Looking for the QCD critical point on the lattice

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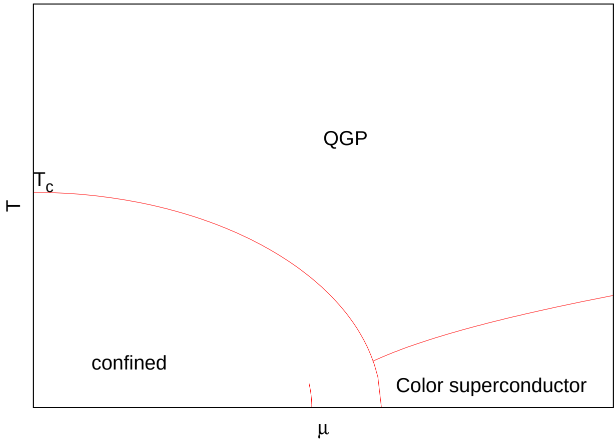
The minimum phase diagram of QCD



- Dictated by perturbation theory at large T or large μ
- Phase transitions or crossovers ?

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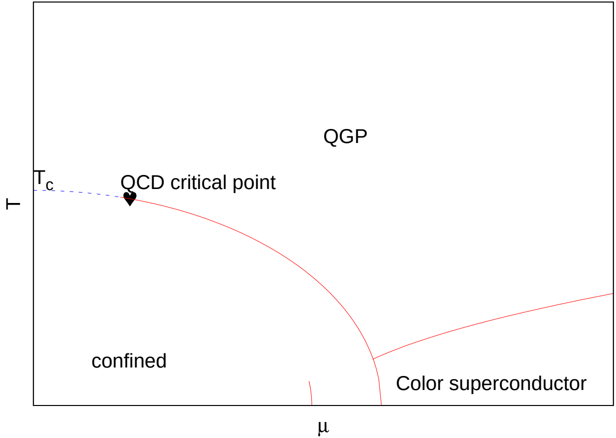

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- Phase transitions or crossovers ?
- Crossover at $(\mu = 0, T_c)$, first-order at small T $\longrightarrow$ **QCD critical point**

The sign problem

- Integrate over fermions: $\det(\not{D} + m + \mu\gamma_0)$ complex *unless* $\mu = 0$ *or* $\mu = i\mu_i$

  $\longrightarrow$ standard importance sampling $\Leftrightarrow$ $\langle\mathrm{Re(baryon\ density)}\rangle = 0$

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- **Reweighting**: - simulate theory with no sign pb., eg. $|\det(\mu)|$
  - reweight each measurement with $\rho(U) = \frac{\det(U,\mu)}{|\det(U,\mu)|}$ complex phase
  - av. "sign" $\langle \rho(U) \rangle = \frac{Z(\mu,\det)}{Z(\mu,|\det|)} \sim \exp(-V\frac{\Delta f(\mu)}{T}) \longrightarrow$ large V ?, large μ ?

1. maintain statistical accuracy on $\langle \rho \rangle$: **sign** pb.

2. ensure that $Z(\mu, \det)$ is properly sampled: **overlap** pb.

1 and **2** require statistics $\propto$ $\exp(+V)$

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- Measure **derivatives** w.r.t. μ at $\mu = 0$: $\langle W(\mu) \rangle = \langle W(\mu = 0) \rangle + \sum_k \mathbf{c_k} \left(\frac{\mu}{\pi T}\right)^k$
  - directly at $\mu = 0$ MILC, TARO, Bielefeld-Swansea, Gavai-Gupta,..
  - by fitting polynomial to $\mu = i\mu_i$ results D'Elia-Lombardo, PdF-Philipsen,..

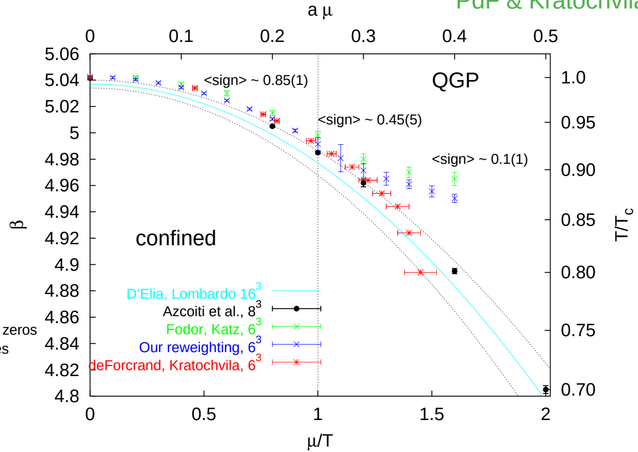
Controlled thermodynamics and continuum limits $\Rightarrow$ **derivatives** only

The good news: curvature of the pseudo-critical line

All with $N_f = 4$ staggered fermions, $am_q = 0.05, N_t = 4 \quad (a \sim 0.3$ fm)

PdF & Kratochvila LAT05

imaginary μ
2 param. imag. μ
dble reweighting, LY zeros
Same, susceptibilities
canonical



Agreement for $\mu/T \lesssim 1$

The good news: curvature of the pseudo-critical line

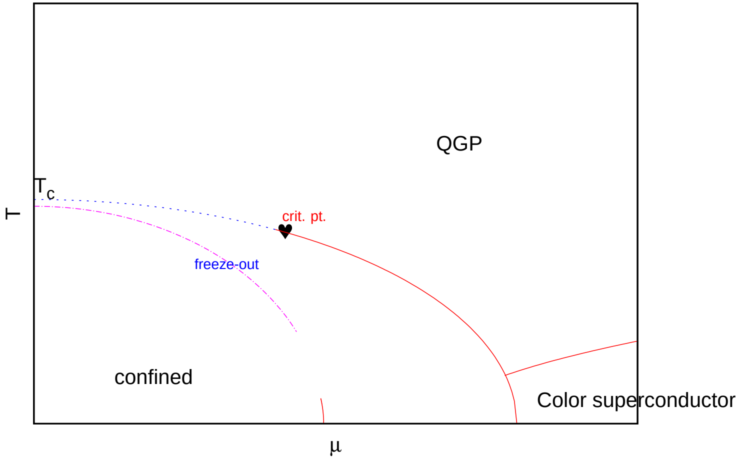
- $T_c(\mu)$ simpler/more precise by analytic continuation of $T_c(i\mu_i)$:
  determine $T_c(\mu = i\mu_i)$ and fit results with polynomial

$$\frac{T_c(\mu)}{T_c(\mu{=}0)} = 1 - \sum_{k=1} \mathbf{t_{2k}} \left(\frac{\mu}{\pi T}\right)^{2k}$$

For $N_t = 4$: curvature $t_2(N_f, m_q)$ varies from ~ 0.3 to ~ 1 Schmidt 06

- Indications (PdF& OP $N_t = 6$; Fodor et al. LAT08): t_2 *decreases* as $a \to 0$
- Extrapolation $m_q \to m_{\rm phys},\ a \to 0$ feasible G. Endrodi LAT09

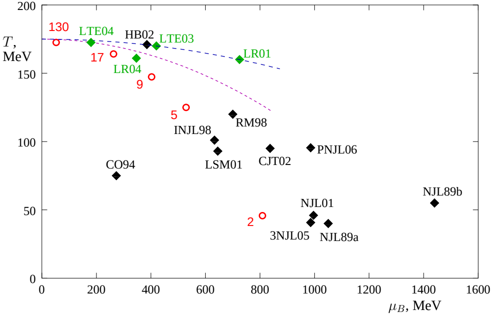
Why is curvature of pseudo-critical line important?



IF $T_c(\mu)$ really flatter than **freeze-out** curve ($a \rightarrow 0$: factor $\sim 2-6$ in $\left.\frac{d^2 T_c}{d\mu^2}\right|_{\mu=0}$?)

$\Longrightarrow$ exp. signal from critical pt. modified/washed out by evolution until freeze-out

The bad news: locating the critical point



M. Stephanov, hep-lat/0701002

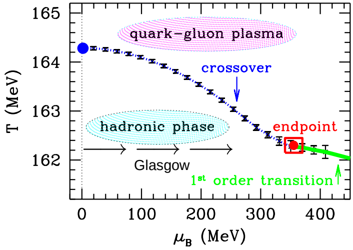
- Challenging task:

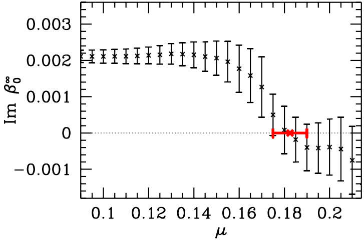
  detect divergent correlation length (2nd order)
  vs finite but large (crossover, 1rst order)
  on small lattice

  Mission impossible?

0. The ultimate reweighting

Fodor & Katz: hep-lat/0402006 ($\sim$ physical quark masses)





$$(\mu_E^q, T_E) = \ (120(13), 162(2))\ \text{MeV}$$

Strategy: reweight from $(\mu = 0, T_c)$ along pseudo-critical line

Legitimate **concerns**:

- Discretization error? $N_t = 4 \implies a \sim 0.3$ fm
- Abrupt qualitative change near μ_E:

  abrupt change of physics or breakdown of algorithm (Splittorff)?

$\longrightarrow$ repeat with conservative approach (**derivative**)

1. Taylor expansion

- Reweighting gives exact μ dependence, **BUT** limited to small V,μ

Error non-Gaussian, analysis subtle $\longrightarrow$ breakdown may go unnoticed (Glasgow meth)

- Instead, obtain reliable $V \rightarrow \infty$ behaviour of Taylor coefficients:

$$P(T,\mu) = \underbrace{P(T,\mu=0)}_{\text{indep. calc.}} + \Delta P(T,\mu), \qquad \boxed{\frac{\Delta P(T,\mu)}{T^4} = \sum_{k=1} c_{2k}(T)\left(\frac{\mu}{T}\right)^{2k}}$$

$$c_{2k} = \langle \mathrm{Tr}(\text{ degree } 2k \text{ polynomial in } \not{D}^{-1}, \frac{\partial \not{D}}{\partial \mu})\rangle_{\mu=0} \longrightarrow \text{vanilla HMC}$$

- From $\{c_{2k}\}$, obtain all thermodynamic info: EOS *and* $T_c(\mu)$ *and* crit. pt. *and* ...
- As $\frac{\mu}{T}$ increases, need higher-order c_{2k}'s to control truncation error

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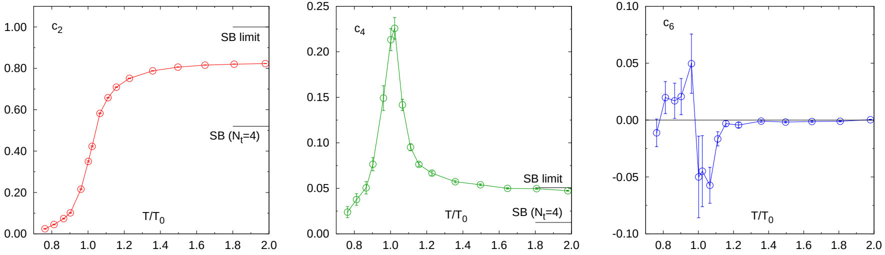

C. Schmidt, hep-lat/0610116

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- As $\frac{\mu}{T}$ increases, need higher-order c_{2k}'s to control truncation error
- Higher order $k \Rightarrow \mathrm{Tr}(\not{D}^{-2k}, ..)$, ie. more noise vectors, more cancellations..
- ..and also larger volumes $\longrightarrow$ work $\sim 36^k$ (Karsch et al.) at least
- Current best: $N_t = 8$, 6th order HOTQCD, $N_t = 6$, 8th order Gavai & Gupta, 0806.2233

Ratio method: ”effective radius of convergence”

$$\frac{P}{T^4} = \sum_{n=0}^{\infty} c_{2n}(T) \left(\frac{\mu}{T}\right)^{2n}$$

Singularity $(\mu_E, T_E) \Rightarrow$ $$\boxed{\frac{\mu_E}{T_E} = \lim_{n\to\infty} \sqrt{\left|\frac{c_{2n}}{c_{2n+2}}\right|}(T_E)}$$

Karsch et al.

Similar, for $\frac{\chi_q}{T^2}$: Gavai & Gupta

“$\mu_E / T_E < 0.6$”

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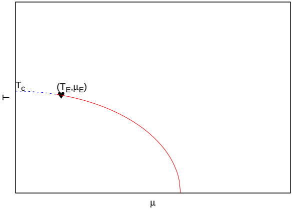
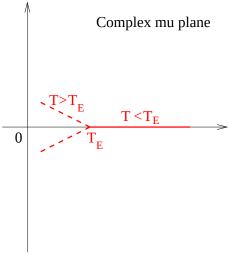
Critique:
- Need $n \to \infty$, not $n = 1, 2, 3$; $\sqrt{\left|\frac{c_2}{c_4}\right|}$ is not a lower or upper bound
- Robust criterion to choose T_E?
- Smallest convergence radius is NOT CEP

Stephanov hep-lat/0603014





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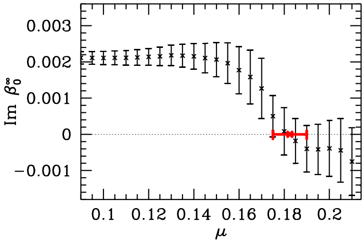
Remember Fodor & Katz:



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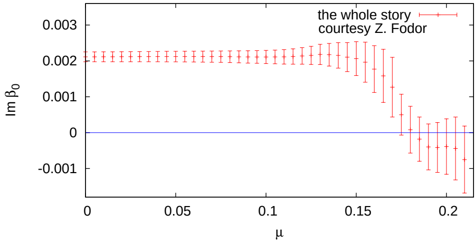
Remember Fodor & Katz:

need high-order derivatives



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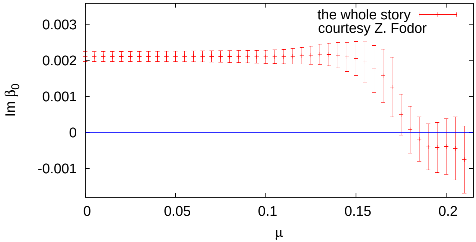
Can systematic error be controlled ?

First goal: discriminate between CEP and no CEP

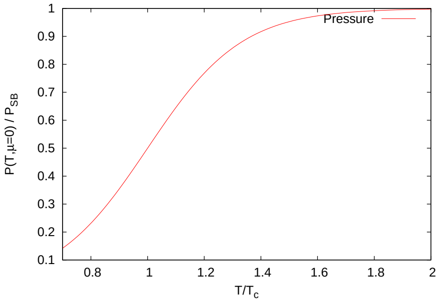


Case study on toy model, with Roger Herrigel

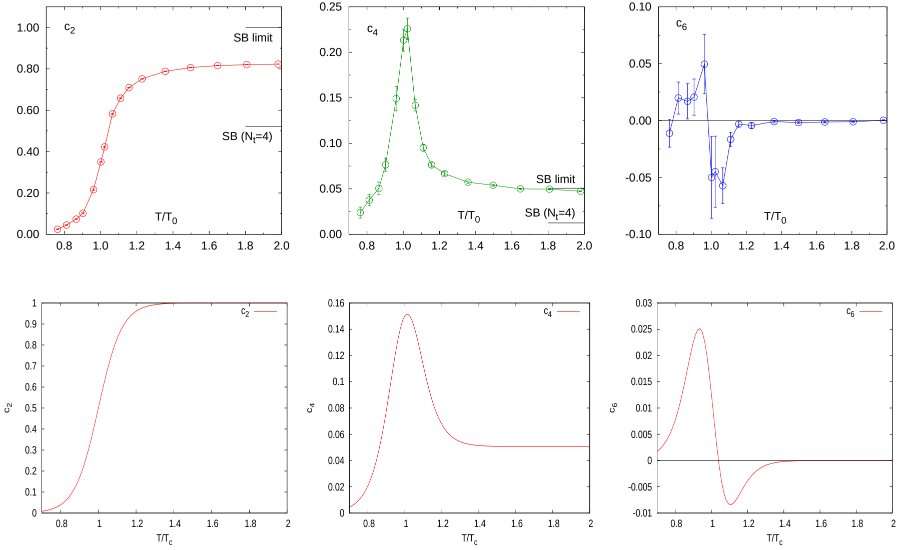
- **Idea**: study Taylor coeffs of $\frac{\Delta P}{T^4}\left(\frac{\mu}{T}\right) \equiv \Delta\hat{P}(\hat{\mu})$
  and "effective radius of convergence" in **controlled** situation
- Ansatz: $\Delta\hat{P}(\hat{\mu}) = \frac{\Delta\hat{P}_{SB}}{2} + \log(\cosh(\lambda(\hat{T}-1) + \frac{\Delta\hat{P}_{SB}}{2})) - \log(\cosh(\lambda(\hat{T}-1))$

Why ? - correct limits high-T, low-T
- $c_2 \approx (1+\tanh(\lambda(\hat{T}-1))$, *sigmoid*, no phase trans., no CEP
- single parameter λ controls width of crossover (rescaling of $\hat{T}$)

- no singularity on real $\hat{\mu}$ axis $\longrightarrow$ test for spurious signals



Toy ansatz: compare with Bielefeld et al.



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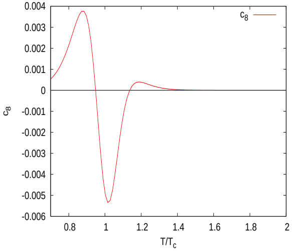

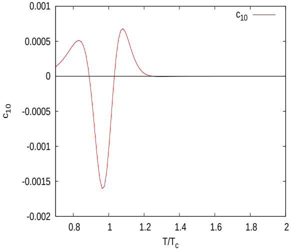

Oscillatory pattern $\longrightarrow c_k = 0$ at lower $\hat{T}$ as k increases

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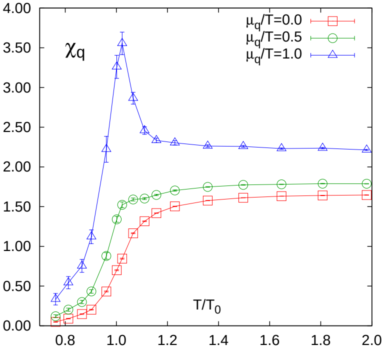

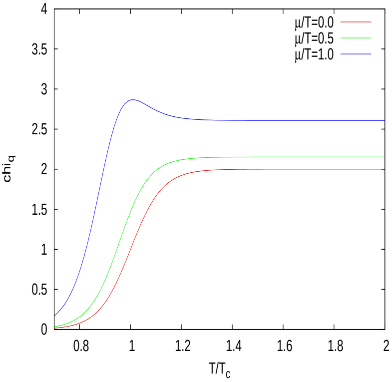

- Quark susceptibility rises, even without phase transition

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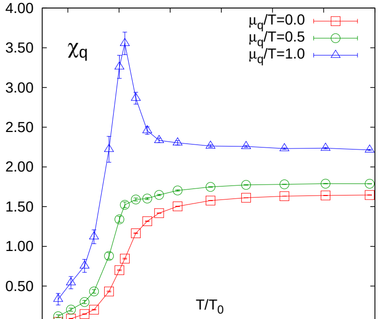
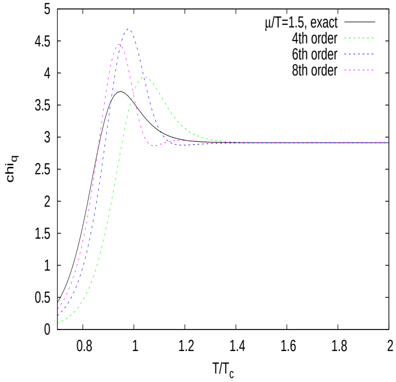
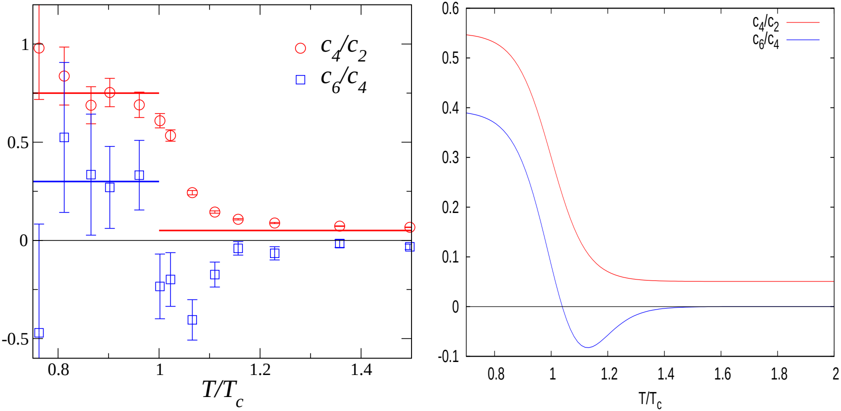




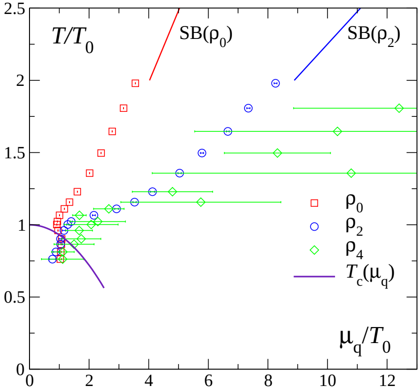
- Quark susceptibility rises, even without phase transition
- Truncation may increase susceptibility

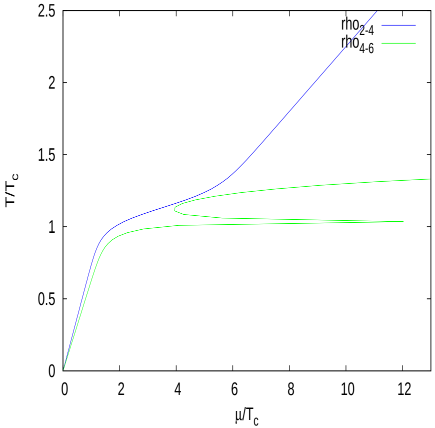
Toy ansatz: compare with Bielefeld et al.



- Qualitative agreement below T_c although HRG not built-in

Toy ansatz: compare with Bielefeld et al.

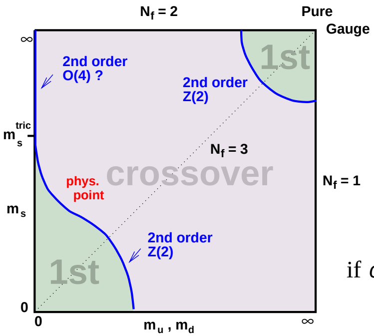




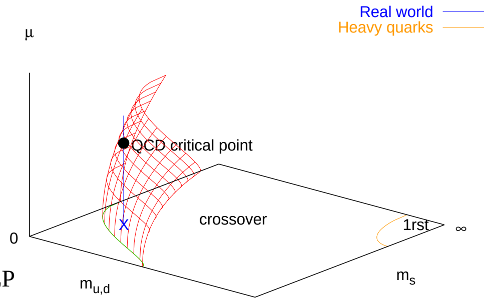
- Qualitative agreement

- **IF** we know that there is a crit. point: how to choose T_E?? BJS et al.
- **Can we predict whether or not there is a critical point ??**

A safer strategy ?



$\underbrace{\longrightarrow}_{\text{if } chiral \text{ CEP}}$



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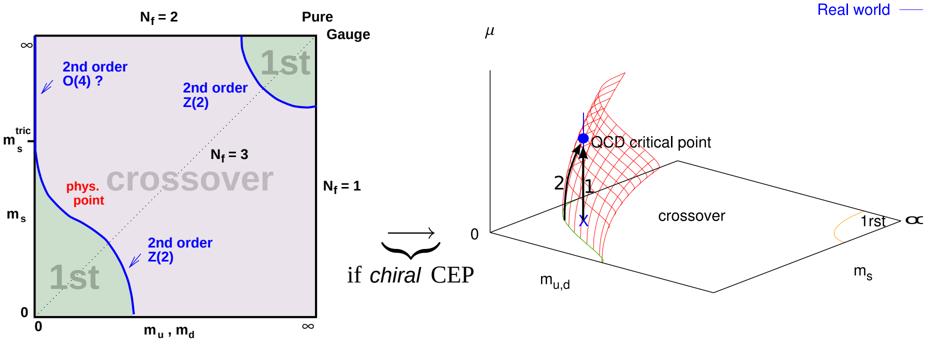
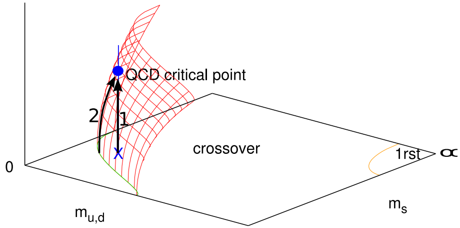


Two strategies:

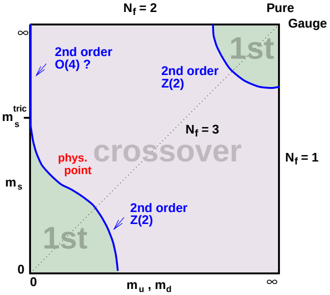
1 follow vertical line: $m = m_{\rm phys}$, turn on $\mu \longrightarrow$ radius of convergence?

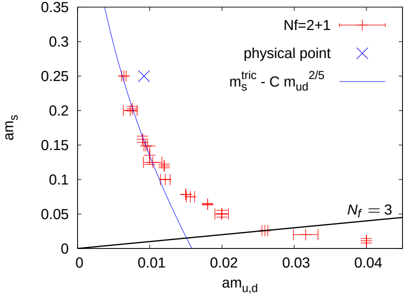
2 follow critical surface: $m = m_{\rm crit}(\mu)$

Strategy 2: follow critical surface PdF & O. Philipsen



(a) Identify $\mu = 0$ critical line:





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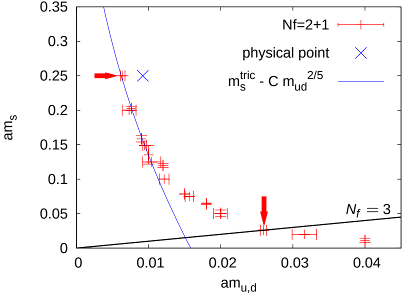
(b) Turn on imaginary μ and measure $\frac{m_c(\mu)}{m_c(0)} = 1 + \sum_{k=1} \mathbf{c_k} \left(\frac{\mu}{\pi T}\right)^{2k}$

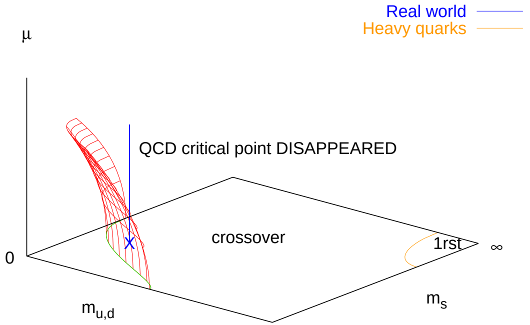


$N_f = 3$	$N_f = 2+1, m_s = m_s^{\rm phys}$
consistent $8^3 \times 4$ and $12^3 \times 4$, $\sim 5 \times 10^6$ traj.	$16^3 \times 4$, Grid computing, $\sim 10^6$ traj.
$\frac{m_c(\mu)}{m_c(0)} = 1 - 3.3(3)\left(\frac{\mu}{\pi T}\right)^2 - 47(20)\left(\frac{\mu}{\pi T}\right)^4 - \ldots$	$\frac{m_c^{u,d}(\mu)}{m_c^{u,d}(0)} = 1 - 39(8)\left(\frac{\mu}{\pi T}\right)^2 - \ldots$

(the μ^4 term: 8th derivative of P)

- Higher order terms? Convergence?

Strategy 2: follow critical surface PdF & O. Philipsen



No chiral crit. pt. at small chem. pot., $\frac{\mu}{T} \lesssim O(1)$, for $N_t = 4$ ($a \sim 0.3$ fm)

cf. Ejiri 0812.1534 $\longrightarrow \left(\frac{\mu}{T}\right)^{\mathrm{CEP}} \sim 2.4$

Consistency with effective models?

- Restoration of $U_A(1)$ anomaly favors first-order finite T transition

Chandrasekharan & Mehta

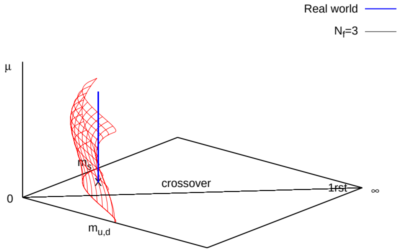
- Progressive restoration of $U_A(1)$ symmetry at finite density?

$$\text{NJL} + [\det \bar{q}_i(1-\gamma_5)q_j + \text{h.c.}] \times \exp(-\mu^2/\mu_0^2)$$

Chen et al. 0901.2407

- backbending of crit. surface
- crit. surf. ends when $T_c = 0$

$\longrightarrow$ still no chiral critical point!

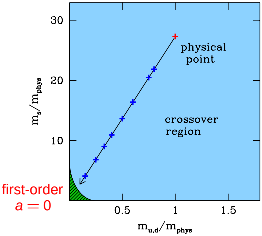


- Same pattern with linear sigma model + fluctuations

Bowman & Kapusta 0810.0042

Towards the continuum limit

- Critical line recedes away from physical point towards origin



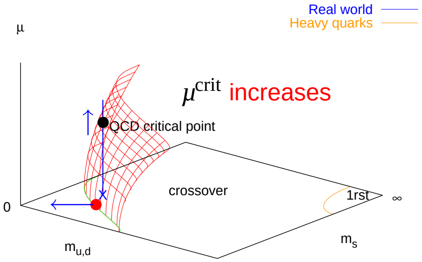
Endrodi, Fodor et al., arXiv:0710.0998

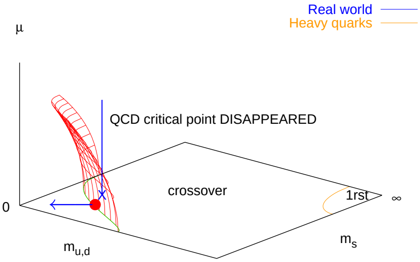
PdF & O. Philipsen, arXiv:0711.0262

Karsch et al., hep-lat/0309116

$N_t = 4 \;\longrightarrow\; N_t = \infty \;\Rightarrow\;$ change $\sim \mathcal{O}(10)$!!

IF curvature of critical surface unchanged...





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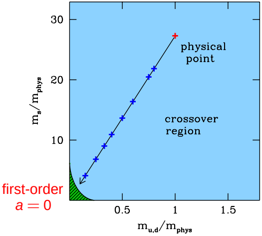

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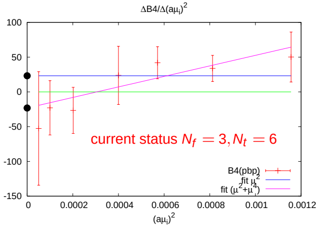


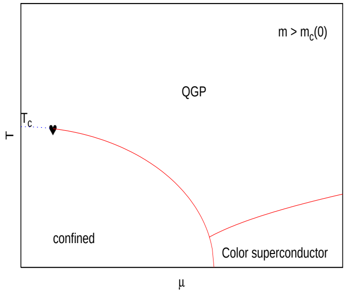
$$\frac{m_c(\mu)}{m_c(0)} = 1 \begin{cases} +12(11) & \mu^2 \text{ fit} \\ -12(15) & (\mu^2+\mu^4) \text{ fit} \end{cases} \left(\frac{\mu}{\pi T}\right)^2 - \ldots$$

$|c_1|$ not large: $\frac{m_c(\mu)}{m_c(0)} \sim 10 \;\longrightarrow\; \mu \sim \pi T$

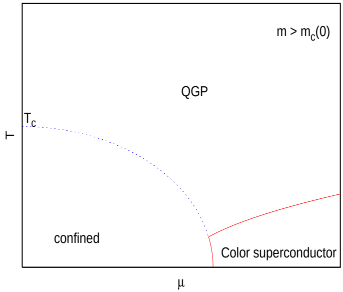
Crit. surface starts almost "vertical" $\longrightarrow$ need higher order to decide on crit. pt.

Resulting phase diagram (simplest possibility)

Standard scenario



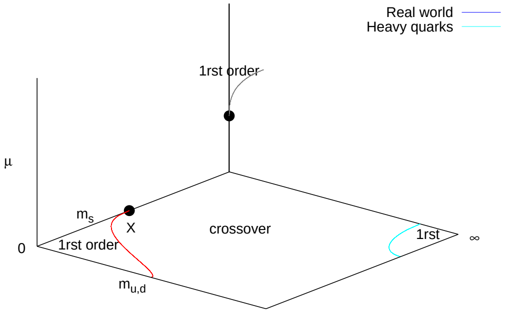
Exotic scenario



Arguments for standard wisdom?

- $O(4)$ transition for 2 massless flavors — Pisarski & Wilczek

$\Rightarrow$ tricritical points $(m_{u,d} = 0, m_s = \infty, \mu = \mu^*)$ and $(m_{u,d} = 0, m_s = m_s^*, \mu = 0)$



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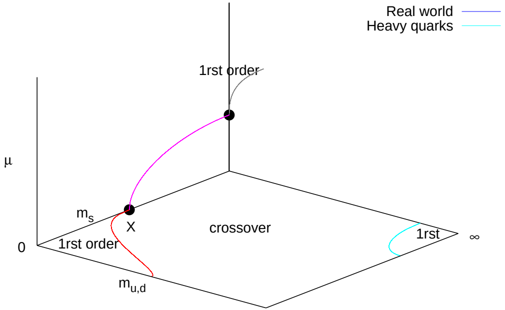
- $N_f = 2$ and $N_f = 2 + 1$ analytically connected



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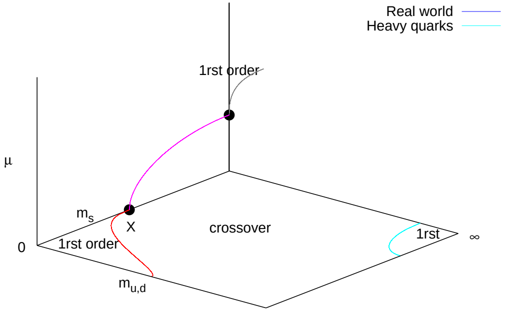

Critique:

- $O(4)$ if strong enough $U_A(1)$ anomaly, otherwise first-order

Chandrasekharan & Mehta

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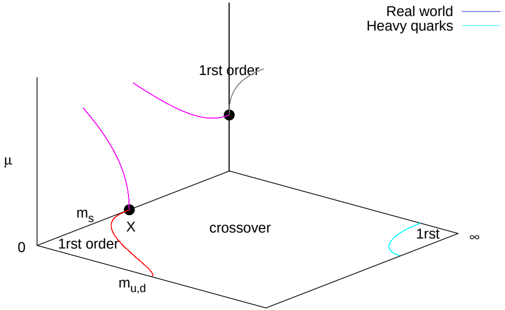


Critique:

- $O(4)$ if strong enough $U_A(1)$ anomaly, otherwise first-order — Chandrasekharan & Mehta
- $N_f = 2$ and $N_f = 2+1$ need not be connected $\longrightarrow$ study $N_f = 2$ crit. surf.

Conclusions

- **Confucius: Real knowledge is to know the extent of one's ignorance**

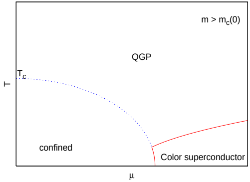
- $\frac{m_c(\mu)}{m_c(0)} = 1 + \mathbf{c_1} \left(\frac{\mu}{\pi T}\right)^2 + \ldots$: *can control systematics*

  Non-standard scenario $c_1 < 0$ for $N_t = 4$

- $a \longrightarrow 0$: critical surface *far* from physical point

  $\Longrightarrow$ need $c_1 > 0$ *and large* for $\frac{\mu_E}{T_E} \lesssim 1$, disfavored by data

- **QCD critical point?**



$\mu_E^B \lesssim 500$ MeV unlikely, or non-chiral

Backup: Gavai & Gupta's critical point

"We find the radius of convergence of the series at various temperatures, and bound the location of the QCD critical point to be $T_E/T_c \approx 0.94$ and $\mu_E/T < 0.6$"

- Arbitrariness in definition of "effective radius of convergence"

$$\frac{\mu_E}{T_E} = \lim_{n\to\infty} \sqrt{\left|\frac{c_{2n}}{c_{2n+2}}\right|}(T_E) \text{ versus } \frac{\chi_q}{T^2} = \sum_{n=1}^{\infty} 2n(2n-1)\, c_{2n} \left(\frac{\mu}{T}\right)^{2n-2}$$

$$\to \frac{\mu_E}{T_E} = \lim_{n\to\infty} \sqrt{\left|\frac{2n(2n-1)c_{2n}}{(2n+2)(2n+1)c_{2n+2}}\right|} \qquad n=1 \to \text{ factor } 1/\sqrt{6}$$

- Criterion for T_E
- Consistency of results with toy model having no critical point
- Endrodi, Fodor & Katz:

  peaks of all susceptibilities ***decrease*** as μ increases