Scaling analysis of the two-flavor phase transition

Bertram Klein

Physik Department, Technische Universität München

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J. Braun and B. Klein, Phys. Rev. D **77** (2008) 096008;
EPJ C (2009), arXiv:0810.0857.

Technische Universität München

Phase transitions in QCD

QCD phase transitions

- de-confinement phase transition
- chiral phase transition

Phase transitions in QCD

QCD phase transitions

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Lattice Gauge Theory

- fully non-perturbative method
- finite simulation volume
- explicit symmetry breaking through quark masses
- phase transition order? → (finite-size) scaling analysis

- $N_f = 2$: second order for $m_q = 0$, crossover for $m_q \neq 0$, O(4)

R. D. Pisarski and F. Wilczek, Phys. Rev. D **29** (1984) 338.

- first order phase transition? (confinement dominates?) (staggered fermions)

M. D'Elia, A. Di Giacomo and C. Pica, Phys. Rev. D **72** (2005) 114510 [arXiv:hep-lat/0503030];

G. Cossu, M. D'Elia, A. Di Giacomo, and C. Pica (2007), arXiv:0706.4470 [hep-lat].

- consistent with O(2) for χQCD (staggered) (but no scaling)

J. B. Kogut and D. K. Sinclair, Phys. Rev. D **73** (2006) 074512 [arXiv:hep-lat/0603021].

- decide by analyzing scaling behavior

A scaling analysis requires *critical exponents* and *scaling functions*

- Scaling functions obtained mostly from O(N) lattice simulations and perturbative RG

  D. Toussaint, Phys. Rev. D55 (1997) 362.

  J. Engels, S. Holtmann, T. Mendes, and T. Schulze, Phys. Lett. **B514** (2001) 299.

  E. Brézin, D. J. Wallace, and K. Wilson, Phys. Rev. B7, 232 (1973).

  F. Parisen Toldin, A. Pelissetto and E. Vicari, JHEP **0307** (2003) 029.

- O(N) critical exponents from FRG calculations

  N. Tetradis and C. Wetterich, Nucl. Phys. **B422** (1994) 541.

  O. Bohr, B.J. Schaefer, and J. Wambach, Int. J. Mod. Phys. **A16** (2001) 3823.

  D. F. Litim and J. M. Pawlowski, Phys. Lett. B **516** (2001) 197.

- few results with explicit symmetry breaking
- no results on finite-size scaling

Functional RG for the O(N)-model

Effective action at scale k ($\Lambda \geq k \geq 0$) $(\eta = 0)$

$$\Gamma_k[\phi] = \int d^d x \, \frac{1}{2}(\partial_\mu \phi)^2 + $$
$$+a_1(k)(\phi^2 - \sigma_0^2(k)) + a_2(k)(\phi^2 - \sigma_0^2(k))^2 + \ldots - H(\sigma - \sigma_0(k))$$

$$\phi = (\sigma, \vec{\pi}) \qquad \phi^2 = \sigma^2 + \vec{\pi}^2 \quad O(N)\text{-symmetric}$$

scale-dependent couplings

$\sigma_0(k), a_n(k), \quad n = 1, \ldots, n_{max} \qquad 2a_1(k)\sigma_0(k) = H \equiv const.$

Inputs:

- choice of initial scale Λ sets all scales
- couplings $\sigma_0(\Lambda)$ and $a_2(\Lambda)$ at initial scale Λ
- $d = 3$: no field-theoretical temperature
- $\sigma_0(\Lambda) - \sigma_0^{\rm crit}(\Lambda) \sim T - T_c$:
  $\sigma_0(\Lambda) > \sigma_0^{\rm crit}(\Lambda) \Rightarrow$ broken phase
  $\sigma_0(\Lambda) < \sigma_0^{\rm crit}(\Lambda) \Rightarrow$ symmetric phase

application to finite volume:

$$\int \frac{d^d p}{(2\pi)^d} \; f(p^2) \longrightarrow \frac{1}{L^d} \sum_{n_1,\ldots,n_d} f(\left(\frac{2\pi}{L}\right)^2 (n_1^2 + \ldots + n_d^2))$$

- local expansion around the minimum
- lowest order in local potential approximation ($\eta = 0$)
- potential with explicit symmetry breaking
- FRG (Wetterich) with optimized cutoff

  D. F. Litim, Phys. Lett. B486 (2000) 92.

  $\Leftrightarrow$ proper-time RG (infinite volume)

  S. B. Liao, Phys. Rev. D53 (1996) 2020.

  D. F. Litim and J. M. Pawlowski, Phys. Lett. B 516 (2001) 197.

Scaling for the singular free energy

there is no length scale at a critical point: $\xi \to \infty$
▷ scale invariance of free energy density at critical point
▷ use behavior close to critical point

$$f_s(t, h) \quad = \quad \ell^{-d} f_s(\ell^{y_t} t, \ell^{y_h} h)$$

$$t = (T - T_c)/T_0, \quad h = H/H_0, \quad y_t = \frac{1}{\nu}, \quad y_h = \frac{\nu}{\beta\delta}$$

idea: keep one of the arguments fixed
▷ becomes function of a single scaling variable

Scaling function in Fisher parameterization

$$M \sim \frac{\partial}{\partial H} f(t, h)$$

$$t = (T - T_c)/T_0, \qquad h = H/H_0, \qquad z = \frac{t}{h^{1/(\beta\delta)}}, \qquad \xi(t) \sim t^{-\nu}$$

Scaling function for the order parameter M

$$\left.\begin{array}{l} M(t, h=0) = (-t)^{\beta} \\ M(t=0, h) = \quad h^{1/\delta} \end{array}\right\} \to M(t,h) = h^{1/\delta} f(z),\ f(z) \underset{z\to-\infty}{\longrightarrow} (-z)^{\beta}$$

critical exponent β enters into asymptotic behavior

Scaling function in Fisher parameterization

Scaling function for the susceptibility χ

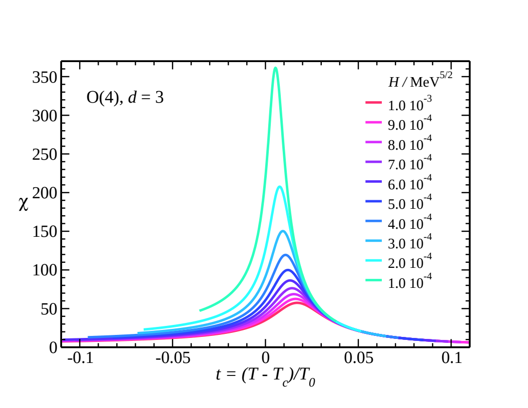
$$\chi = \frac{\partial M}{\partial H} \Rightarrow \chi = \frac{1}{H_0} h^{1/\delta - 1} \frac{1}{\delta} \left[f(z) - \frac{z}{\beta} f'(z) \right] = \frac{1}{H_0} h^{1/\delta - 1} f_\chi(z)$$

▷ consequence of scaling:

$f_\chi(z)$ completely specified by $f(z)$

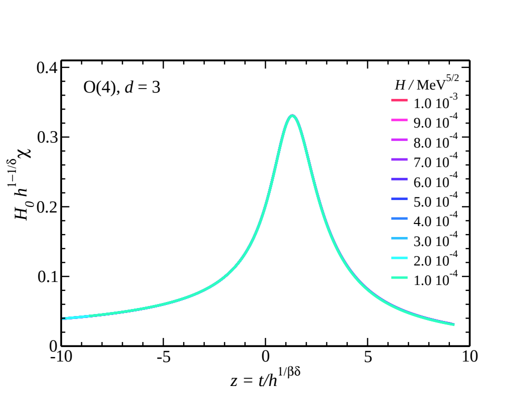
Susceptibility χ as a function of t

for small values of H (note trajectory $t_p = z_p h^{1/(\beta\delta)}$)



Rescaled Susceptibility $H_0 h^{1-1/\delta}\chi$ as a function of z

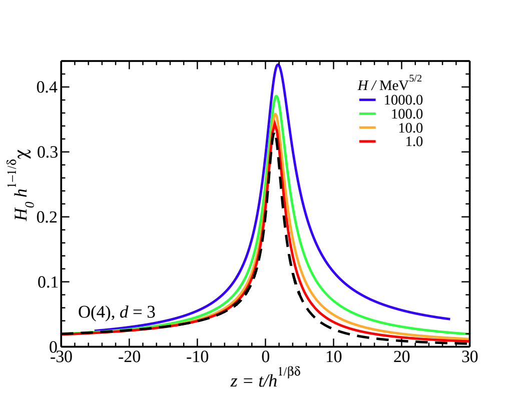
for small values of H



Rescaled Susceptibility $H_0 h^{1-1/\delta}\chi$ as a function of z

for large values of H:

▷ large scaling corrections (but no intersections!)



Finite-Size Scaling

- Universal behavior requires divergence of the correlation length ξ
- Finite volume size L cuts off the critical fluctuations
- Universal scaling behavior is therefore affected if correlation length $\xi \sim L$, depends on ratio ξ/L

Finite-Size Scaling hypothesis (Fisher): The ratio of thermodynamic quantities $(M, \chi, \ldots)$
in the finite-size system and the infinite-size system is a function of *only* the ratio ξ/L:

$$\frac{M_L(t)}{M_\infty(t)} = \mathcal{F}\left(\frac{L}{\xi(t)}\right)$$

Finite-Size Scaling Functions

Idea for obtaining the universal Finite-Size Scaling functions:

- keep $L/\xi = \text{const.} \quad \rightarrow \quad \xi(t) \sim t^{-\nu} \quad \rightarrow$ vary $t \sim L^{1/\nu}$
- keep $z = t/h^{1/(\beta\delta)} = \text{const.} \quad \rightarrow$ vary $h \sim L^{-\beta\delta/\nu}$

$\Rightarrow$ form finite-size scaling variable $\bar{h} = hL^{\beta\delta/\nu}$

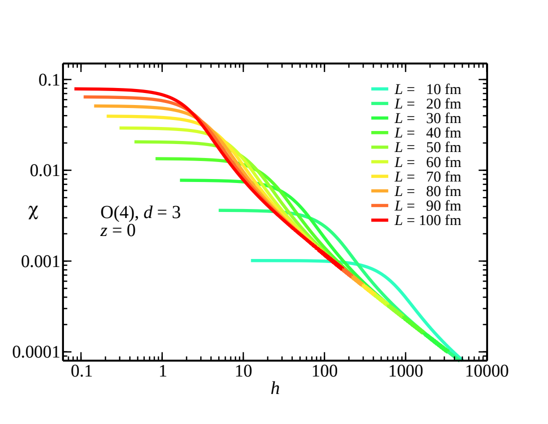
$$M(t,h) = h^{1/\delta} f(z) \quad \rightarrow \quad L^{-\beta/\nu}(hL^{\beta\delta/\nu})^{1/\delta} f(z)$$

Finite-Size Scaling Functions depend only on $hL^{\beta\delta/\nu}$
(for any given value of z):

$$\begin{aligned} L^{\beta/\nu} M &= Q_M(z, hL^{\beta\delta/\nu}) \\ L^{-\gamma/\nu} \chi &= Q_\chi(z, hL^{\beta\delta/\nu}) \end{aligned}$$

Finite-Size Scaling

Susceptibility $\chi(h)$ vs. h for $L = 10 - 100$ fm

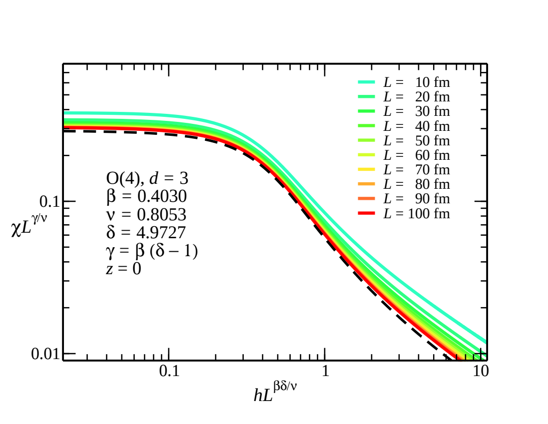


at the critical temperature ($z = 0$).

- ξ small for large h
- deviations for $\xi \sim L$
- asymptotic behavior given by $1/\delta - 1$

Finite-Size Scaling

Finite-size scaled susceptibility $\chi L^{-\gamma/\nu}$ vs. $hL^{\beta\delta/\nu}$



for $L = 10 - 100$ fm.

- scaling deviations for large fields h
- controlled by sub-leading operator
- consistent with RG prediction for ω
- extrapolate to obtain scaling function

Analysis of lattice QCD results

exploit *universality* near a critical point:

- long-range fluctuations determine behavior
- depends only on symmetry properties and dimensionality

▷ only comparison of universal scaling functions possible

- only IR quantities can be compared, different UV physics
- determine non-universal normalization factors *or*
- determine dimensionless ratios

▷ locate finite-size scaling regions

Scale normalizations

- for every coupling $(T, H, \dots)$ there is a normalization scale $(T_0, H_0, \dots)$
- $\bar{h} = hL^{\beta\delta/\nu}$ is not dimensionless $\Rightarrow$ not universal!
- Length scale normalization also required

best candidate:

- dimensionless ratio/product of IR quantities

$$\frac{L}{\xi(t, h, L)} = M_\sigma(t, h, L)L \geq M_\pi(t, h, L)L$$

- ratios of quantities in systems with fixed system size ratio, e.g. $\xi(2L)/\xi(L)$

Application to lattice simulation results

comprehensive lattice calculation with $N_f = 2$ staggered fermions for careful analysis of scaling and finite-size scaling

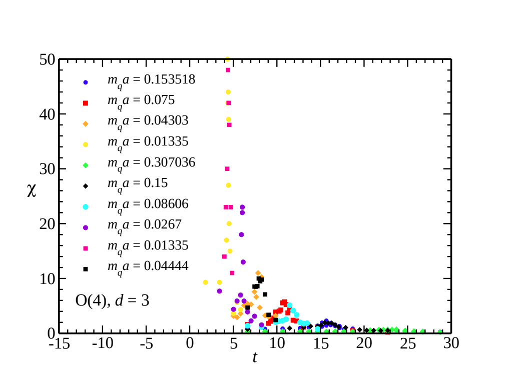
M. D'Elia, A. Di Giacomo and C. Pica, Phys. Rev. D **72** (2005) 114510 [arXiv:hep-lat/0503030];

G. Cossu, M. D'Elia, A. Di Giacomo, and C. Pica (2007), arXiv:0706.4470 [hep-lat].

- $L_t = 4$ throughout
- $L_s = 12, 16, 20, 24, 32$ varied, $L_s/L_t > 3$
- effectively three-dimensional
- $L_s\, am_\pi \gtrsim 10 \Rightarrow$ expect small finite-size effects
- one exception: $m_q a = 0.01335$, $L_s = 16$ has $L_s\, am_\pi = 4.5$

Scaling of the susceptibility on the lattice?

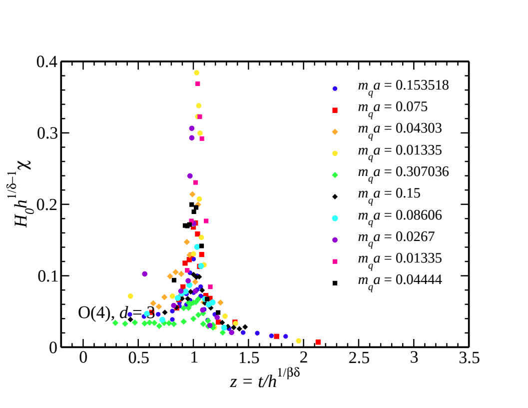
Normalized susceptibility of DiGiacomo *et al.*



- large $m_q\,a$ effects
- no discernible finite-size effects
- susceptibility curves for different $m_q\,a$ intersect!
- expect that results do not match after rescaling

Scaling of the susceptibility on the lattice?

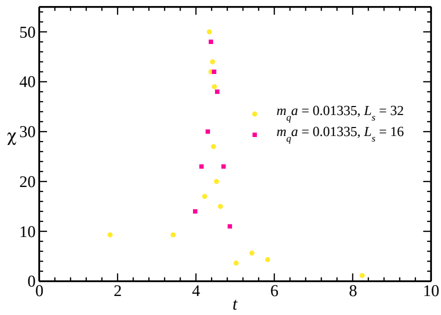
Rescaled susceptibility of DiGiacomo *et al.*



- peak position agrees due to normalization
- scaled peak heights not the same
- clearly no good scaling behavior
- large mass results strongly suppressed over expectations

Finite-size scaling of the susceptibility?

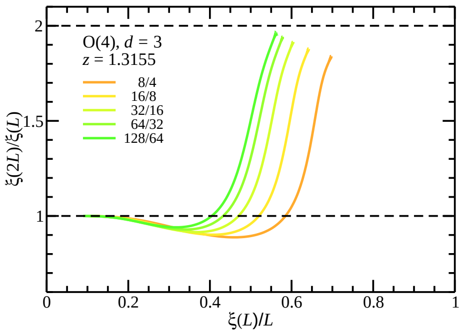
Can we observe finite-size scaling for the smallest quark masses ($am_q = 0.01335$)?



- estimate correlation length: $\chi \sim \xi^2 \lesssim 1/m_\pi^2$
- for $L_s = 16$ $\Rightarrow am_\pi L_s = 4.5$
- for $L_s = 32$ $\Rightarrow am_\pi L_s = 8.9$
- $\xi(L_s = 16)/L_s \approx 7.1/16 \approx 0.44 > 1/(am_\pi L_s) \approx 0.22$

Ratios of results from systems with fixed size ratio

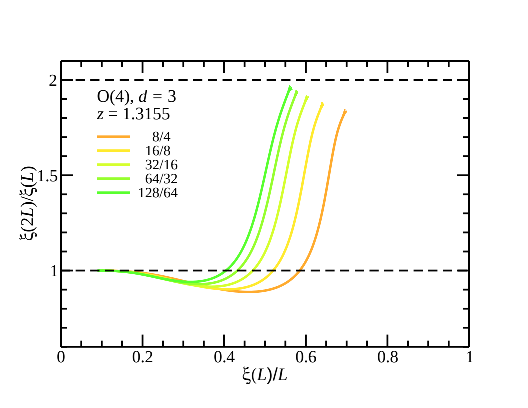
$\xi(2L)/\xi(L)$ as a function of $\xi(L)/L$



- no normalization problems
- but large corrections
- correlation length bounded $\xi(L) \leq k_0 L$
- ξ-ratio approaches volume ratio for $\xi(L) \to L$
- ξ-ratio approaches 1 for $L \to \infty$

Ratios of results from systems with fixed size ratio

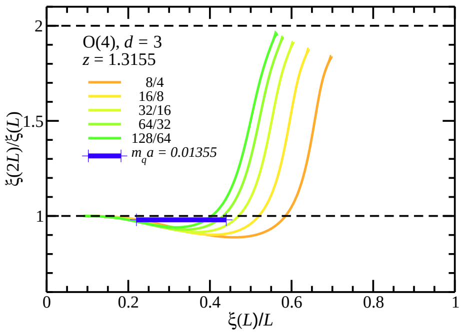
$\xi(2L)/\xi(L)$ as a function of $\xi(L)/L$



- works only with results for both $L(=16)$ and $2L(=32)$
- use in first approximation $\xi(L) \sim \sqrt{\chi(L)}$

Ratios of results from systems with fixed size ratio

$\xi(2L)/\xi(L)$ as a function of $\xi(L)/L$



- compare only result for peak position ($z_p = 1.3155$)
- $\chi(16)/\chi(32) \approx 0.96$
- $\xi(L=16)/16 \sim \sqrt{\chi(L=16)}/16 = 7.1/16 \approx 0.44 > 1/(am_\pi L_s) \approx 0.22$
- no meaningful finite-size effects

Conclusions

- (Finite-Size) Scaling functions important for comparison
- no discernible finite-size effects for current values of lattice size and pion mass
- large corrections to scaling for large symmetry breaking
- O(4) scaling not observed in these results, but not yet ruled out either
- *even smaller* quark masses on the lattice necessary?

Outlook

- comparison to O(2) scaling functions: scaling region?
- scaling with actual dimensional reduction