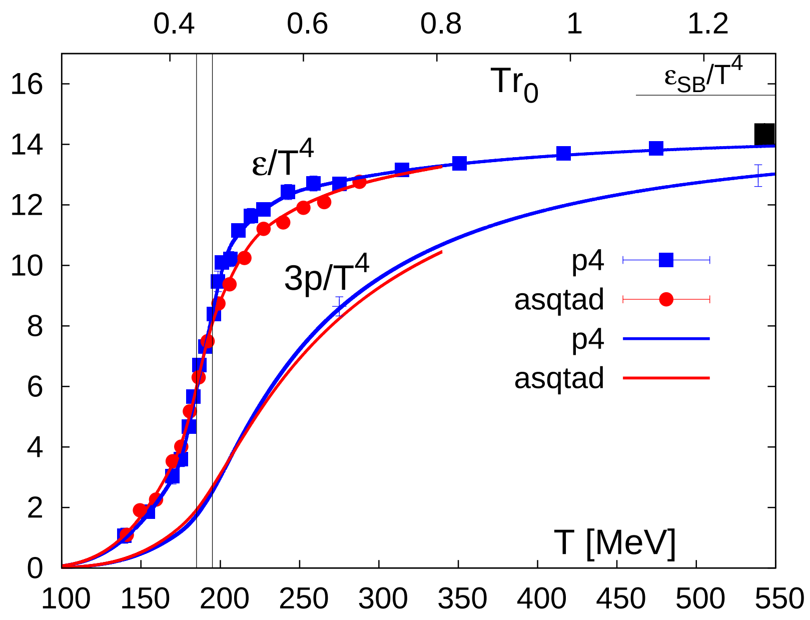
Kajantie et al.; Laine, Schröder

good agreement with resummed pert.theory

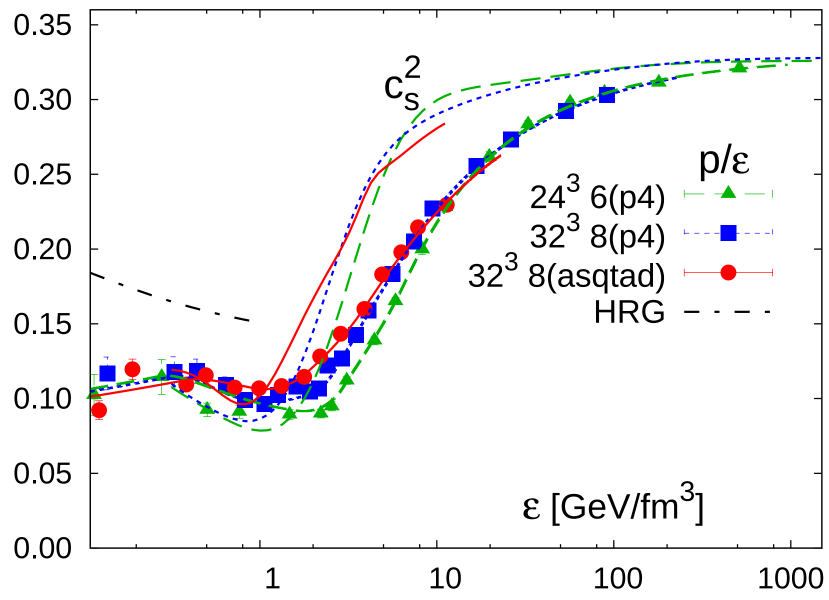
Blaizot et al.

the results:

pressure and energy density



speed of sound $c_s^2 = \frac{p}{\epsilon} + \epsilon \frac{d(p/\epsilon)}{d\epsilon}$



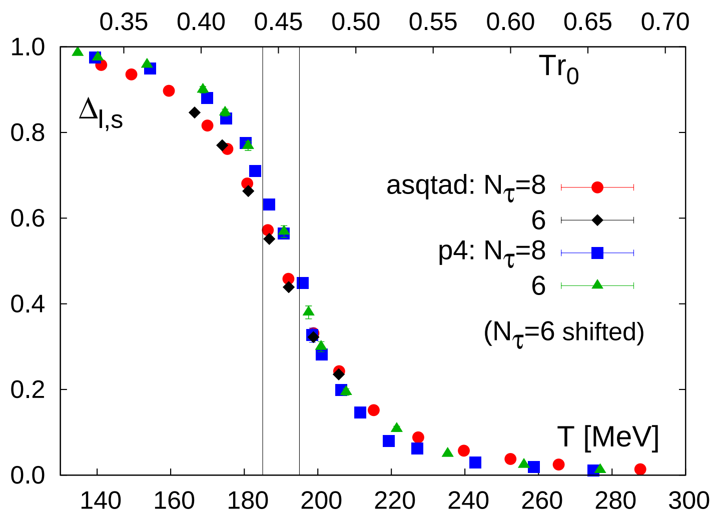
- ⋆ discretization errors small, good agreement between **asqtad** and **p4fat3** actions
- ⋆ integration error: see little bar to the right
- ⋆ in comparison with **Stefan-Boltzmann**: 10 % below at 2 - 3 T_c
- ⋆ softest point corresponds to $\epsilon \simeq 1$ GeV/fm^3

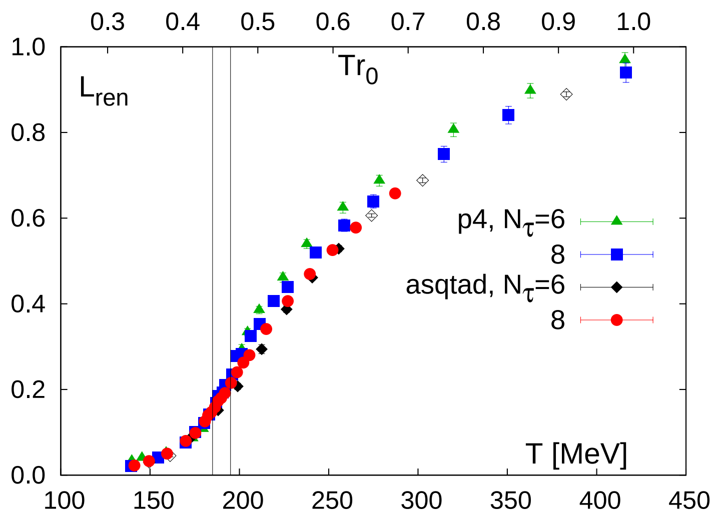
vertical lines denote transition region 185 MeV $< T <$ 195 MeV

comparison with chiral condensate and Polyakov loop

– chiral condensate $\langle\bar{\psi}\psi\rangle$ true order parameter in the chiral limit $m_q \to 0$

– Polyakov limit $L \sim \exp(-F_q/T)$ true order parameter in the static quark limit $m_q \to \infty$





– both need renormalization

$$\Delta_{l,s}(T) = \frac{\langle\bar{\psi}\psi\rangle_{l,T} - \frac{m_l}{m_s}\langle\bar{\psi}\psi\rangle_{s,T}}{\langle\bar{\psi}\psi\rangle_{l,0} - \frac{m_l}{m_s}\langle\bar{\psi}\psi\rangle_{s,0}}$$

removes additive as well as multiplicative UV divergencies

UV renormalization via normalization of static quark potential

significant changes in the same temperature regime as EoS

II Fluctuations and Correlations

⋆ probe of deconfinement and chiral aspects of the QCD transitions at $\mu_X = 0$

⋆ related to event-by-event fluctuations in HICs

Taylor expansion

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(T, \mu_u, \mu_d, \mu_s) = \sum_{i,j,k} c_{ijk}(T) \left(\frac{\mu_u}{T}\right)^i \left(\frac{\mu_d}{T}\right)^j \left(\frac{\mu_s}{T}\right)^k$$

with $(\hat{\mu} = \mu/T)$

$$c_{ijk} = \frac{1}{i!j!k!} \frac{\partial^i}{\partial \hat{\mu}_u^i} \frac{\partial^j}{\partial \hat{\mu}_d^j} \frac{\partial^k}{\partial \hat{\mu}_s^k} \left(\frac{p}{T^4}\right) |_{\vec{\mu}=0}$$

for instance

$$c_{200} = \frac{N_\tau}{2N_\sigma^3} \left(\frac{1}{4}\left\langle\frac{\partial^2 \ln \det M}{\partial \hat{\mu}_u^2}\right\rangle + \frac{1}{16}\left\langle\left(\frac{\partial \ln \det M}{\partial \hat{\mu}_u}\right)^2\right\rangle\right)$$

with

$$\frac{\partial^2 \ln \det M}{\partial \hat{\mu}_u^2} = \mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \hat{\mu}_u^2}\right) - \mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \hat{\mu}_u} M^{-1}\frac{\partial M}{\partial \hat{\mu}_u}\right)$$

For degenerate u and d quarks: $\mu_u = \mu_d \equiv \mu_q \quad \Rightarrow$ e.g. $\quad c_{20}^{qs} = c_{200} + c_{020} + c_{110}$

note: $c_{ijk} = 0$ for $i + j + k$ odd because of charge symmetry closer to experiment in terms of B, S, Q quantum numbers

$$\frac{p}{T^4} = \sum_{i,j,k} c_{ijk}^{BSQ}(T) \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_S}{T}\right)^j \left(\frac{\mu_Q}{T}\right)^k$$

where $\mu_u = \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q \qquad \mu_d = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q \qquad \mu_s = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S$

for instance $c_{400}^{BSQ} = \frac{1}{81}\left(c_{40}^{qs} + c_{31}^{qs} + c_{22}^{qs} + c_{13}^{qs} + c_{04}^{qs}\right)$

in both cases:

fluctuations

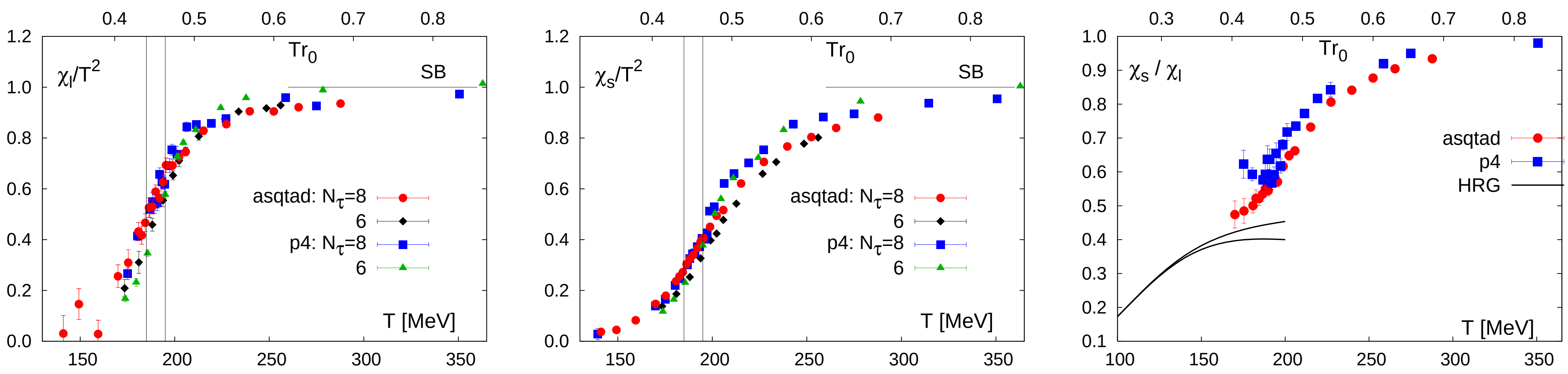
$$\chi_2^X \sim \langle N_X^2 \rangle - \langle N_X \rangle^2 \sim c_2$$

$$\chi_4^X \sim \langle N_X^4 \rangle - 3\langle N_X^2 \rangle^2 \sim c_4$$

correlations

$$\chi_{11}^{XY} \sim \langle N_X N_Y \rangle - \langle N_X \rangle \langle N_Y \rangle \sim c_{11}$$

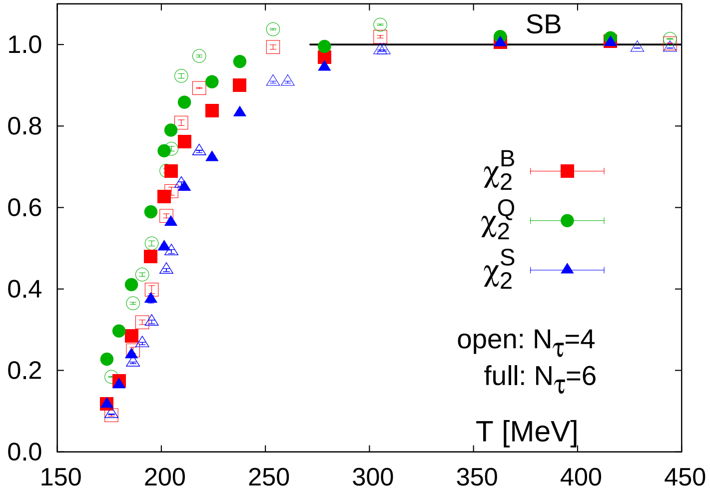
quark number susceptibilities/fluctuations

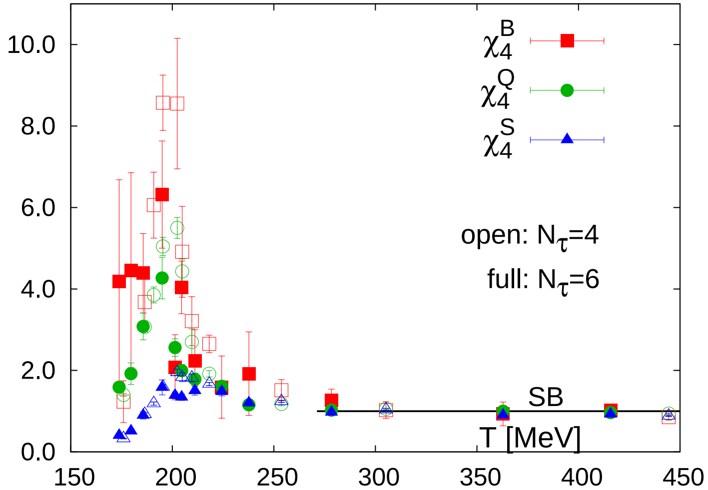


- ⋆ some but no large discretization effects
- ⋆ approaching SB limit, in case of light quarks quickly
- ⋆ rapid rise in χ_l signalling liberation of light quarks
- ⋆ slower rise in χ_s, in fact in transition region $\chi_s \simeq 1/2\chi_l$

BQS quantum number susceptibilities/fluctuations

normalized to SB limits $\langle B^2 \rangle = 1/3$, $\langle Q^2 \rangle = 2/3$, $\langle S^2 \rangle = 1$





⋆ some but no large discretization effects

⋆ approaching SB limit

⋆ rapid rise in χ_2, all quantum numbers B, Q, S

⋆ quartic fluctuations χ_4 develop peaks at transition

in fact,

with reduced temperature:

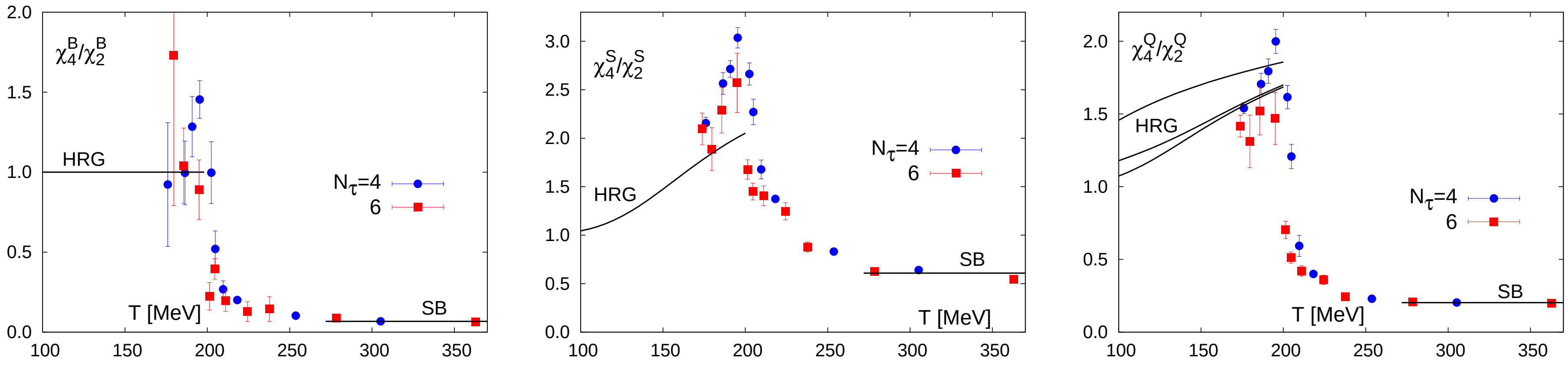
$$\tau = \left|\frac{T - T_c}{T_c}\right| - \left(\frac{\mu}{T}\right)^2$$

$$\chi_{2n}^B(\mu = 0) \sim \left|\frac{T - T_c}{T_c}\right|^{2-n-\alpha}$$

from scaling of f_{singular}

criticality of nearby phase transition at $m_q \to 0$

fluctuation ratios



in comparison with hadron resonance gas

$$p^{HRG} \sim \sum_{i \in \, mesons} \ln \mathcal{Z}^M_{m_i}(T, V, \mu_B, \mu_Q, \mu_S) + \sum_{i \in \, baryons} \ln \mathcal{Z}^B_{m_i}(T, V, \mu_B, \mu_Q, \mu_S)$$

with $\ln \mathcal{Z}^{M/B}_{m_i} \sim \int_0^\infty dk k^2 \ln(1 \mp z_i e^{-\varepsilon_i/T})$ and fugacities $z_i = \exp\left((B_i\mu_B + Q_i\mu_Q + S_i\mu_S)/T\right)$

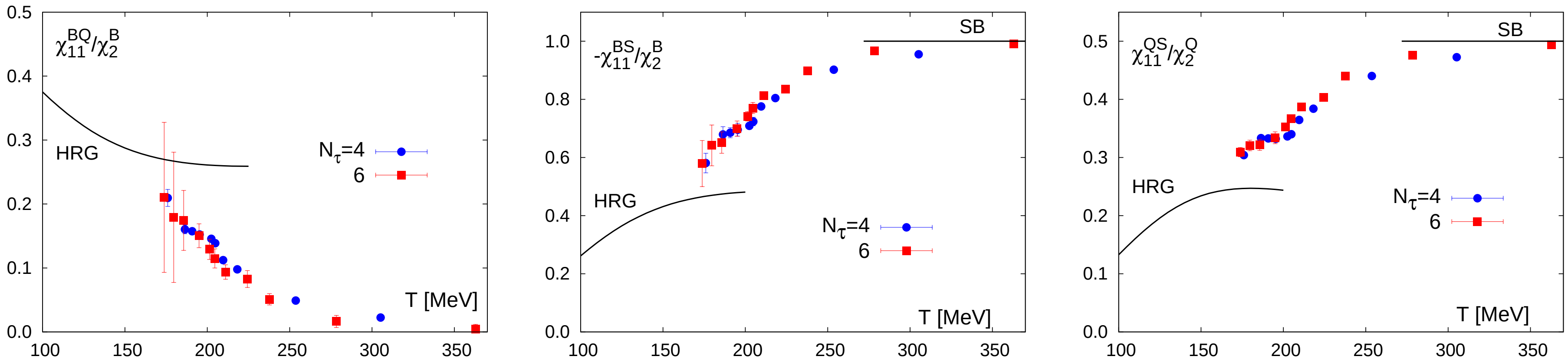
e.g. for B: $\ln Z^B_m \sim \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \ell^{-2} K_2(\ell m/T) \cosh(\ell \mu_B/T) \quad \rightsquigarrow \quad \frac{\chi_4^B}{\chi_2^B} = 1$

for Q,S: hadrons with $Q, S > 1$ contribute, more sensitivity to quark mass because of πs

general picture: right ballpark at low T

pronounced cusps at transition

correlations



at low T: HRG qualitatively captures physics: magnitude, drop in $\langle BQ \rangle$ and rise in $\langle BS \rangle$, $\langle QS \rangle$

e.g. BQ: numerator (anti) protons only
denominator also (anti) neutrons $\Rightarrow$ $\langle BQ \rangle / \langle B^2 \rangle \longrightarrow 1/2$ at $T = 0$

BS: lightest baryons don't carry strangeness $\Rightarrow$ $\langle BS \rangle \longrightarrow 0$ at $T = 0$

at high T: for fully uncorrelated quarks

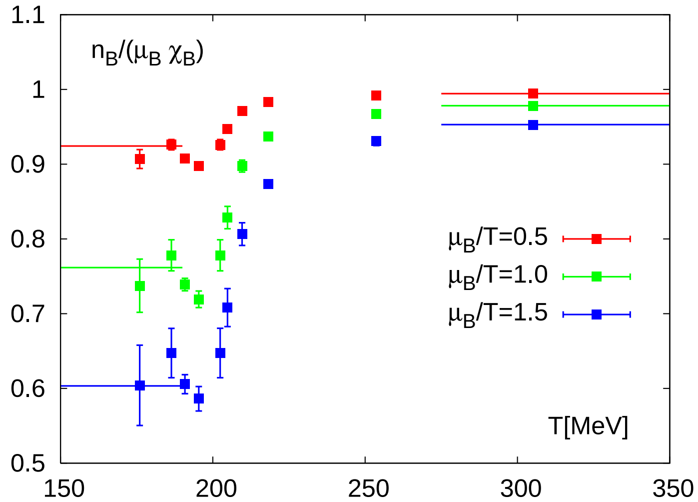
$$\langle BQ \rangle = \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right) + \left(\frac{1}{3}\right) \cdot \left(-\frac{1}{3}\right) + \left(\frac{1}{3}\right) \cdot \left(-\frac{1}{3}\right) = 0$$

$$\langle BS \rangle = \left(\frac{1}{3}\right) \cdot 0 + \left(\frac{1}{3}\right) \cdot 0 + \left(\frac{1}{3}\right) \cdot (-1) = -\frac{1}{3}$$

further, Taylor coefficients provide a window to, at least, small $\mu \neq 0$ $(\rightarrow$ RHIC, LHC)

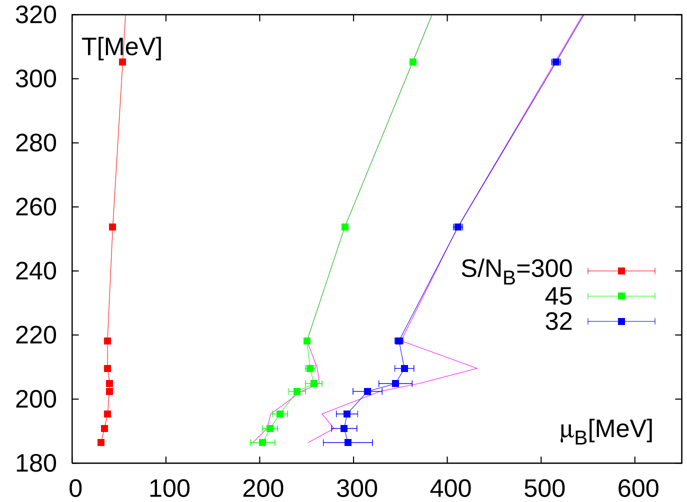
e.g.

compressibility



$$\frac{1}{\kappa_T} \sim \frac{n_B}{\mu_B \chi_B} \rightarrow 0 \quad \text{at critical point.}$$

isentropic lines



RHIC
SPS
FAIR

$$\frac{S}{N_B}(T, \mu_B) = const$$

Summary and Outlook

- For the EoS
  - we are getting to robust numbers
  - further improvements : $\longrightarrow$ physical light quark mass $m_q = m_s/20$
    $\longrightarrow$ $aT = 1/12$ to control continuum limit (R_m !)

- Fluctuations and correlations show
  - qualitative agreement with a hadron resonance gas below transition
  - quark degrees of freedom dominate rather rapidly above the transition
  - Taylor coeff. useful for QCD at (small) finite density (relevant for RHIC, LHC, early universe):
    $\rightsquigarrow$ get systematics under better control $\longrightarrow$ higher orders in Taylor
    $\longrightarrow$ comprehensive attack of the various approaches

- in the long run
  - comparison with chiral fermion discretizations
  - addressing the critical end point (?) in the $T - \mu$ phase diagram ($\longrightarrow$ FAIR)