Complete second-order dissipative relativistic fluid dynamics

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Preliminaries (I)

Tensor decomposition of net charge current and energy-momentum tensor:

1. **Net charge current:** $\boxed{N^\mu = n\, u^\mu + \nu^\mu}$

   u^μ fluid 4-velocity, $u^\mu u_\mu = u^\mu g_{\mu\nu} u^\nu = 1$

   $g_{\mu\nu} \equiv \mathrm{diag}(+,-,-,-)$ (West coast!!) metric tensor,

   $n \equiv u^\mu N_\mu$ net charge density in fluid rest frame

   $\nu^\mu \equiv \Delta^{\mu\nu} N_\nu$ diffusion current (flow of net charge relative to u^μ), $\nu^\mu u_\mu = 0$

   $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ projector onto 3-space orthogonal to u^μ, $\Delta^{\mu\nu} u_\nu = 0$

2. **Energy-momentum tensor:** $\boxed{T^{\mu\nu} = \epsilon\, u^\mu u^\nu - (p + \Pi)\, \Delta^{\mu\nu} + 2\, q^{(\mu} u^{\nu)} + \pi^{\mu\nu}}$

   $\epsilon \equiv u^\mu T_{\mu\nu} u^\nu$ energy density in fluid rest frame

   p pressure in fluid rest frame

   Π bulk viscous pressure, $p + \Pi \equiv -\frac{1}{3}\, \Delta^{\mu\nu} T_{\mu\nu}$

   $q^\mu \equiv \Delta^{\mu\nu} T_{\nu\lambda} u^\lambda$ heat flux current (flow of energy relative to u^μ), $q^\mu u_\mu = 0$

   $\pi^{\mu\nu} \equiv T^{<\mu\nu>}$ shear stress tensor, $\pi^{\mu\nu} u_\mu = \pi^{\mu\nu} u_\nu = 0\,,\ \pi^\mu{}_\mu = 0$

   $a^{(\mu\nu)} \equiv \frac{1}{2}\, (a^{\mu\nu} + a^{\nu\mu})$ symmetrized tensor

   $a^{<\mu\nu>} \equiv \left(\Delta_\alpha^{(\mu} \Delta^{\nu)}{}_\beta - \frac{1}{3}\, \Delta^{\mu\nu} \Delta_{\alpha\beta} \right) a^{\alpha\beta}$ symmetrized, traceless spatial projection

Preliminaries (II)

Fluid dynamical equations:

1. **Net charge (e.g., strangeness) conservation:**

$$\boxed{\partial_\mu N^\mu = 0} \iff \boxed{\dot{n} + n\,\theta + \partial \cdot \nu = 0}$$

$\dot{a} \equiv u^\mu \partial_\mu a$ convective (comoving) derivative

(fluid rest frame, $u^\mu_{\rm RF} \equiv g^\mu{}_0 \implies$ time derivative, $\dot{a}_{\rm RF} \equiv \partial_t a$)

$\theta \equiv \partial_\mu u^\mu$ expansion scalar

2. **Energy-momentum conservation:**

$\boxed{\partial_\mu T^{\mu\nu} = 0} \iff$ energy conservation:

$$\boxed{u_\nu\, \partial_\mu T^{\mu\nu} = \dot{\epsilon} + (\epsilon + p + \Pi)\,\theta + \partial \cdot q - q \cdot \dot{u} - \pi^{\mu\nu}\, \partial_\mu u_\nu = 0}$$

acceleration equation:

$$\Delta^{\mu\nu}\, \partial^\lambda T_{\nu\lambda} = 0 \iff$$
$$(\epsilon + p)\dot{u}^\mu = \nabla^\mu(p + \Pi) - \Pi \dot{u}^\mu - \Delta^{\mu\nu}\dot{q}_\nu - q^\mu\theta - q^\nu\partial_\nu u^\mu - \Delta^{\mu\nu}\, \partial^\lambda \pi_{\nu\lambda}$$

$\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ 3-gradient (spatial gradient in fluid rest frame)

Preliminaries (III)

Problem:

5 equations, but 15 unknowns (for given u^μ): $\epsilon\,,\; p\,,\; n\,,\; \Pi\,,\; \nu^\mu\,(3)\,,\; q^\mu\,(3)\,,\; \pi^{\mu\nu}\,(5)$

Solution:

1. clever choice of frame (Eckart, Landau,...): eliminate ν^μ or q^μ

   $\Longrightarrow$ does not help! Promotes u^μ to dynamical variable!

2. ideal fluid limit: all dissipative terms vanish, $\Pi = \nu^\mu = q^\mu = \pi^{\mu\nu} = 0$

   $\Longrightarrow$ 6 unknowns: $\epsilon\,,\; p\,,\; n\,,\; u^\mu\,(3)$ (not quite there yet...)

   $\Longrightarrow$ fluid is in local thermodynamical equilibrium

   $\Longrightarrow$ provide equation of state (EOS) $p(\epsilon, n)$ to close system of equations

3. provide additional equations for dissipative quantities

   $\Longrightarrow$ dissipative relativistic fluid dynamics

   (a) First-order theories: e.g. generalization of Navier-Stokes (NS) equations to the relativistic case (Eckart, Landau-Lifshitz)

   (b) Second-order theories: e.g. Israel-Stewart (IS) equations

Preliminaries (IV)

Navier-Stokes (NS) equations:

1. **bulk viscous pressure:** $\boxed{\Pi_{\mathrm{NS}} = -\zeta\,\theta}$

   ζ bulk viscosity

2. **heat flux current:** $\boxed{q^{\mu}_{\mathrm{NS}} = \frac{\kappa}{\beta}\,\frac{n}{\beta(\epsilon + p)}\,\nabla^{\mu}\alpha}$

   $\beta \equiv 1/T$ inverse temperature,

   $\alpha \equiv \beta\,\mu$, μ chemical potential,

   κ thermal conductivity

3. **shear stress tensor:** $\boxed{\pi^{\mu\nu}_{\mathrm{NS}} = 2\,\eta\,\sigma^{\mu\nu}}$

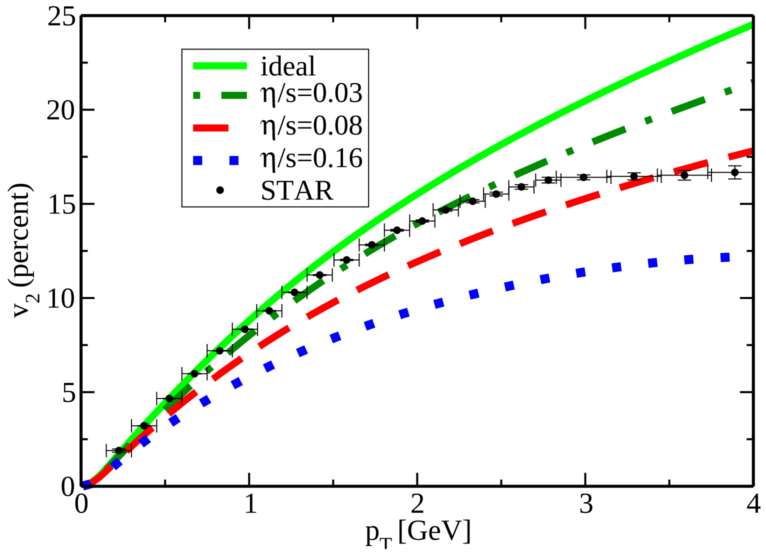
   η shear viscosity,

   $\sigma^{\mu\nu} = \nabla^{<\mu}u^{\nu>}$ shear tensor

$\Longrightarrow$ **algebraic expressions in terms of thermodynamic and fluid variables**

$\Longrightarrow$ **simple... but: unstable and acausal equations of motion!!**

Motivation (I)

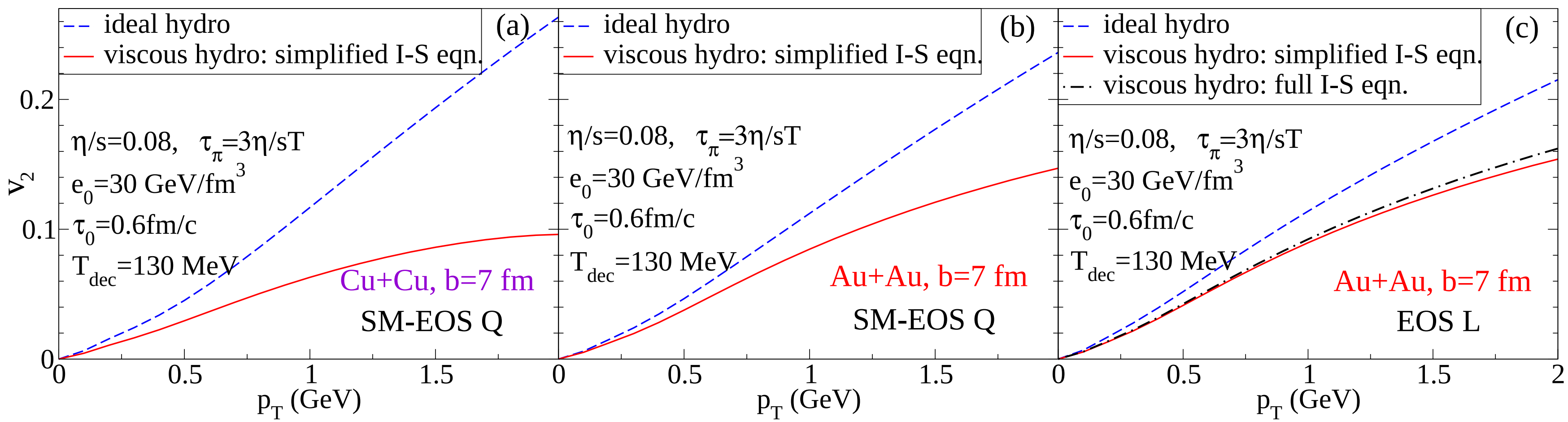


P. Romatschke, U. Romatschke, PRL 99 (2007) 172301

Au+Au @ $\sqrt{s} = 200$ AGeV

charged particles, min. bias

H. Song, U.W. Heinz, PRC 78 (2008) 024902



Motivation (II)

Israel-Stewart (IS) equations: second-order, dissipative relativistic fluid dynamics

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

"Simplified" IS equations: e.g. shear stress tensor

$$\boxed{\tau_\pi \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\rm NS}}$$

$\Longrightarrow$ dynamical (instead of algebraic) equations for dissipative terms!

$\Longrightarrow$ $\pi^{\mu\nu}$ relaxes to its NS value $\pi^{\mu\nu}_{\rm NS}$ on the time scale τ_π

$\Longrightarrow$ stable and causal fluid dynamical equations of motion!

"Full" IS equations:

$$\boxed{\tau_\pi \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\rm NS} - \frac{\eta}{2\beta}\, \pi^{\mu\nu}\, \partial_\lambda \left(\frac{\tau_\pi}{\eta}\, \beta\, u^\lambda \right) + 2\, \tau_\pi\, \pi_\lambda{}^{<\mu} \omega^{\nu>\lambda}}$$

$$\omega^{\mu\nu} \equiv \frac{1}{2} \Delta^{\mu\alpha} \Delta^{\nu\beta} \left(\partial_\alpha u_\beta - \partial_\beta u_\alpha \right) \qquad \text{vorticity}$$

Motivation (III)

$\Longrightarrow$ **Difference between "simplified" and "full" IS equations: the latter include higher-order terms?**

For instance, if $\dfrac{\pi^{\mu\nu}}{\epsilon} \sim \delta \ll 1\,,\;\; \tau_\pi\, \omega^{\mu\nu} \sim \delta \ll 1 \quad \Longrightarrow \quad \tau_\pi\, {\omega_\lambda}^{<\mu} \pi^{\nu>\lambda}\, \dfrac{1}{\epsilon} \sim \delta^2$

$\Longrightarrow$ **Goals:**

1. **What are the correct equations of motion for the dissipative quantities?**
   $\Longrightarrow$ **develop consistent power counting scheme**
2. **Generalization to $\mu \neq 0$ (relevant for FAIR physics!)**
   $\Longrightarrow$ **include heat flux q^μ**
3. **Generalization to non-conformal fluids (relevant near T_c!)**
   $\Longrightarrow$ **include bulk viscous pressure Π**

Results (I)

Power counting:

3 length scales: 2 microscopic, 1 macroscopic

- thermal wavelength $\lambda_{\text{th}} \sim \beta \equiv 1/T$
- mean free path $\ell_{\text{mfp}} \sim (\langle\sigma\rangle n)^{-1}$

  $\langle\sigma\rangle$ averaged cross section, $n \sim T^3 = \beta^{-3} \sim \lambda_{\text{th}}^{-3}$
- length scale over which macroscopic fluid fields vary L_{hydro} , $\partial_\mu \sim L_{\text{hydro}}^{-1}$

Note: since $\eta \sim (\langle\sigma\rangle\lambda_{\text{th}})^{-1}$ $\Longrightarrow$ $\boxed{\dfrac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \dfrac{1}{\langle\sigma\rangle n}\dfrac{1}{\lambda_{\text{th}}} \sim \dfrac{\lambda_{\text{th}}^3}{\langle\sigma\rangle\lambda_{\text{th}}} \sim \dfrac{\lambda_{\text{th}}^3}{\langle\sigma\rangle\lambda_{\text{th}}} \sim \dfrac{\eta}{s}}$

s entropy density, $s \sim n \sim T^3 = \beta^{-3} \sim \lambda_{\text{th}}^{-3}$

$\Longrightarrow$ $\dfrac{\eta}{s}$ solely determined by the 2 microscopic length scales!

Note: similar argument holds for $\dfrac{\zeta}{s}$, $\dfrac{\kappa}{\beta\, s}$

Results (II)

3 regimes:

- **dilute gas limit** $\dfrac{\ell_{\rm mfp}}{\lambda_{\rm th}} \sim \dfrac{\eta}{s} \gg 1 \iff \langle\sigma\rangle \ll \lambda_{\rm th}^2 \implies$ **weak-coupling limit**
- **viscous fluids** $\dfrac{\ell_{\rm mfp}}{\lambda_{\rm th}} \sim \dfrac{\eta}{s} \sim 1 \iff \langle\sigma\rangle \sim \lambda_{\rm th}^2$
  interactions happen on the scale $\lambda_{\rm th}$ $\implies$ **moderate coupling**
- **ideal fluid limit** $\dfrac{\ell_{\rm mfp}}{\lambda_{\rm th}} \sim \dfrac{\eta}{s} \ll 1 \iff \langle\sigma\rangle \gg \lambda_{\rm th}^2 \implies$ **strong-coupling limit**

gradient (derivative) expansion: $\boxed{\ell_{\rm mfp}\,\partial_\mu \sim \dfrac{\ell_{\rm mfp}}{L_{\rm hydro}} \equiv K \sim \delta \ll 1}$

K **Knudsen number**

$\implies$ **equivalent to $k\,\ell_{\rm mfp} \ll 1\,,\quad k$ typical momentum scale**

R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100

$\implies$ **separation of macroscopic fluid dynamics (large scale $\sim L_{\rm hydro}$)**
from microscopic particle dynamics (small scale $\sim \ell_{\rm mfp}$)

Results (III)

Primary quantities: $\epsilon\,,\ p\,,\ n\,,\ s$ $\Longleftrightarrow$ **Dissipative quantities:** $\Pi\,,\ q^\mu\,,\ \pi^{\mu\nu}$

$$\boxed{\text{If}\quad K \sim \ell_{\rm mfp}\,\partial_\mu \sim \delta \ll 1\ ,\ \text{then}\quad \frac{\Pi}{\epsilon} \sim \frac{q^\mu}{\epsilon} \sim \frac{\pi^{\mu\nu}}{\epsilon} \sim \delta \ll 1}$$

Dissipative quantities are small compared to primary quantities

$\Longrightarrow$ **small deviations from local thermodynamical equilibrium!**

Note: statement independent of value of $\dfrac{\zeta}{s}\,,\ \dfrac{\kappa}{\beta\, s}\,,\ \dfrac{\eta}{s}$ **!**

Proof: **Gibbs relation:** $\epsilon + p = Ts + \mu n \quad \Longrightarrow \quad \beta\,\epsilon \sim s$ **!**

Estimate dissipative terms by their Navier-Stokes values:

$$\Pi \sim \Pi_{\rm NS} = -\zeta\,\theta\,, \quad q^\mu \sim q^\mu_{\rm NS} = \frac{\kappa}{\beta}\,\frac{n}{\beta(\epsilon+p)}\,\nabla^\mu\alpha\,, \quad \pi^{\mu\nu} \sim \pi^{\mu\nu}_{\rm NS} = 2\,\eta\,\sigma^{\mu\nu}$$

$$\Longrightarrow \quad \frac{\Pi}{\epsilon} \sim -\frac{\zeta}{\beta\,\epsilon}\,\beta\,\theta \sim -\frac{\zeta}{s}\,\frac{\beta}{\lambda_{\rm th}}\,\frac{\lambda_{\rm th}}{\ell_{\rm mfp}}\,\ell_{\rm mfp}\,\theta \sim \ell_{\rm mfp}\,\partial_\mu u^\mu \sim \delta\,,$$

$$\frac{q^\mu}{\epsilon} \sim \frac{\kappa}{\beta}\,\frac{1}{\beta\,\epsilon}\,\frac{n}{\beta(\epsilon+p)}\,\beta\,\nabla^\mu\alpha \sim \frac{\kappa}{\beta\,s}\,\frac{\beta}{\lambda_{\rm th}}\,\frac{\lambda_{\rm th}}{\ell_{\rm mfp}}\,\ell_{\rm mfp}\,\nabla^\mu\alpha \sim \ell_{\rm mfp}\,\nabla^\mu\alpha \sim \delta\,,$$

$$\frac{\pi^{\mu\nu}}{\epsilon} \sim 2\,\frac{\eta}{\beta\,\epsilon}\,\beta\,\sigma^{\mu\nu} \sim 2\,\frac{\eta}{s}\,\frac{\beta}{\lambda_{\rm th}}\,\frac{\lambda_{\rm th}}{\ell_{\rm mfp}}\,\ell_{\rm mfp}\,\sigma^{\mu\nu} \sim \ell_{\rm mfp}\,\nabla^{<\mu}u^{\nu>} \sim \delta\,, \qquad \text{q.e.d.}$$

Results (IV)

IS equations:

$$
\begin{aligned}
\tau_\Pi \dot{\Pi} + \Pi &= \Pi_{NS} + \tau_{\Pi q}\, q \cdot \dot{u} - \ell_{\Pi q}\, \partial \cdot q \\
&\quad - \tau_\Pi \frac{\hat{\zeta}_1}{\zeta} \Pi^2 - \tau_\Pi \frac{\hat{\zeta}_2 \beta}{\kappa} q \cdot q - \tau_\Pi \frac{\hat{\zeta}_3}{2\eta} \pi^{\mu\nu}\pi_{\mu\nu} \\
\tau_q \Delta^{\mu\nu} \dot{q}_\nu + q^\mu &= q^\mu_{NS} - \tau_{q\Pi}\, \Pi\, \dot{u}^\mu - \tau_{q\pi}\, \pi^{\mu\nu} \dot{u}_\nu \\
&\quad + \ell_{q\Pi} \nabla^\mu \Pi - \ell_{q\pi} \Delta^{\mu\nu} \partial^\lambda \pi_{\nu\lambda} + \tau_q\, \omega^{\mu\nu} q_\nu \\
&\quad - \tau_q \frac{\hat{\kappa}_1}{\zeta} q^\mu \Pi - \tau_q \frac{\hat{\kappa}_2}{2\eta} \pi^{\mu\nu} q_\nu \\
\tau_\pi \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} &= \pi^{\mu\nu}_{NS} + 2\,\tau_{\pi q}\, q^{<\mu} \dot{u}^{\nu>} + 2\,\ell_{\pi q} \nabla^{<\mu} q^{\nu>} + 2\,\tau_\pi\, \pi_\lambda{}^{<\mu} \omega^{\nu>\lambda} \\
&\quad - 2\,\tau_\pi \frac{\hat{\eta}_1}{2\eta} \pi_\lambda{}^{<\mu} \pi^{\nu>\lambda} - 2\,\tau_\pi \frac{\hat{\eta}_2 \beta}{\kappa} q^{<\mu} q^{\nu>} - 2\,\tau_\pi \frac{\hat{\eta}_3}{\zeta} \Pi\, \pi^{\mu\nu}
\end{aligned}
$$

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

A. Muronga, PRC 76 (2007) 014909 (and parts of $\hat{\zeta}_1$, $\hat{\kappa}_1$, $\hat{\eta}_3$)

B. Betz, D. Henkel, DHR, Prog. Part. Nucl. Phys. 62 (2009) 556

B. Betz, T. Koide, H. Niemi, DHR, in preparation

Results (V)

Remarks:

1. Structure of second-order terms follows exclusively from Lorentz covariance
2. Coefficients can be computed from kinetic theory and Grad's 14-moment method B. Betz, H. Niemi, T. Koide, DHR, in preparation
3. R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100: second-order fluid dynamics for conformal fluids (AdS/CFT correspondence) $\Longrightarrow$ second-order term $\sim \frac{\lambda_1}{\eta^2}\, \pi_\lambda{}^{<\mu}\pi^{\nu>\lambda}$ $\Longrightarrow$ $\lambda_1 \equiv \tau_\pi\, \eta\, \hat{\eta}_1$
   Note: second-order terms from collision integral $\Longrightarrow$ $\eta_1 \neq 1$!
   cf. M.A. York, G.D. Moore, arXiv:0811.0729
4. Coefficients $\hat{\zeta}_1$, $\hat{\zeta}_2$, $\hat{\zeta}_3$, $\hat{\kappa}_1$, $\hat{\kappa}_2$, $\hat{\eta}_1$, $\hat{\eta}_2$, $\hat{\eta}_3$ are (complicated) dimensionless functions of α, β
5. Viscosities and thermal conductivity ζ, η, κ, relaxation times τ_Π, τ_q, τ_π, coefficients $\tau_{\Pi q}$, $\tau_{q\Pi}$, $\tau_{q\pi}$, $\tau_{\pi q}$, $\ell_{\Pi q}$, $\ell_{q\Pi}$, $\ell_{q\pi}$, $\ell_{\pi q}$ are (complicated) functions of α, β, divided by tensor coefficients of second moment of collision integral:
   $\sim \chi_i(\alpha, \beta)/\langle\sigma\rangle \longrightarrow 0$ as cross section $\sigma \longrightarrow \infty$ ("strong coupling limit"!)
   $\Longrightarrow$ $\Pi = q^\mu = \pi^{\mu\nu} \longrightarrow 0$ ideal fluid limit!

Results (VI)

6. IS equations are formally independent of calculational frame (Eckart, Landau,...), but ...

7. Values of coefficients are frame dependent! We have analyzed:

   (a) Eckart (N) or (net) charge frame: $\boxed{\nu^\mu = 0\,,\ \epsilon = \epsilon_0\,,\ n = n_0}$

   $\epsilon_0\,,\ n_0$: energy density and charge density in local thermodyn. equilibrium

   (b) Landau (E) or energy frame: $\boxed{q^\mu = 0\,,\ \epsilon = \epsilon_0\,,\ n = n_0}$

   Note: in IS equations $q^\mu \equiv -\dfrac{\epsilon + p}{n}\,\nu^\mu$

   (c) Tsumura-Kunihiro-Ohnishi (TKO) frame: $\boxed{\nu^\mu = 0\,,\ \epsilon = \epsilon_0 - 3\,\Pi\,,\ n = n_0}$

   We have checked agreement with the results of IS for most coefficients computed by IS...

8. R.h.s.: all terms except NS terms are of second order, $\sim \delta^2$

   $\Longrightarrow$ $t < \tau_\Pi \sim \tau_q \sim \tau_\pi$: dissipative terms relax towards their NS values,

   $t > \tau_\Pi \sim \tau_q \sim \tau_\pi$: last terms on r.h.s. and NS terms on l.h.s. largely cancel, second-order terms govern evolution!

Conclusions and open problems

1. Derived Israel-Stewart (IS) equations from kinetic theory via Grad's 14-moment method $\Longrightarrow$ new second-order terms!

2. Results consistent with

   R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100

   M.A. York, G.D. Moore, arXiv:0811.0729

3. Coefficients of terms in IS equations are frame dependent

   $\Longrightarrow$ have (not yet completely) been computed in various frames (Eckart, Landau, TKO)

4. Generalization to a system of various particle species (done: quarks, antiquarks, gluons), various conserved charges

   cf. M. Prakash, M. Prakash, R. Venugopalan, G. Welke, Phys. Rept. 227 (1993) 321

   G. Denicol, DHR, in preparation

5. Numerical implementation

   E. Molnar, H. Niemi, DHR, in preparation