Inhomogeneous phases of strongly interacting matter

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Massachusetts Institute of Technology

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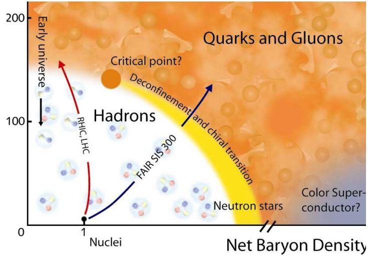
Motivation



- QCD phase diagram

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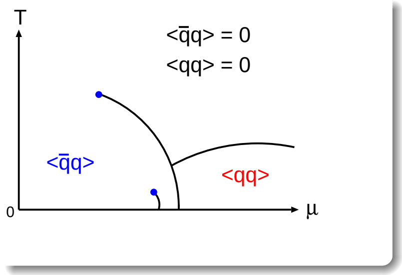

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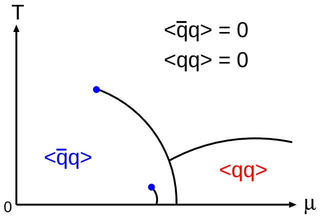

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- frequent assumption:
  $\langle \bar{q}q \rangle$, $\langle qq \rangle$ constant in space

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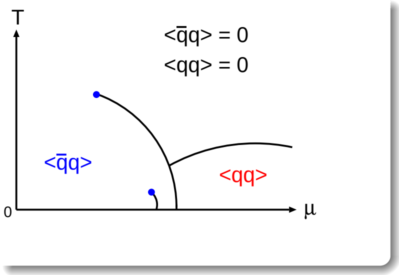

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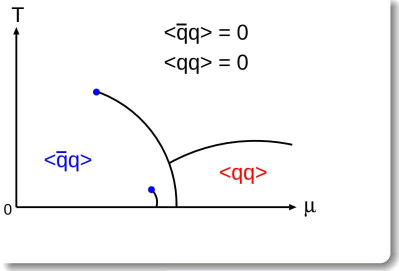

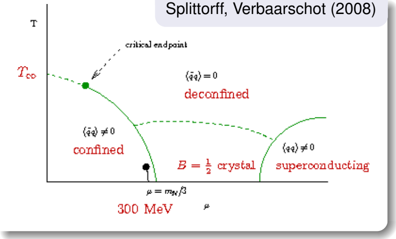

Splittorff, Verbaarschot (2008)

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- inhomogeneous phases:
  - Skyrme crystal

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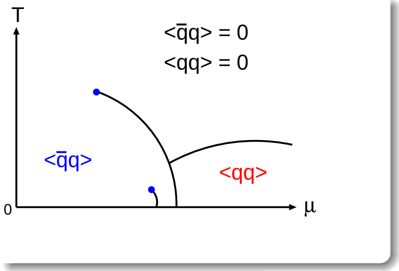

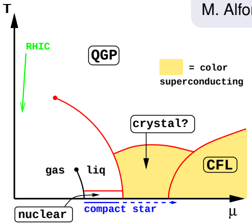

M. Alford (2003)

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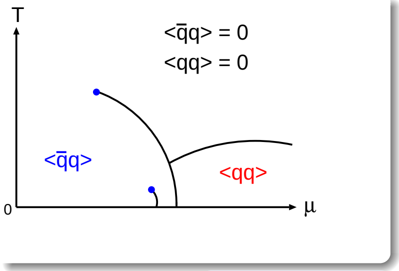

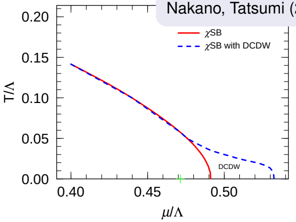

Nakano, Tatsumi (2005)

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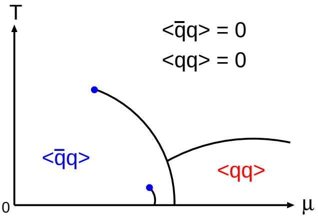
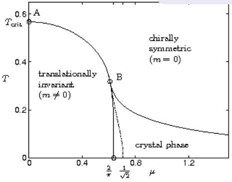


Thies, Urlichs (2003)



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  - chiral density wave
  - 1+1 D Gross-Neveu model

Outline

Topics to be discussed:

- Solitonic ground states in color superconductivity

  (based on: D. Nickel, M.B., PRD 79, 054009 (2009))

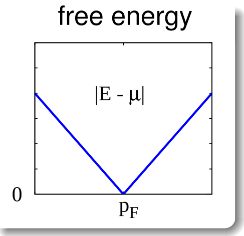
- Chiral phase transitions with inhomogeneous phases

  (based on: D. Nickel, arXiv:0902.1778, arXiv:0906.5295)

BCS pairing

- Cooper instability:

  Fermi gas



BCS pairing

- Cooper instability:

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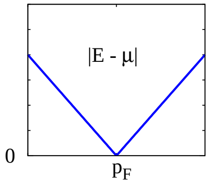
  ➜ condensation of Cooper pairs

free energy



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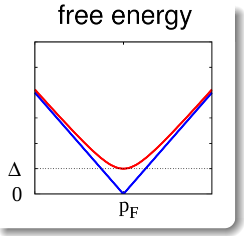

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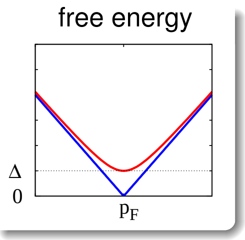

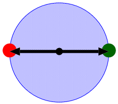

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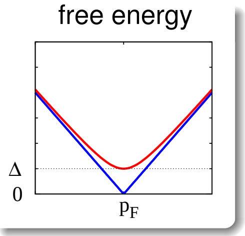
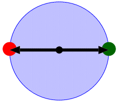
  ➜ works only if $p_F^a = p_F^b$





Stressed pairing

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    - strange-nonstrange systems
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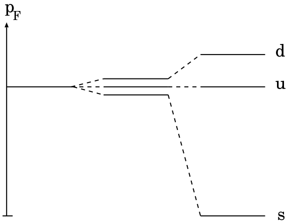

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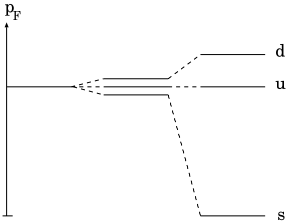

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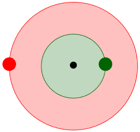

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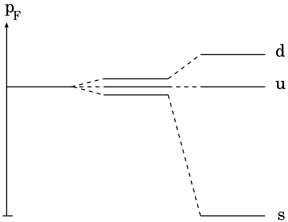

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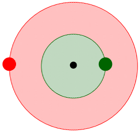

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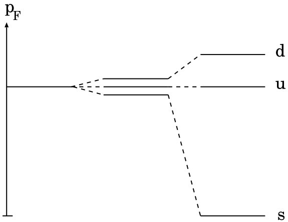

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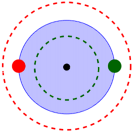

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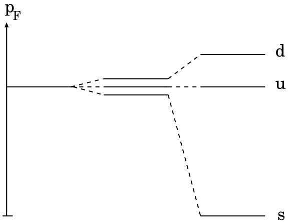

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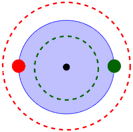

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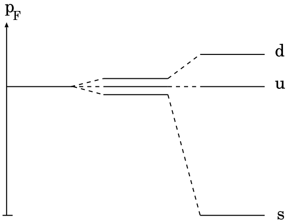

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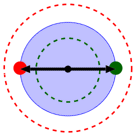

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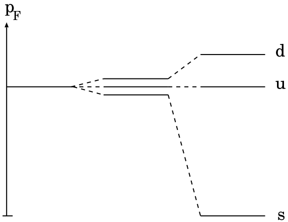

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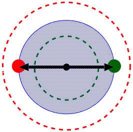
  - first equalize Fermi momenta
  - then pair
  - favored if $\delta p_F \lesssim \frac{\Delta}{\sqrt{2}}$

(Chandrasekhar, Clogston (1962))



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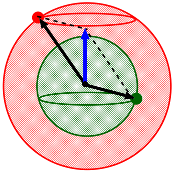
- FF (Fulde, Ferrell, 1964):
  - single plane wave
    $\langle q(\vec{x})q(\vec{x})\rangle \sim \Delta\, e^{2i\vec{q}\cdot\vec{x}}$ for fixed $\vec{q}$
  - disfavored by phase space



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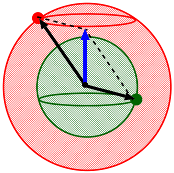

- LO (Larkin, Ovchinnikov, 1964):
  - multiple plane waves (e.g., $\cos(2\vec{q}\cdot\vec{x})$)
  - Ginzburg-Landau expansion

Inhomogeneous phases in color superconductivity

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  - single plane wave (Alford, Bowers, Rajagopal, 2001)

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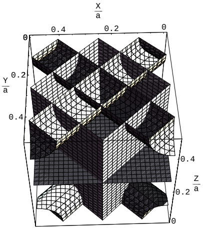
    (e.g., Bowers, Rajagopal, 2002; Casalbuoni et al., 2006 . . .)
    - different directions, but equal wave lengths
    - mostly Ginzburg-Landau



(Rajagopal, Sharma, 2006)

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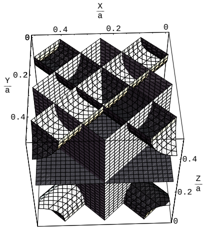

(Rajagopal, Sharma, 2006)

- aim of our work:
  - NJL model for inhomogeneous pairing beyond Ginzburg-Landau
  - superimpose different wave lengths

Model

- Model Lagrangian with NJL-type qq interaction:

$$\mathcal{L} = \bar{q}\,(i\partial\!\!\!/ + \hat{\mu}\gamma^0)\,q + \mathcal{L}_{int}$$

$$\mathcal{L}_{int} = H \sum_{A,A'=2,5,7} (\bar{q}\, i\gamma_5\tau_A\lambda_{A'}\, q_c)(\bar{q}_c\, i\gamma_5\tau_A\lambda_{A'}\, q)\,, \qquad q_c = C\bar{q}^T$$

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- bosonize: $\varphi_{AA'}(x) \;=\; -2H\,\bar{q}_c(x)\,\gamma_5\tau_A\lambda_{A'}\,q(x)$

$$\Rightarrow \quad \mathcal{L}_{int} = \frac{1}{2}\sum_{A,A'}\Big\{\,(\bar{q}\,\gamma_5\tau_A\lambda_{A'}\,q_c)\,\varphi_{AA'} + \;h.c.\; - \frac{1}{2H}\,\varphi^{\dagger}_{AA'}\,\varphi_{AA'}\Big\}$$

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- mean-field approximation:

$$\langle\,\varphi_{AA'}(x)\rangle = \Delta_A(x)\,\delta_{AA'}\,, \qquad \langle\,\varphi^\dagger_{AA'}(x)\rangle = \Delta^*_A(x)\,\delta_{AA'}$$

  - $\Delta_A(x)$ *classical* fields
  - retain space-time dependence!

Mean-field model

- Lagrangian:

$$\mathcal{L}_{MF}(x) = \bar{\Psi}(x)\, S^{-1}(x)\, \Psi(x) \;-\; \frac{1}{4H} \sum_A |\Delta_A(x)|^2$$

  - Nambu-Gor'kov bispinors: $\Psi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} q(x) \\ q_c(x) \end{pmatrix}$
  - inverse dressed propagator:

$$S^{-1}(x) = \begin{pmatrix} i\partial\!\!\!/ + \hat{\mu}\gamma^0 & \hat{\Delta}(x)\,\gamma_5 \\ -\hat{\Delta}^*(x)\,\gamma_5 & i\partial\!\!\!/ - \hat{\mu}\gamma^0 \end{pmatrix}, \quad \hat{\Delta}(x) := \sum_A \Delta_A(x)\, \tau_A \lambda_A$$

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- Thermodynamic potential:

$$\Omega_{MF}(T,\mu) = -\frac{1}{2}\frac{T}{V} \mathrm{Tr} \ln \frac{S^{-1}}{T} + \frac{T}{V} \sum_A \int_{[0,\frac{1}{T}]\otimes V} d^4x \, \frac{|\Delta_A(x)|^2}{4H}$$

  - $\mathrm{Tr} \ln S^{-1}$ nontrivial because of x-dependent gap functions!

Crystalline ansatz

- gap functions: time-independent, periodic in space
  $\hat{\Delta}(x) \equiv \hat{\Delta}(\vec{x}) = \hat{\Delta}(\vec{x} + \vec{a}_i)\,, \quad i = 1, 2, 3$

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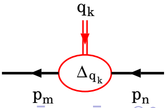
  - condensates couple different momenta!



Crystalline ansatz

- gap functions: time-independent, periodic in space

  $$\hat{\Delta}(x) \equiv \hat{\Delta}(\vec{x}) = \hat{\Delta}(\vec{x} + \vec{a}_i)\,, \quad i = 1, 2, 3$$

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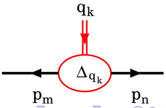

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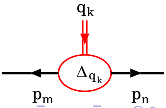
  - diagonal in energy ➜ Matsubara sum as usual



Hamiltonian

- The inverse propagator can be put into the form

$$S^{-1}_{p_m,p_n} = \gamma^0 \left(i\omega_{p_n} - \mathcal{H}_{\vec{p}_m,\vec{p}_n} \right) \delta_{\omega_{p_m},\omega_{p_n}}$$

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- thermodynamic potential:

$$\Omega_{MF} = -\frac{1}{4V}\sum_{\lambda}\left[E_\lambda + 2T\,\ln\left(1+2e^{-E_\lambda/T}\right)\right] + \sum_A \sum_{q_k} \frac{|\Delta_{A,q_k}|^2}{4H}$$

    - E_λ: eigenvalues of $\mathcal{H}$

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- remaining problem: diagonalize $\mathcal{H}$

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➜ We finally obtain:

$$\Omega_{MF} = -\frac{1}{4} \int_{B.Z.} \frac{d^3k}{(2\pi)^3} \sum_{\lambda} \left[E_\lambda(\vec{k}) + 2T \ln\left(1 + 2e^{-E_\lambda(\vec{k})/T}\right)\right] + \sum_A \sum_{q_k} \frac{|\Delta_{A,q_k}|^2}{4H}$$

  - $E_\lambda(\vec{k})$: eigenvalues of $\mathcal{H}(\vec{k})$.

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$$(\mathcal{H}_{\Delta,\delta\mu})_{\vec{p}_m,\vec{p}_n} = \begin{pmatrix} (p_m - \bar{\mu} - \delta\mu)\,\delta_{\vec{p}_m,\vec{p}_n} & \Delta_{p_m - p_n} \\ \Delta^*_{p_n - p_m} & -(p_m - \bar{\mu} + \delta\mu)\,\delta_{\vec{p}_m,\vec{p}_n} \end{pmatrix}$$

Regularization

- unregularized expression for Ω_{MF} divergent
  - ➜ needs to be regularized
- 3-momentum cutoff ?
- inhom. phases: $\mathcal{H}$ depends on two momenta!
  - cut off both of them, e.g., $|\vec{p}_m|, |\vec{p}_n| \leq \Lambda$?
  - ➜ strong regularization artifacts:
    - large $|\vec{q}|$ suppressed, e.g., $|\vec{q}| < 2\Lambda$
    - violates “model independent” low-energy results

➜ Pauli-Villars-like scheme:

$$F(E_\lambda) \rightarrow \sum_{j=0}^{2} F(E_{\lambda,j}) \,, \qquad E_{\lambda,j}(\vec{k}) = \sqrt{E_\lambda^2(\vec{k}) + j\Lambda^2}$$

Numerical calculations

- external parameters: T, $\bar{\mu}$, $\delta\mu$
  - here: $T = 0$, $\bar{\mu} = 400$ MeV ➜ only $\delta\mu$ is varied

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- model parameters: $H,\ \Lambda$
  - $\Lambda = 400$ MeV
  - $H \ \leftrightarrow\ \Delta_{BCS} = 80$ MeV
- crystal structure:
  - general problem (too) difficult
  - ➜ consider one-dimensional modulations (in 3+1 D):

$$\Delta(z) = \sum_k \Delta_k \, e^{2ikqz}$$

  - further restriction: $\Delta(z) = \text{real} \quad \Leftrightarrow \quad \Delta_k = \Delta^*_{-k}$

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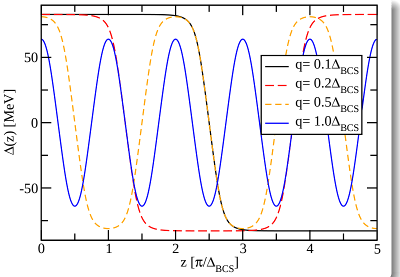

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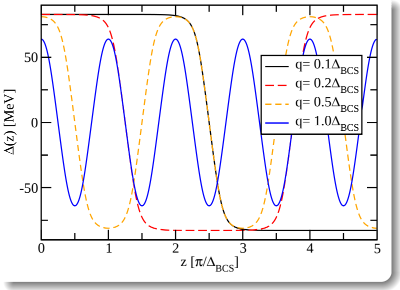

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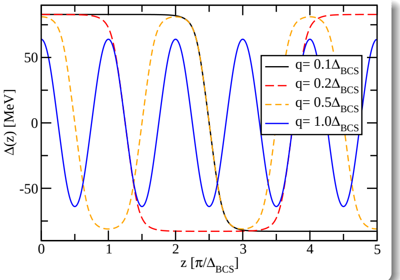

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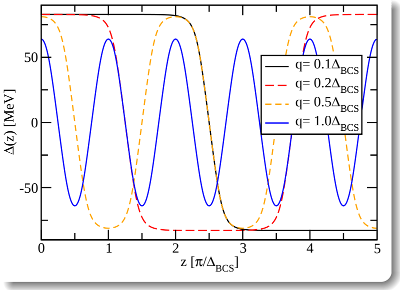

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- parametrization of the gap function:

$$\Delta(z) = A\,\mathrm{sn}\big(\kappa(z - z_0); \nu\big)$$

(works extremely well)

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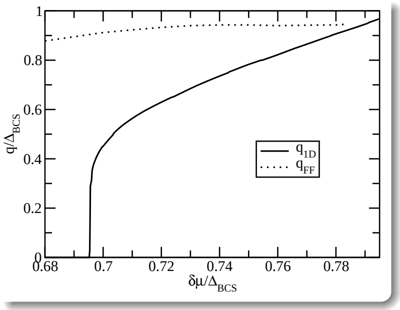

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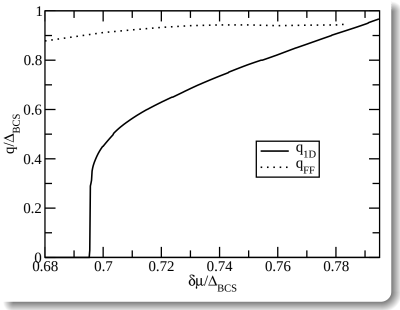

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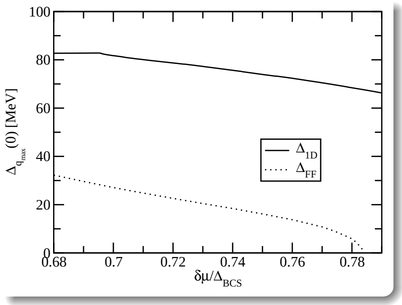

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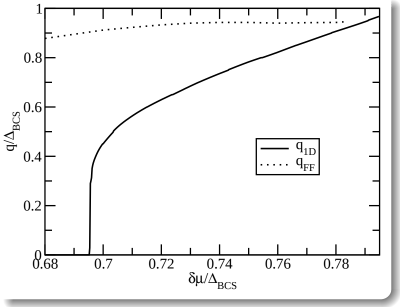

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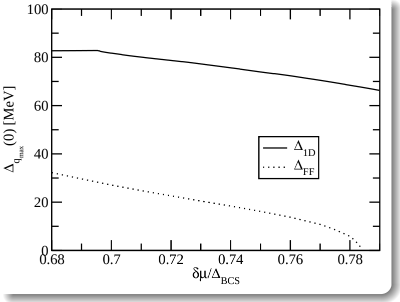

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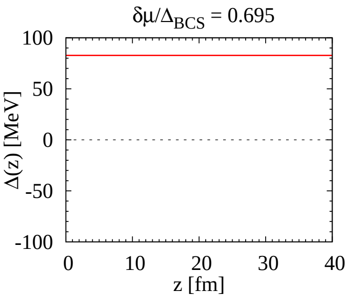
- $\Delta_{inhom.} \gg \Delta_{FF}$

Gap functions

- putting everything together:



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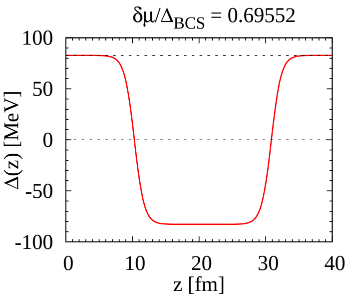

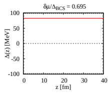

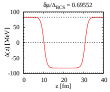

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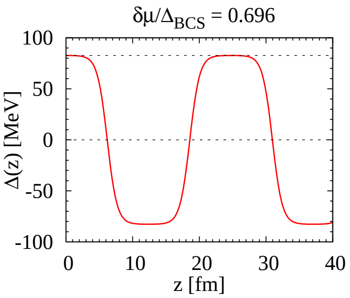

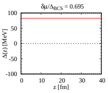

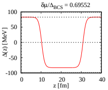

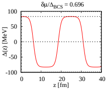

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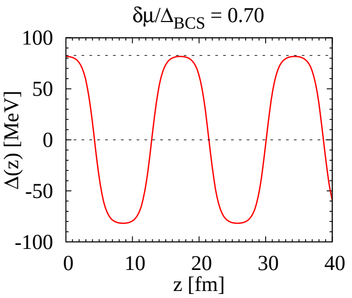

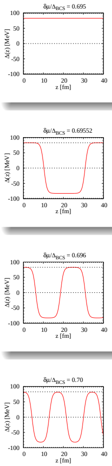

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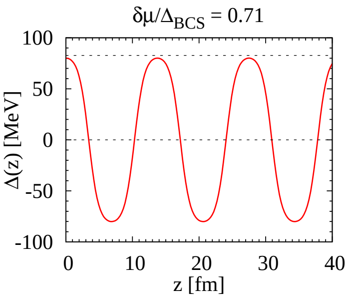

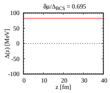

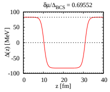

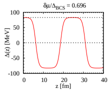

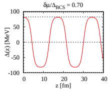

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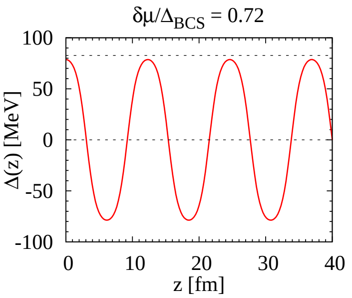

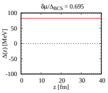

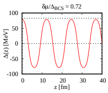

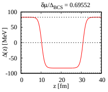

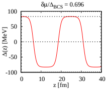

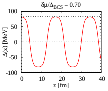

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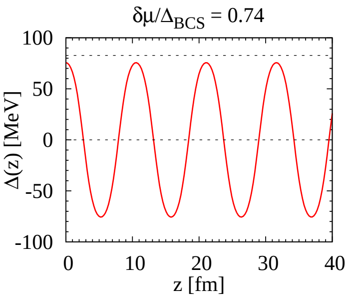

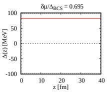

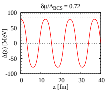

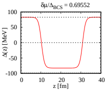

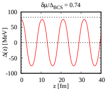

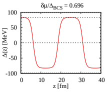

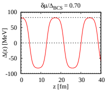

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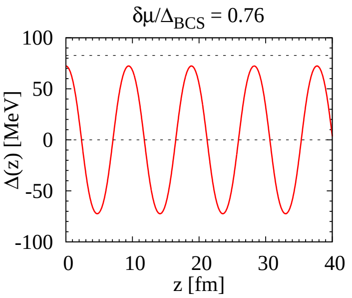
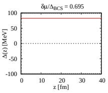
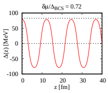
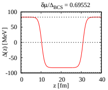
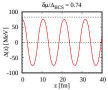
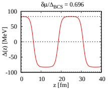


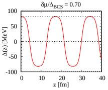






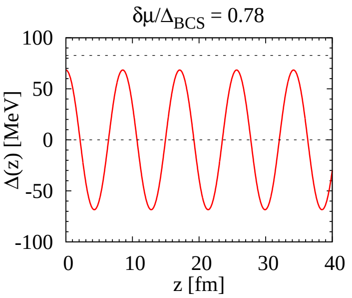


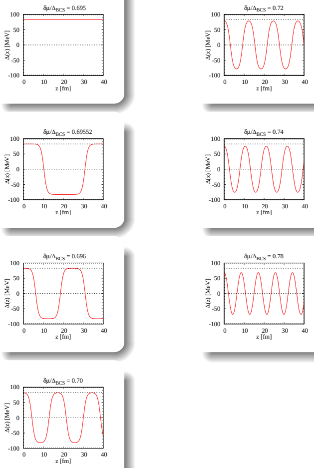




Gap functions

- putting everything together:





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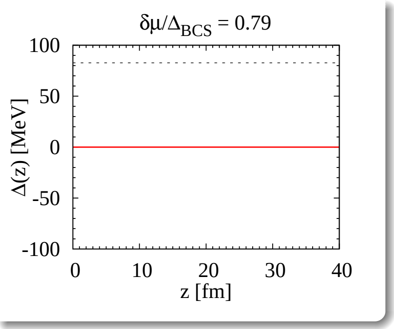
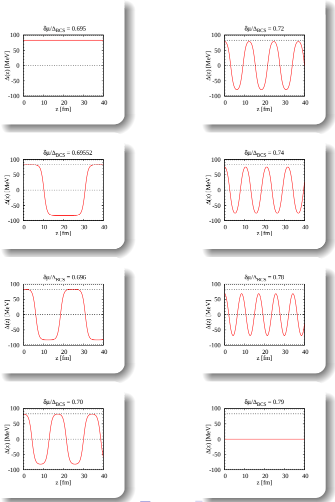
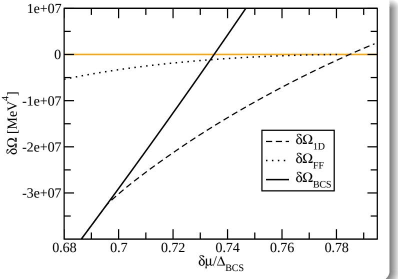




General one-dimensional solutions

- free-energy gain:



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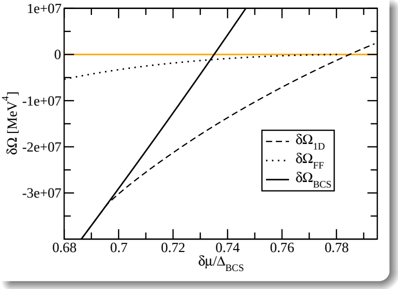

- LO window
  $\sim 2 \times$ FF window

Quasiparticle spectra

- anisotropic dispersion relations: $E_\lambda(\vec{k}) = E_\lambda(\vec{k}_\perp, k_z)$

Quasiparticle spectra

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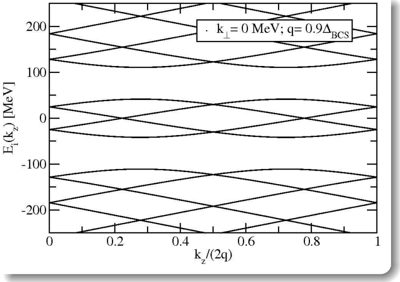
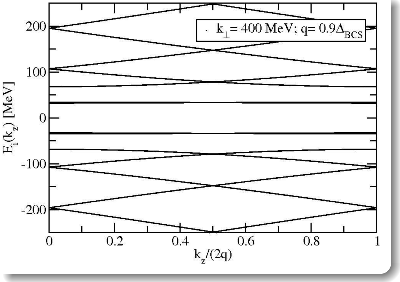
- typical examples at fixed $k_\perp$:

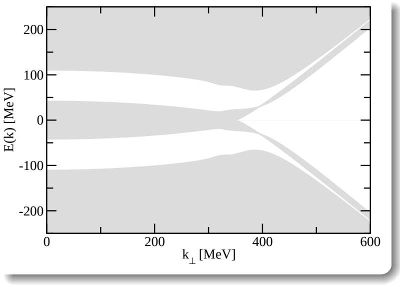




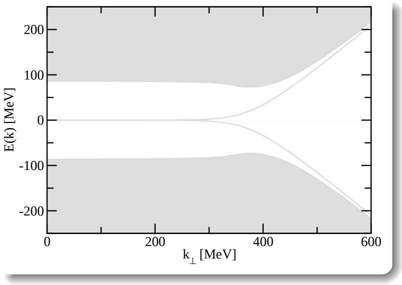
Band structure

- superposition of the eigenvalue spectra of all k_z:

sinusoidal ($q = 0.9\ \Delta_{BCS}$)



soliton lattice ($q = 0.2\ \Delta_{BCS}$)



- "almost gapped" regions between low- and high-lying modes
- low-lying modes related to solitons: $q \to 0$ ➜ $E \to 0$

Outlook I

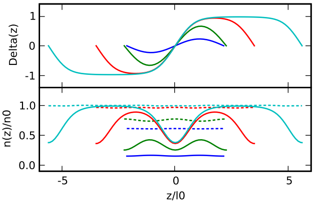
- to be done:
  - globally neutral two-flavor quark matter in beta equilibrium
  - density profiles

(example:

A. Bulgac, M.M. Forbes, PRL (2008),

Density Functional Theory calculation for

polarized Fermi systems at unitarity)



  - three flavors (including masses)
  - finite temperature, phase diagram
  - two- and three-dimensional crystals

Chiral phase transition (D. Nickel, arXiv:0902.1778, arXiv:0906.5295)

- “constituent quark mass” $M(z)$ with 1 dimensional modulations

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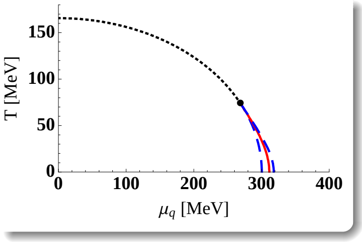

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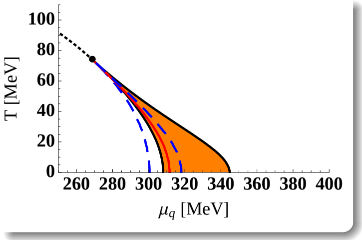

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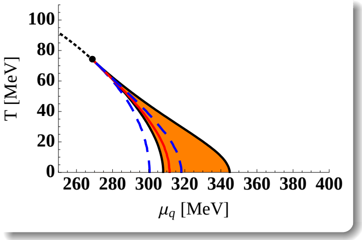

- 1st-order line covered by the inhomogeneous phase

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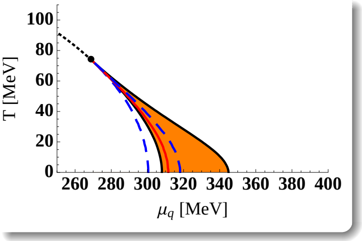
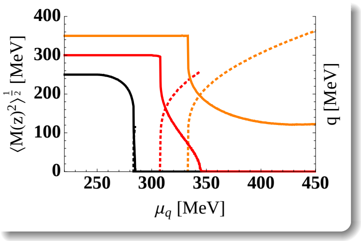


- $\sqrt{\langle M^2(z)\rangle}$, wave vector q



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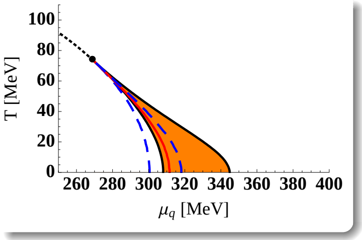

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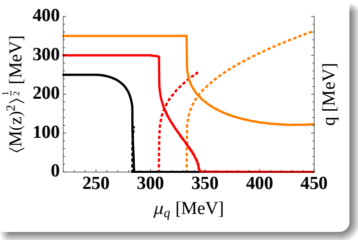

- 1st-order line covered by the inhomogeneous phase
- all phase boundaries 2nd order

Outlook II

- to be done:
  - include Polyakov loop
  - three flavors
  - two- and three-dimensional crystals
  - combine with color superconductivity