Three-Loop HTL Free Energy for QED

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Three Days of Strong Interactions, Wroclaw, Poland
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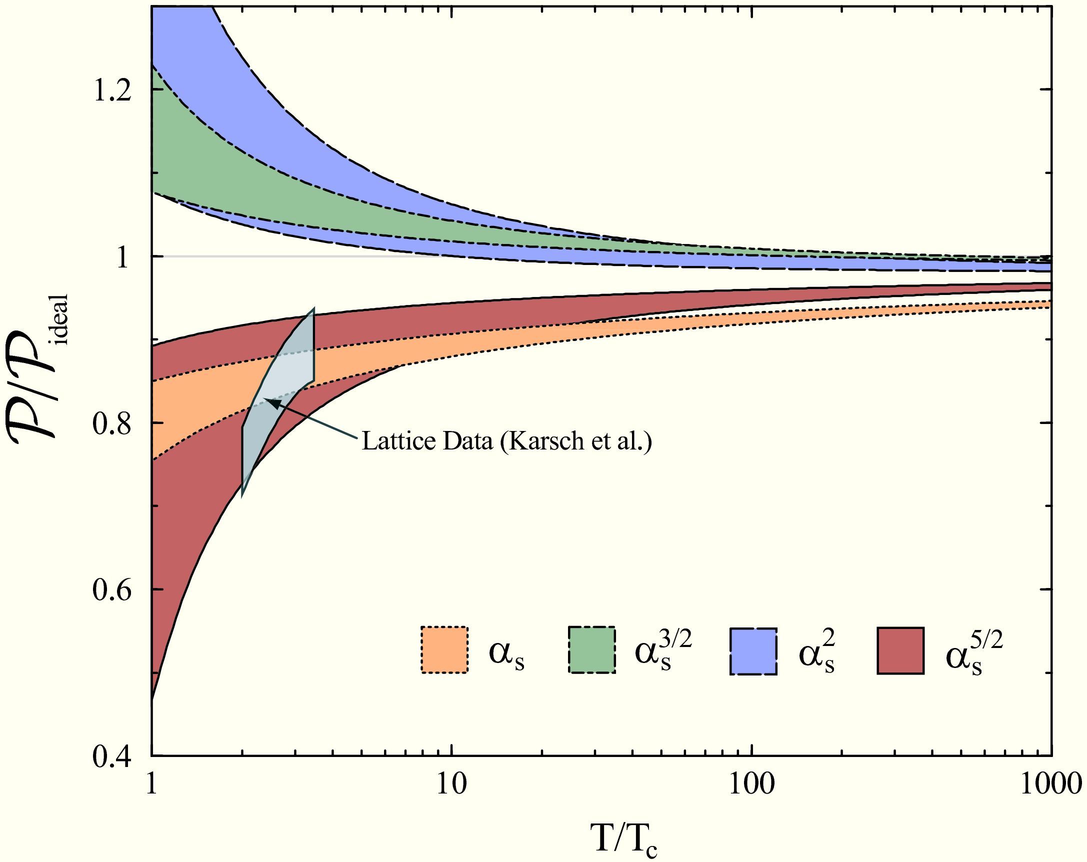
Introduction



Perturbative QCD free energy vs temperature. ($\pi T \leq \mu \leq 4\pi T$)
QCD with $N_c = 3$ and $N_f = 2$.
4-d lattice results from Karsch et al, 03.

(Here $\alpha_s = g_s^2/4\pi$)

- The weak-coupling expansion of the QCD free energy, $\mathcal{F}$, has been calculated to order $\alpha_s^3 \log \alpha_s$. [1,2,3,4]
- At temperatures expected at RHIC energies, $T \sim 0.3$ GeV, the running coupling constant $\alpha_s(2\pi T)$ is approximately 1/3, or $g_s \sim 2$.
- The successive terms contributing to $\mathcal{F}$ can strictly only form a decreasing series if $\alpha_s \lesssim 1/20$ which corresponds to $T \sim 10^5$ GeV.

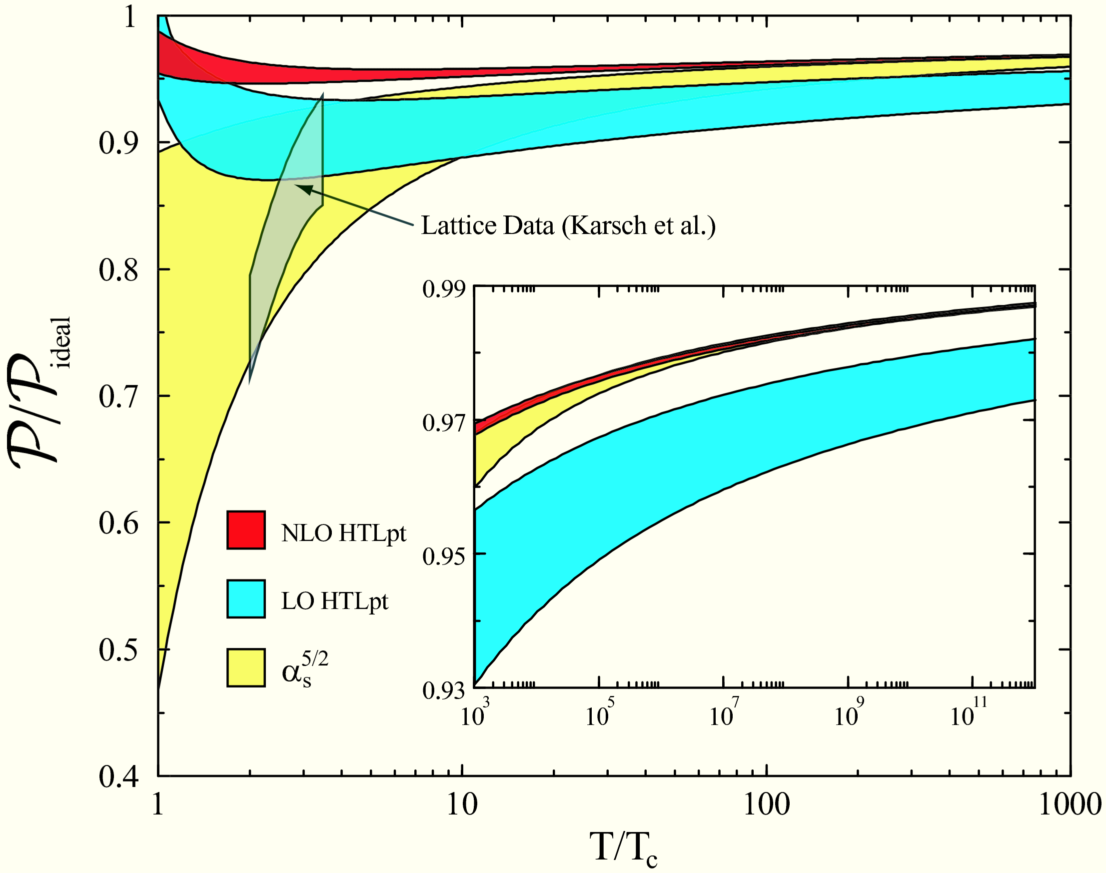
[1] Arnold and Zhai, 94/95.

[2] Kastening and Zhai, 95.

[3] Braaten and Nieto, 96.

[4] Kajantie, Laine, Rummukainen and Schröder, 02.

Introduction



LO and NLO HTLpt free energy of QCD with $N_c = 3$ and $N_f = 2$ together with the perturbative prediction accurate to g^5.

- Hard-thermal-loop (HTL) perturbation theory [4,5] is a systematic, self-consistent and gauge-invariant reorganization of thermal quantum fields.
- Hard-thermal-loop perturbation theory is formulated in Minkowski space, therefore it is in principle possible to carry out real time calculations.
- Interested in $T > 2 - 3\,T_c$.

[4] Andersen, Braaten, Strickland, 99/99/99.

[5] Andersen, Braaten, Petitgirard, Strickland, 02; Andersen, Petitgirard, Strickland, 03.

But there is still work to do!

- Problems remain:
  - g^4 and g^5 terms can't be fully fixed at NLO.
  - For example, when the NLO HTLpt is expanded in a truncated series in g, it is found that the g^5 term has approximately the right magnitude, but the **wrong sign** when comparing to the known weak-coupling expansion.
  - Running coupling doesn't enter at NLO. At this order, running coupling needs to be put in by hand.
- Can be fixed by going to NNLO.

Time to roll up your sleeves . . .

Anharmonic Oscillator

- Consider the perturbation series for the ground state energy, E, of a simple anharmonic oscillator with potential

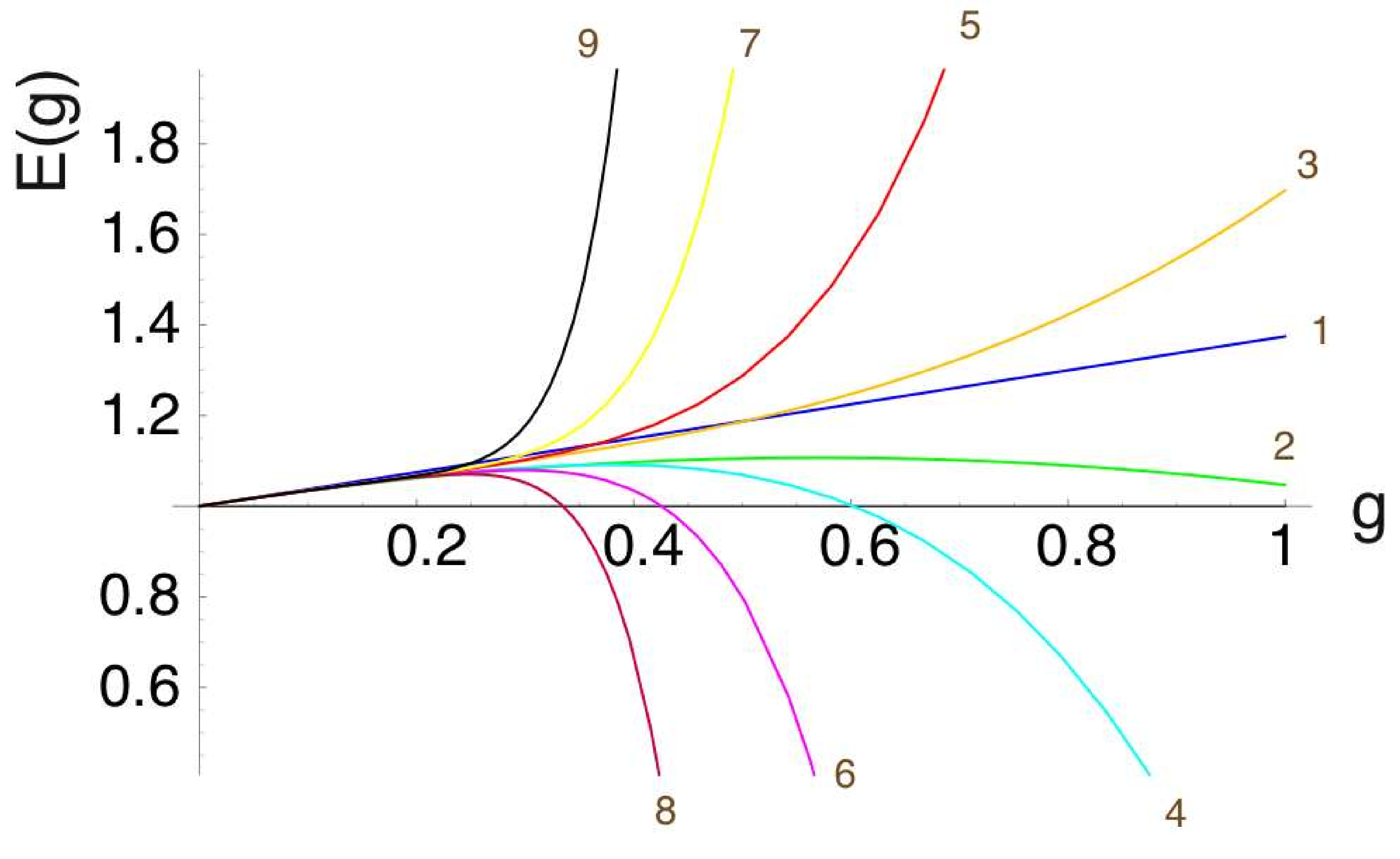
$$V(x) = \frac{1}{2}\omega^2 x^2 + \frac{g}{4}x^4 \qquad (\omega^2, g > 0)$$

- Weak-coupling expansion of the ground state energy $E(g)$ is known to all orders (Bender and Wu 69/73)

$$E(g) = \omega \sum_{n=0}^{\infty} c_n^{\mathrm{BW}} \left(\frac{g}{4\omega^3}\right)^n, \quad c_n^{\mathrm{BW}} = \left\{\frac{1}{2}, \frac{3}{4}, -\frac{21}{8}, \frac{333}{16}, -\frac{30885}{16}, \ldots\right\}$$

- $\lim_{n\to\infty} c_n^{\mathrm{BW}} = (-1)^{n+1}\sqrt{\frac{6}{\pi^3}}3^n(n-\frac{1}{2})!$
- Because of the factorial growth, the expansion is an asymptotic series with zero radius of convergence!

Anharmonic Oscillator



Variational Perturbation Theory (Janke and Kleinert 95/97)

- Split the harmonic term into two pieces and treat the second as part of the interaction

$$\omega^2 \to \Omega^2 + \left(\omega^2 - \Omega^2\right) \quad \Longrightarrow \quad E_N(g,r) = \Omega \sum_{n=0}^{N} c_n(r) \left(\frac{g}{4\Omega^3}\right)^n$$

where $r \equiv \frac{2}{g}\left(\omega^2 - \Omega^2\right)$

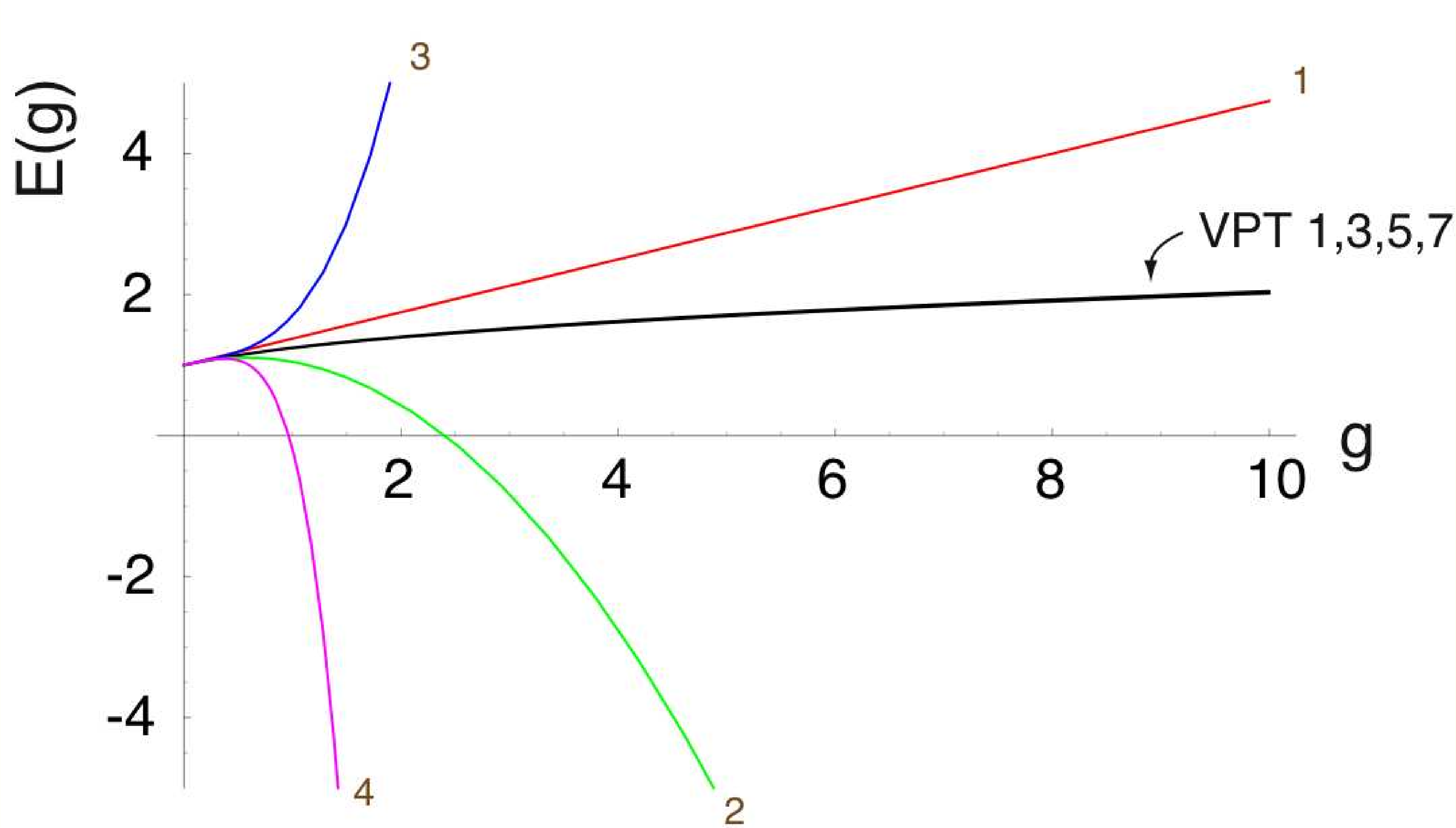
- The new coefficients c_n can be obtained by

$$c_n(r) = \sum_{j=0}^{n} c_j^{\mathrm{BW}} \begin{pmatrix} (1-3j)/2 \\ n-j \end{pmatrix} (2r\Omega)^{n-j}$$

- Fix Ω_N by requiring that at each order N

$$\left.\frac{\partial E_N}{\partial \Omega}\right|_{\Omega=\Omega_N} = 0$$

Variational Perturbation Theory



Hard-Thermal-Loop Perturbation Theory (HTLpt)

- Hard-thermal-loop perturbation theory is a reorganization of the perturbative series for QCD which is similar in spirit to variational perturbation theory

$$\mathcal{L}_{\rm HTLpt} = \left(\mathcal{L}_{\rm QCD} + \mathcal{L}_{\rm HTL}\right)\Big|_{g\to\sqrt{\delta}g} + \Delta\mathcal{L}_{\rm HTL}(g, m_D^2(1-\delta))$$

The HTL "improvement" term is

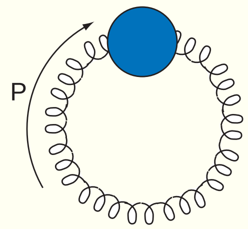
$$\mathcal{L}_{\rm HTL} = -\frac{1}{2}(1-\delta)m_D^2 \,{\rm Tr}\left(G_{\mu\alpha}\left\langle \frac{y^\alpha y^\beta}{(y\cdot D)^2}\right\rangle_y G^\mu{}_\beta\right)$$

where $\langle\cdots\rangle_y$ indicates angle average

HTLpt: 1-loop free energy for pure glue

- Separation into hard and soft contributions ($d = 3 - 2\epsilon$)

$$\mathcal{F}_g = -\frac{1}{2} \sum\!\!\!\!\!\!\!\int_P \{(d-1)\log[-\Delta_T(P)] \;+\; \log\Delta_L(P)\}$$



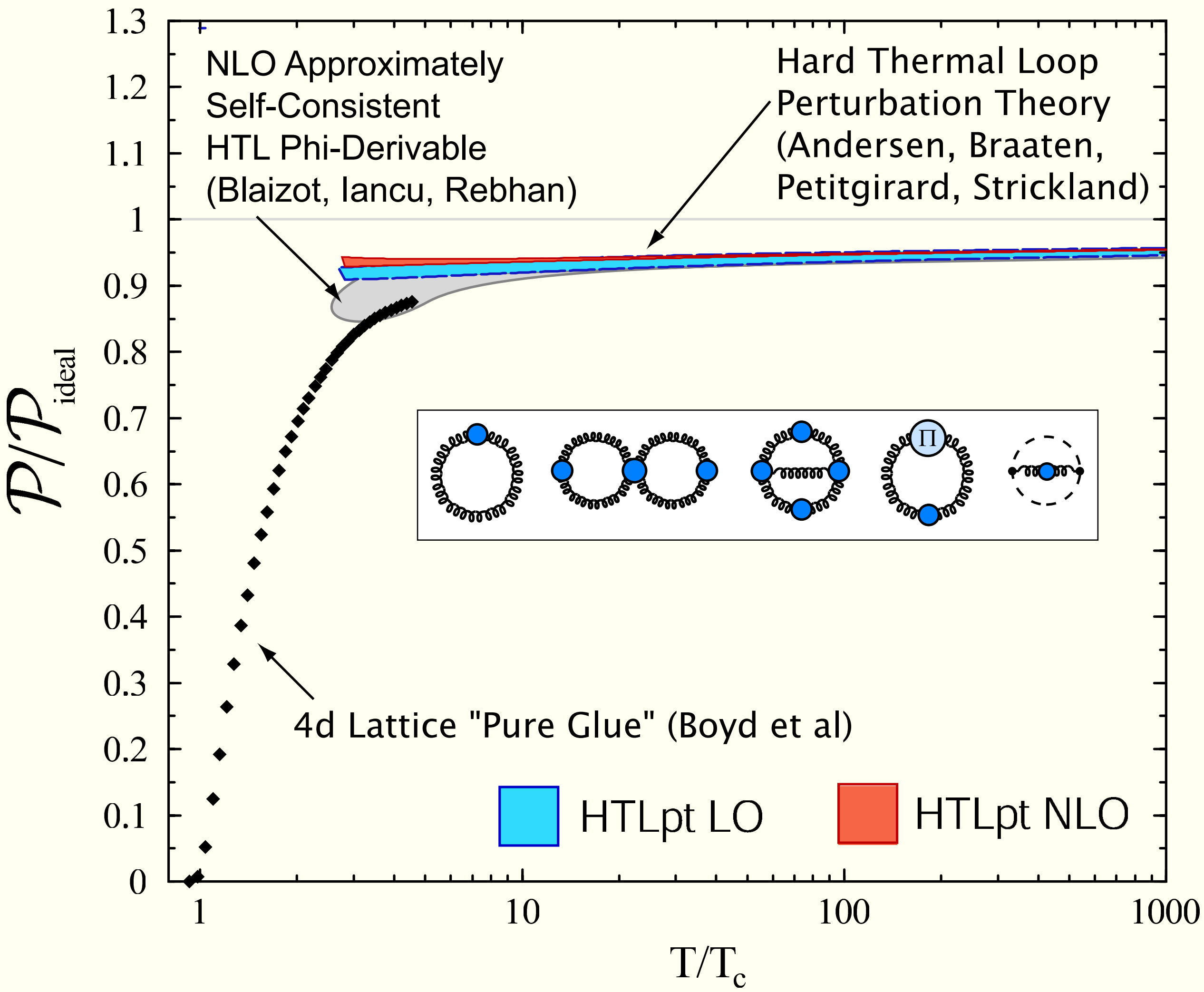
  - Hard momenta $(\omega, \mathbf{p} \sim T)$

$$\mathcal{F}_g^{(h)} \;=\; \frac{d-1}{2} \sum\!\!\!\!\!\!\!\int_P \log(P^2) + \frac{1}{2} m_D^2 \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2} - \frac{1}{4(d-1)} m_D^4 \sum\!\!\!\!\!\!\!\int_P \left[\frac{1}{(P^2)^2} - 2\,\frac{1}{p^2 P^2} - 2d\,\frac{1}{p^4}\mathcal{T}_P + \;2\,\frac{1}{p^2 P^2}\mathcal{T}_P + d\,\frac{1}{p^4}(\mathcal{T}_P)^2 \right] + \mathcal{O}(m_D^6)$$

  - Soft momenta $(\omega, \mathbf{p} \sim gT)$

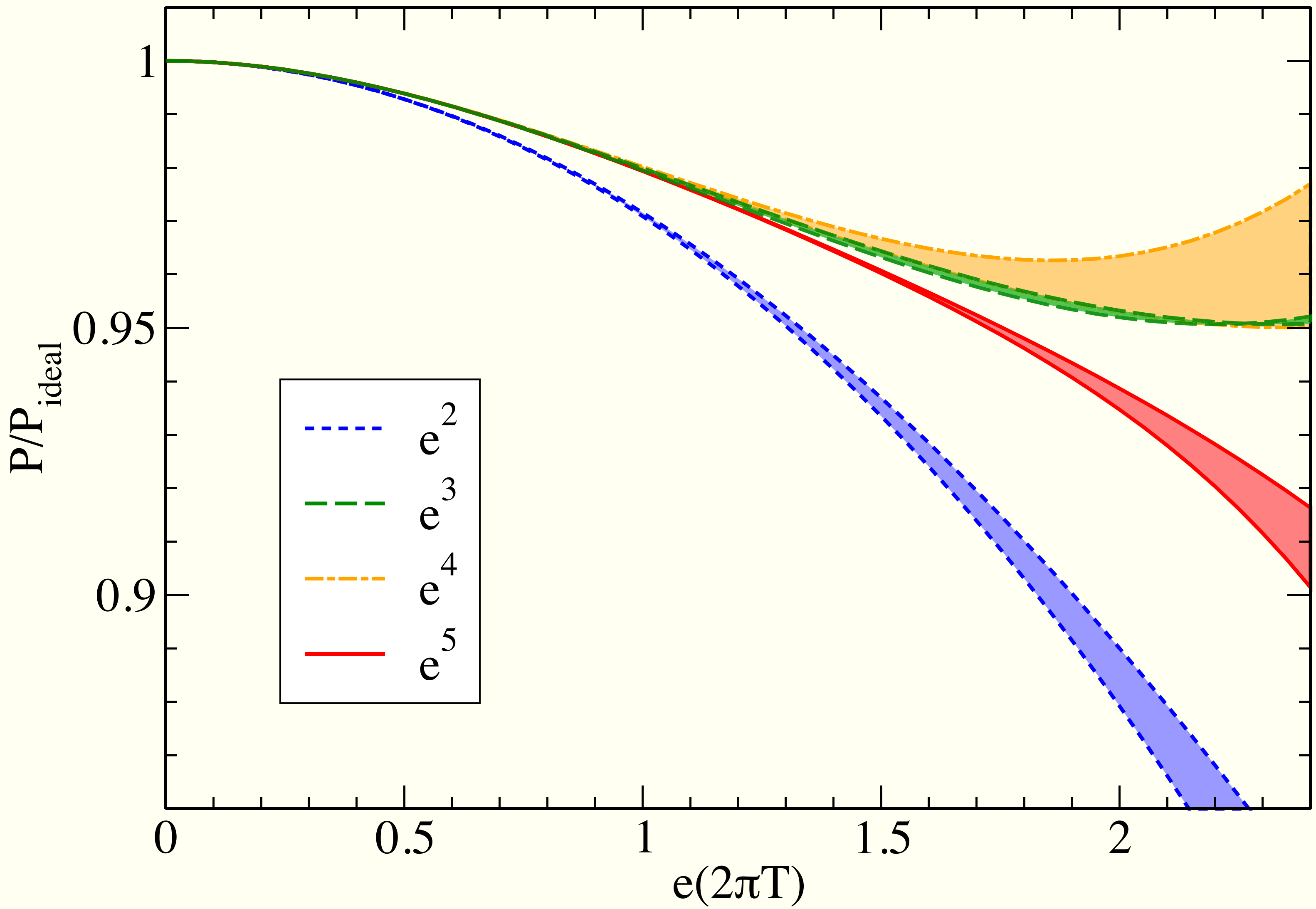
$$\mathcal{F}_g^{(s)} \;=\; \frac{1}{2} T \int_{\mathbf{p}} \log(p^2 + m_D^2)$$

HTLpt: 1- and 2-loop free energy for pure glue



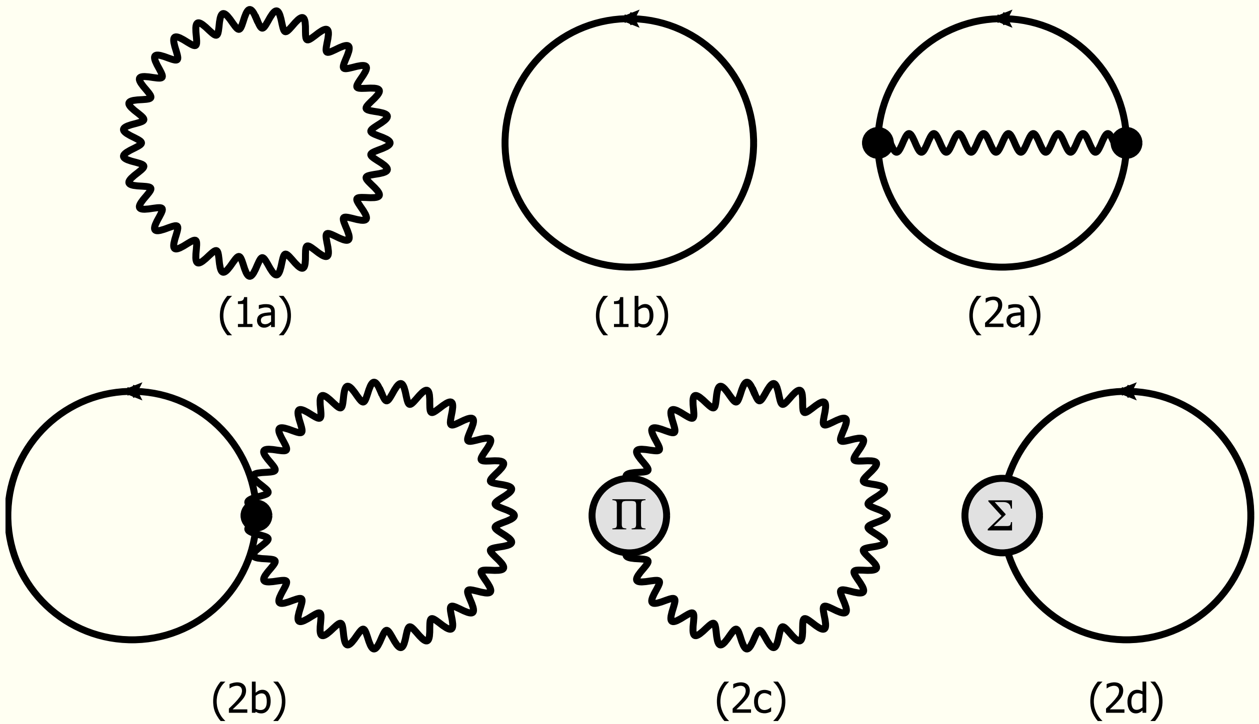
LO and NLO HTLpt free energy of pure glue vs temperature
Andersen, Braaten, Petitgirard, Strickland, 02.

HTLpt: naive pert. expansion of QED free energy



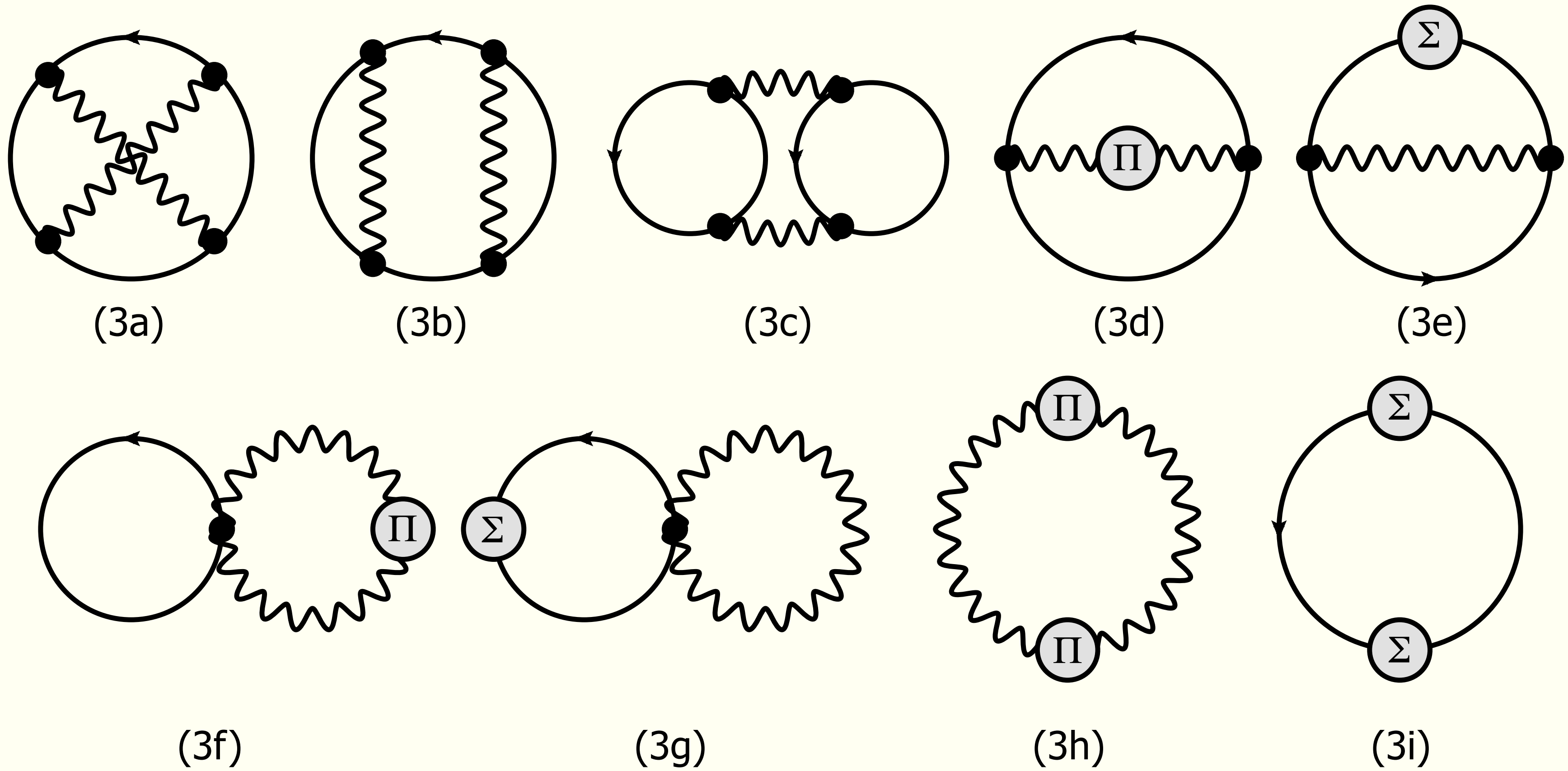
Perturbative QED free energy (Kastening and Zhai, 95)

HTLpt: 1- and 2-loop diagrams for QED



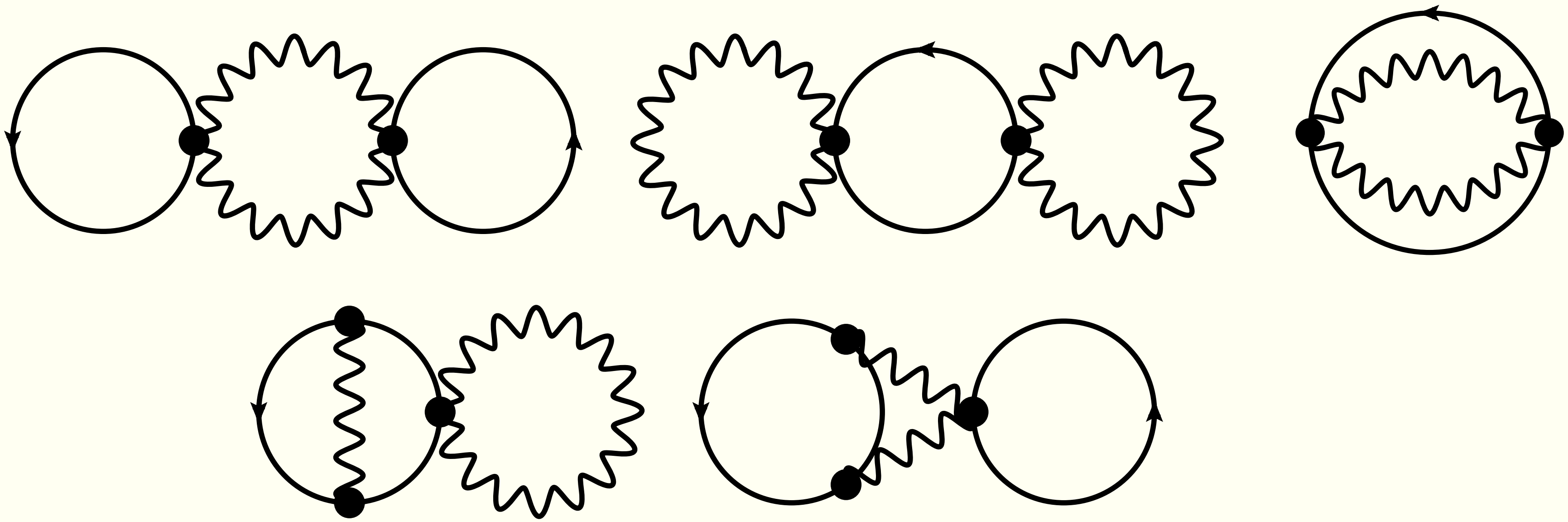
1- and 2-loop QED diagrams contributing to HTLpt

HTLpt: 3-loop diagrams for QED



3-loop QED diagrams contributing to HTLpt

HTLpt: 3-loop diagrams for QED



3-loop HTLpt QED diagrams which can be neglected in our approach

HTLpt: 3-loop thermodynamic potential for QED

- The NNLO thermodynamic potential reads

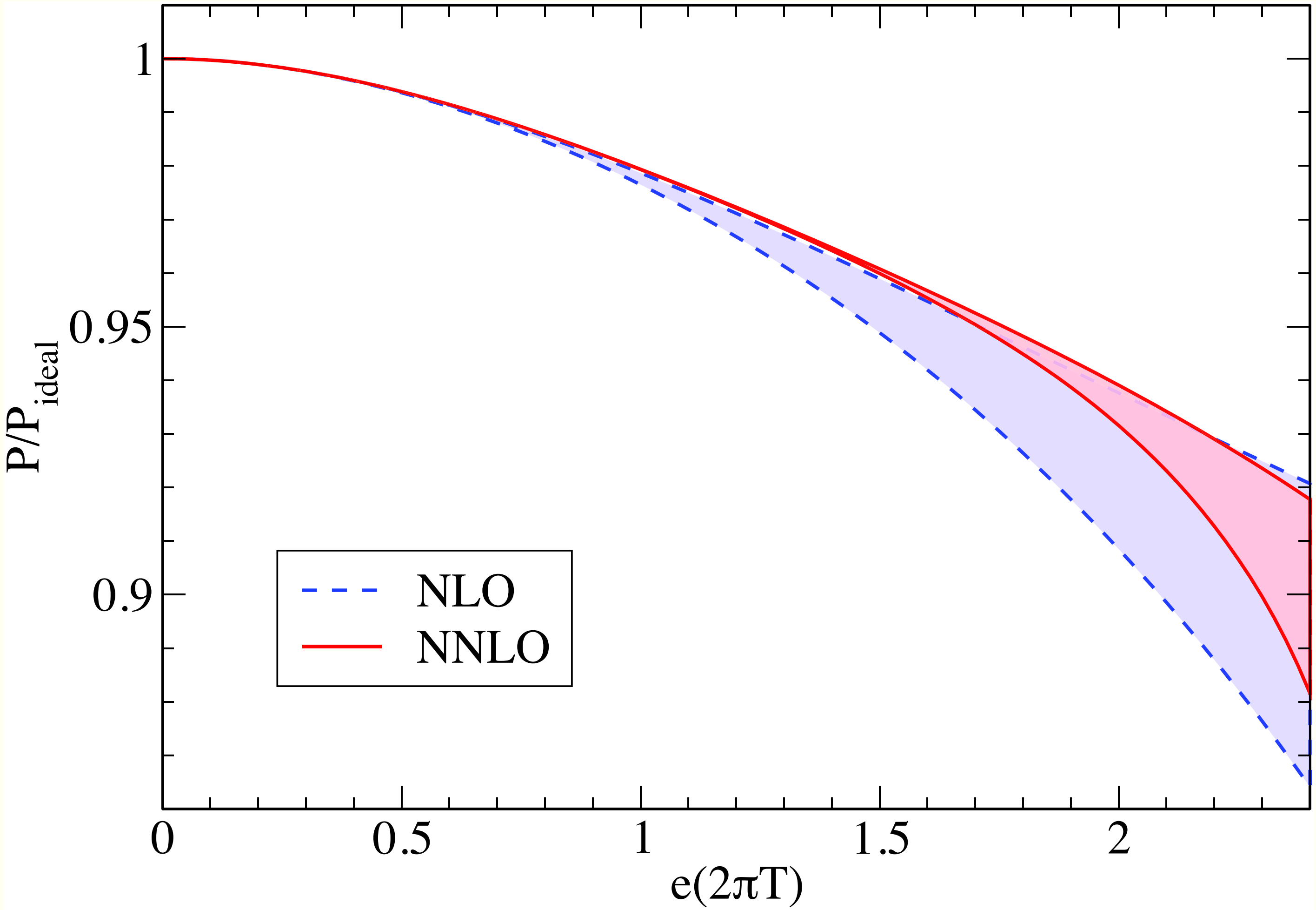
$$
\begin{aligned}
\Omega_{\rm NNLO} \quad = \quad & -\frac{\pi^2 T^4}{45}\Big\{1+\tfrac{7}{4}N_f-\tfrac{15}{4}\hat{m}_D^3 \\
& +N_f\frac{\alpha}{\pi}\Big[-\tfrac{25}{8}+\tfrac{15}{2}\hat{m}_D+15\Big(\log\tfrac{\hat{\mu}}{2}-\tfrac{1}{2}+\gamma+2\log 2\Big)\hat{m}_D^3-90\hat{m}_D\hat{m}_f^2\Big] \\
& +N_f\Big(\frac{\alpha}{\pi}\Big)^2\Big[\tfrac{15}{64}(35-32\log 2)-\tfrac{45}{2}\hat{m}_D\Big] \\
& +N_f^2\Big(\frac{\alpha}{\pi}\Big)^2\Big[\tfrac{25}{12}\Big(\log\tfrac{\hat{\mu}}{2}+\tfrac{1}{20}+\tfrac{3}{5}\gamma-\tfrac{66}{25}\log 2+\tfrac{4}{5}\frac{\zeta'(-1)}{\zeta(-1)}-\tfrac{2}{5}\frac{\zeta'(-3)}{\zeta(-3)}\Big) \\
& \qquad +\tfrac{5}{4}\frac{1}{\hat{m}_D}-15\Big(\log\tfrac{\hat{\mu}}{2}-\tfrac{1}{2}+\gamma+2\log 2\Big)\hat{m}_D\Big]\Big\}
\end{aligned}
$$

PURELY ANALYTIC!

- To eliminate the m_D and m_f dependence, the gap equations are imposed

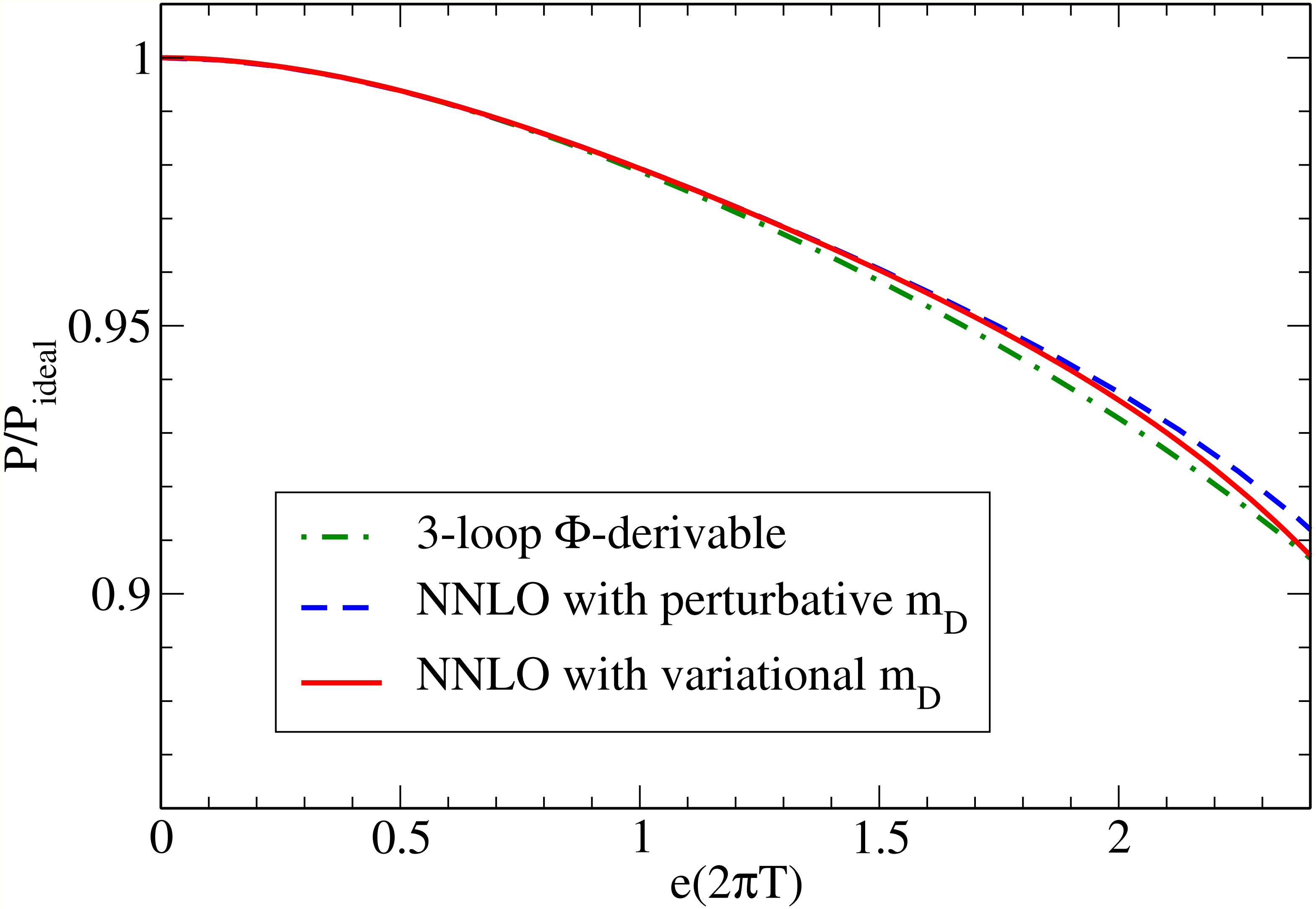
$$
\begin{aligned}
\frac{\partial}{\partial m_D}\Omega(T,\alpha,m_D,m_f,\delta=1) &= 0 \\
\frac{\partial}{\partial m_f}\Omega(T,\alpha,m_D,m_f,\delta=1) &= 0
\end{aligned}
$$

HTLpt: 2- and 3-loop free energy for QED



NLO and NNLO HTLpt predictions for QED free energy

HTLpt: comparison of different methods/schemes



Comparison of three different predictions for the QED free energy at $\mu = 2\pi T$

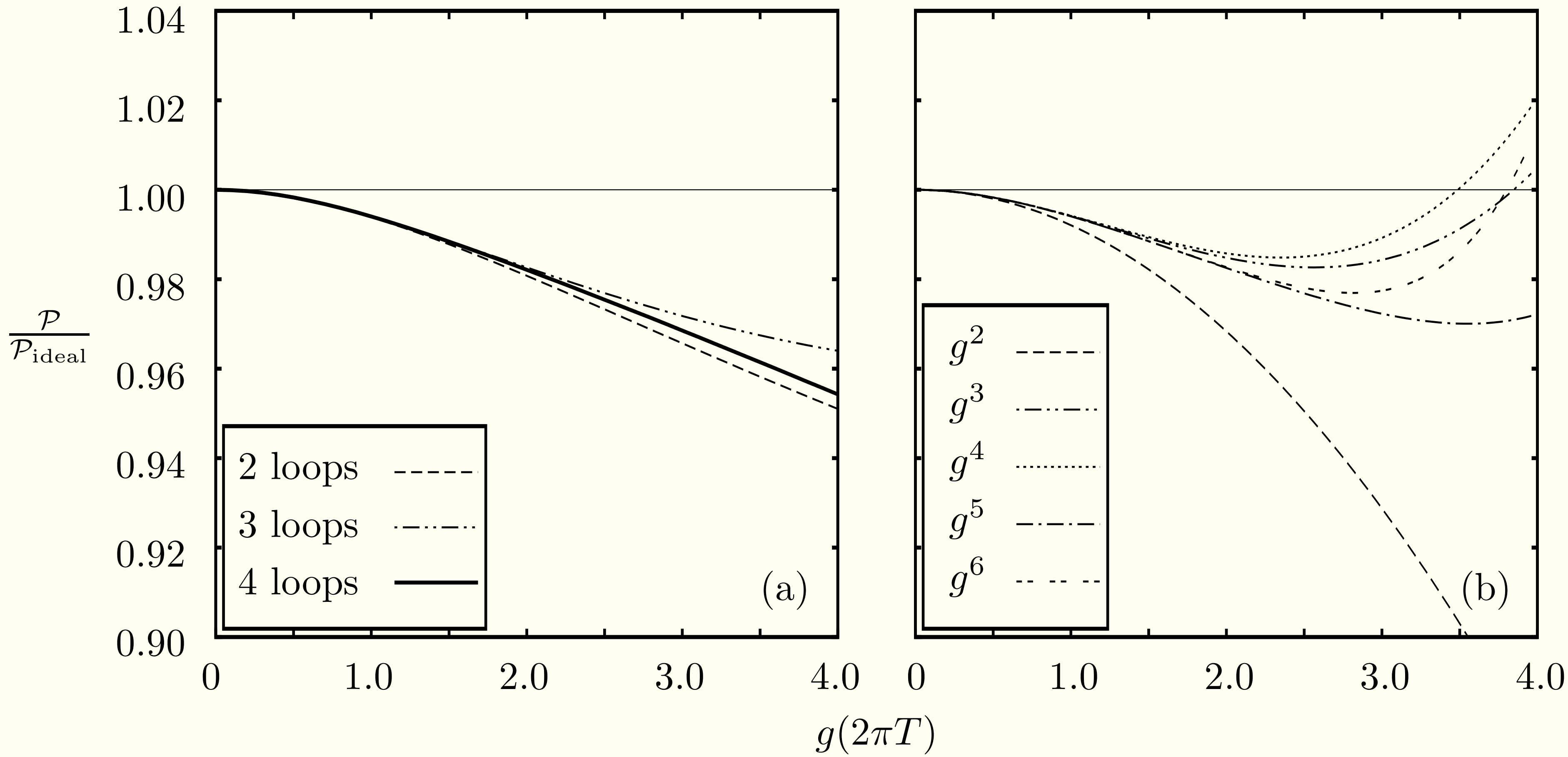
3-loop Φ-derivable result is taken from Andersen and Strickland, 05

Conclusions and Outlook

- The problem of bad convergence of finite temperature weak-coupling expansion is generic.
- It does not just occur in gauge theories, but also in scalar theories, and even in quantum mechanics.
- Variational perturbation theory and hard-thermal-loop perturbation theory can improve the convergence of perturbative calculations in a gauge-invariant manner which is formulated in Minkowski space.
- By pushing forward, hopefully, hard-thermal-loop perturbation theory can give convergent gauge theory calculations at high temperatures, $T > 2 - 3\,T_c$
- Once the NNLO QCD thermodynamics is obtained, running coupling fixed self-consistenly, and m_D fixed via NNLO gap equation we can begin to calculate dynamic quatities.

Back-up

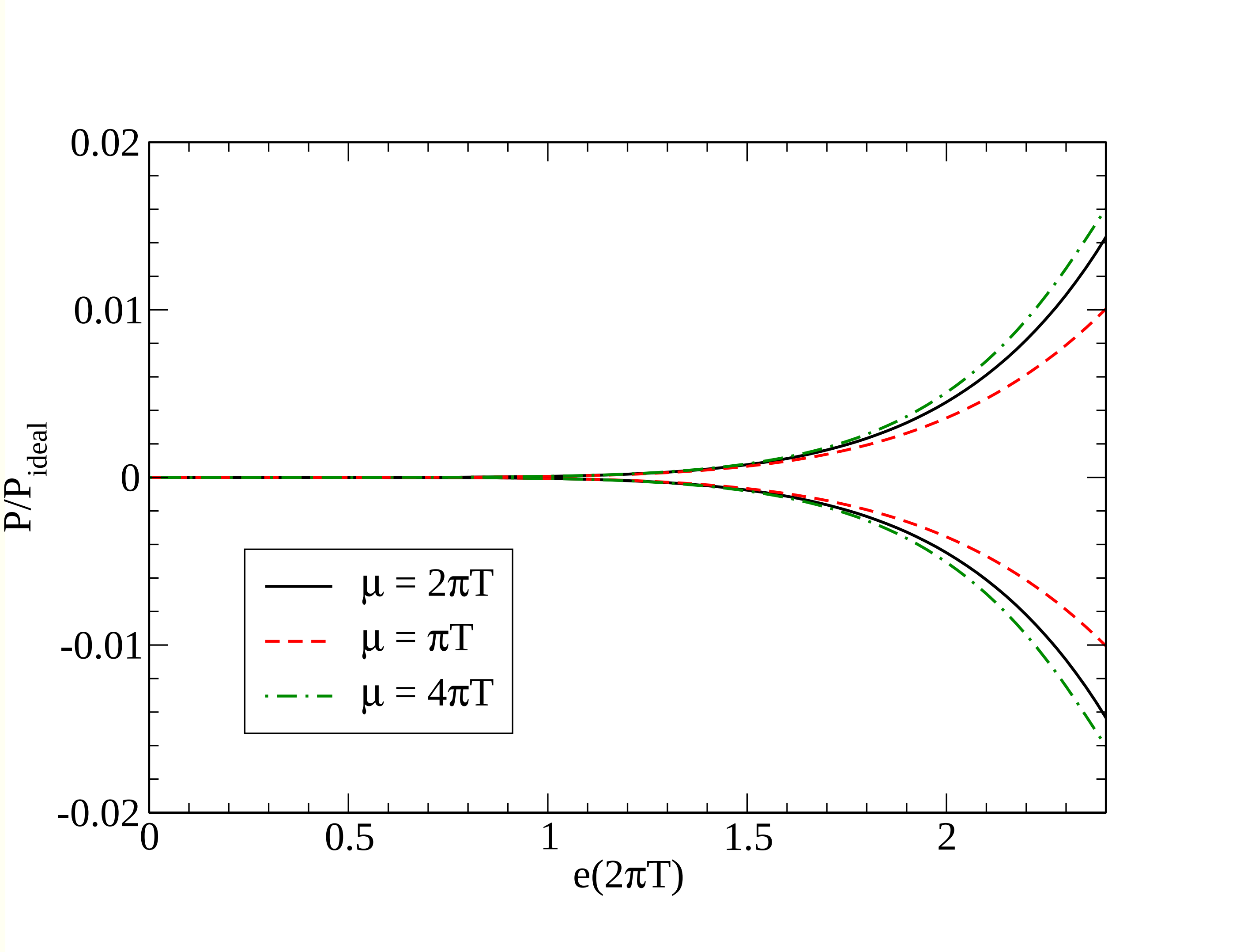
Screened Perturbation Theory



4-loop SPT pressure vs weak-couping pressure

Andersen, Braaten and Strickland, 00. Andersen and Strickland, 01.
Andersen and Kyllingstad, 08.

HTLpt: 3-loop free energy for QED



The imaginary part of NNLO HTLpt predictions for QED free energy

Weinberg and Wu, 87