

Complete second-order dissipative relativistic fluid dynamics

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Preliminaries (I)

Tensor decomposition of net charge current and energy-momentum tensor:

1. **Net charge current:**

$$\boxed{N^\mu = n u^\mu + \nu^\mu}$$

u^μ **fluid 4-velocity**, $u^\mu u_\mu = u^\mu g_{\mu\nu} u^\nu = 1$

$g_{\mu\nu} \equiv \text{diag}(+, -, -, -)$ (**West coast!!**) **metric tensor**,

$n \equiv u^\mu N_\mu$ **net charge density in fluid rest frame**

$\nu^\mu \equiv \Delta^{\mu\nu} N_\nu$ **diffusion current** (flow of net charge relative to u^μ), $\nu^\mu u_\mu = 0$

$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ **projector onto 3-space orthogonal to u^μ** , $\Delta^{\mu\nu} u_\nu = 0$

2. **Energy-momentum tensor:**

$$\boxed{T^{\mu\nu} = \epsilon u^\mu u^\nu - (p + \Pi) \Delta^{\mu\nu} + 2 q^{(\mu} u^{\nu)} + \pi^{\mu\nu}}$$

$\epsilon \equiv u^\mu T_{\mu\nu} u^\nu$ **energy density in fluid rest frame**

p **pressure in fluid rest frame**

Π **bulk viscous pressure**, $p + \Pi \equiv -\frac{1}{3} \Delta^{\mu\nu} T_{\mu\nu}$

$q^\mu \equiv \Delta^{\mu\nu} T_{\nu\lambda} u^\lambda$ **heat flux current** (flow of energy relative to u^μ), $q^\mu u_\mu = 0$

$\pi^{\mu\nu} \equiv T^{<\mu\nu>}$ **shear stress tensor**, $\pi^{\mu\nu} u_\mu = \pi^{\mu\nu} u_\nu = 0$, $\pi^\mu{}_\mu = 0$

$a^{(\mu\nu)} \equiv \frac{1}{2} (a^{\mu\nu} + a^{\nu\mu})$ **symmetrized tensor**

$a^{<\mu\nu>} \equiv \left(\Delta_\alpha^{(\mu} \Delta^{\nu)}_\beta - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} \right) a^{\alpha\beta}$ **symmetrized, traceless spatial projection**

Preliminaries (II)

Fluid dynamical equations:

1. **Net charge** (e.g., baryon, strangeness, etc.) **conservation:**

$$\partial_\mu N^\mu = 0 \iff \dot{n} + n\theta + \partial \cdot \nu = 0$$

$\dot{a} \equiv u^\mu \partial_\mu a$ convective (comoving) derivative

(fluid rest frame, $u_{\text{RF}}^\mu \equiv g^\mu_0 \implies$ time derivative, $\dot{a}_{\text{RF}} \equiv \partial_t a$)

$\theta \equiv \partial_\mu u^\mu$ expansion scalar

2. **Energy-momentum conservation:**

$$\partial_\mu T^{\mu\nu} = 0 \iff \text{energy conservation:}$$

$$u_\nu \partial_\mu T^{\mu\nu} = \dot{\epsilon} + (\epsilon + p + \Pi)\theta + \partial \cdot q - q \cdot \dot{u} - \pi^{\mu\nu} \partial_\mu u_\nu = 0$$

acceleration equation:

$$\Delta^{\mu\nu} \partial^\lambda T_{\nu\lambda} = 0 \iff$$

$$(\epsilon + p)\dot{u}^\mu = \nabla^\mu(p + \Pi) - \Pi\dot{u}^\mu - \Delta^{\mu\nu}\dot{q}_\nu - q^\mu\theta - q^\nu\partial_\nu u^\mu - \Delta^{\mu\nu}\partial^\lambda\pi_{\nu\lambda}$$

$\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ 3-gradient (spatial gradient in fluid rest frame)

Preliminaries (III)

Problem:

5 equations, **but** 15 unknowns (for given u^μ): ϵ , p , n , Π , ν^μ (3), q^μ (3), $\pi^{\mu\nu}$ (5)

Solution:

1. **clever choice of frame (Eckart, Landau,...):** eliminate ν^μ or q^μ
 \implies does not help! Promotes u^μ to dynamical variable!
2. **ideal fluid limit:** all dissipative terms vanish, $\Pi = \nu^\mu = q^\mu = \pi^{\mu\nu} = 0$
 \implies 6 unknowns: ϵ , p , n , u^μ (3) (not quite there yet...)
 \implies fluid is in local thermodynamical equilibrium
 \implies provide **equation of state (EOS)** $p(\epsilon, n)$ to close system of equations
3. **provide additional equations for dissipative quantities**
 \implies **dissipative** relativistic fluid dynamics
 - (a) **First-order** theories: e.g. generalization of **Navier-Stokes (NS)** equations to the relativistic case (Eckart, Landau-Lifshitz)
 - (b) **Second-order** theories: e.g. **Israel-Stewart (IS)** equations

Preliminaries (IV)

Navier-Stokes (NS) equations:

1. **bulk viscous pressure:** $\Pi_{\text{NS}} = -\zeta \theta$

ζ **bulk viscosity**

2. **heat flux current:**

$$q_{\text{NS}}^{\mu} = \frac{\kappa}{\beta} \frac{n}{\beta(\epsilon + p)} \nabla^{\mu} \alpha$$

$\beta \equiv 1/T$ **inverse temperature,**

$\alpha \equiv \beta \mu,$ **μ chemical potential,**

κ **thermal conductivity**

3. **shear stress tensor:** $\pi_{\text{NS}}^{\mu\nu} = 2\eta \sigma^{\mu\nu}$

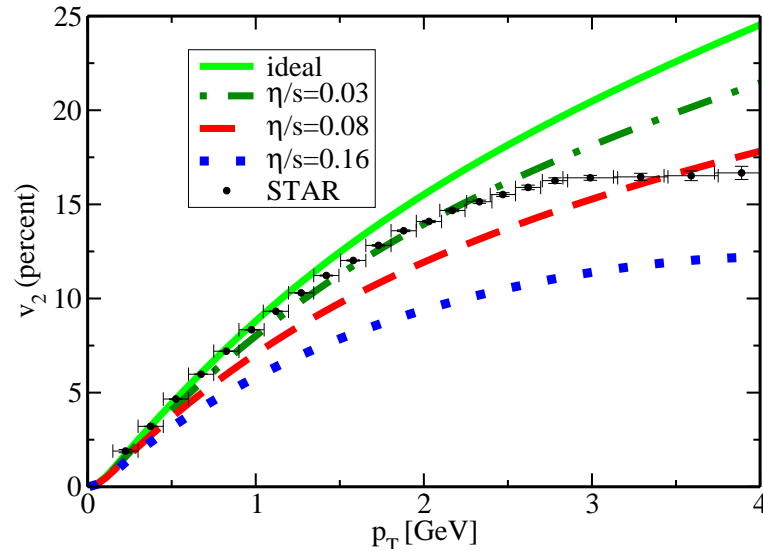
η **shear viscosity,**

$\sigma^{\mu\nu} = \nabla^{\langle\mu} u^{\nu\rangle}$ **shear tensor**

\Rightarrow **algebraic** expressions in terms of thermodynamic and fluid variables

\Rightarrow **simple... but: unstable and acausal** equations of motion!!

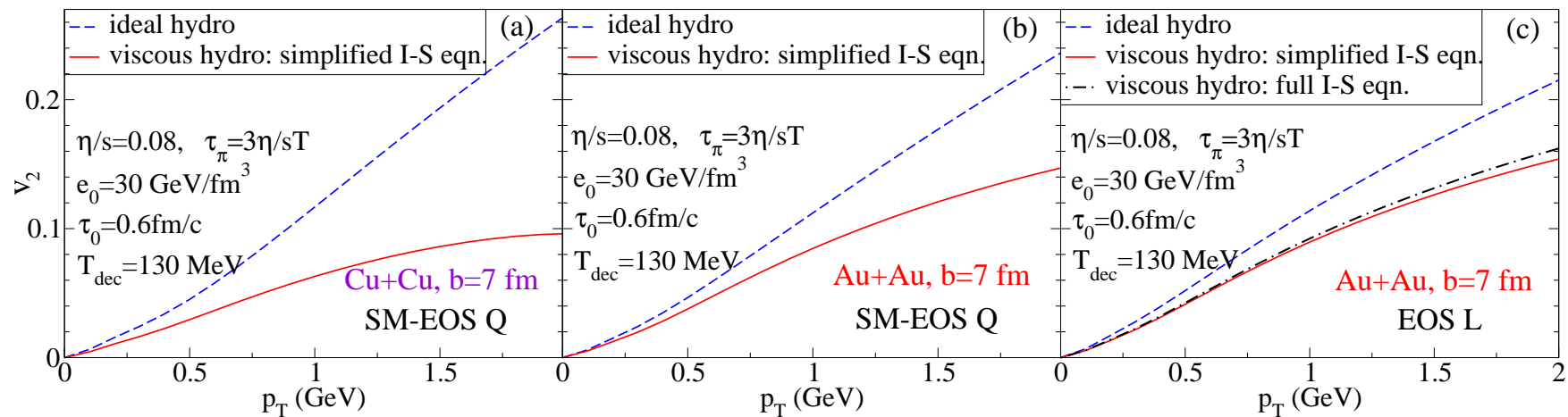
Motivation (I)



P. Romatschke, U. Romatschke, PRL 99 (2007) 172301

Au+Au @ $\sqrt{s} = 200$ AGeV
charged particles, min. bias

H. Song, U.W. Heinz, PRC 78 (2008) 024902



Motivation (II)

Israel-Stewart (IS) equations: second-order, dissipative relativistic fluid dynamics

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

“Simplified” IS equations: e.g. shear stress tensor

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = \pi_{\text{NS}}^{\mu\nu}$$

⇒ **dynamical** (instead of **algebraic**) equations for dissipative terms!

⇒ $\pi^{\mu\nu}$ relaxes to its **NS** value $\pi_{\text{NS}}^{\mu\nu}$ on the time scale τ_π

⇒ **stable** and **causal** fluid dynamical equations of motion!

“Full” IS equations:

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = \pi_{\text{NS}}^{\mu\nu} - \frac{\eta}{2\beta} \pi^{\mu\nu} \partial_\lambda \left(\frac{\tau_\pi}{\eta} \beta u^\lambda \right) + 2 \tau_\pi \pi_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda}$$

$$\omega^{\mu\nu} \equiv \frac{1}{2} \Delta^{\mu\alpha} \Delta^{\nu\beta} (\partial_\alpha u_\beta - \partial_\beta u_\alpha) \quad \text{vorticity}$$

Motivation (III)

⇒ Difference between “simplified” and “full” IS equations:
the latter include higher-order terms?

For instance, if $\frac{\pi^{\mu\nu}}{\epsilon} \sim \delta \ll 1$, $\tau_\pi \omega^{\mu\nu} \sim \delta \ll 1 \implies \tau_\pi \omega_\lambda^{<\mu} \pi^{\nu>\lambda} \frac{1}{\epsilon} \sim \delta^2$

⇒ **Goals:**

1. What are the correct equations of motion for the dissipative quantities?
⇒ develop consistent **power counting scheme**
2. Generalization to $\mu \neq 0$ (relevant for FAIR physics!)
⇒ include **heat flux** q^μ
3. Generalization to non-conformal fluids (relevant near T_c !)
⇒ include **bulk viscous pressure** Π

Results (I)

Power counting:

3 length scales: 2 microscopic, 1 macroscopic

- thermal wavelength $\lambda_{\text{th}} \sim \beta \equiv 1/T$
- mean free path $\ell_{\text{mfp}} \sim (\langle \sigma \rangle n)^{-1}$
 $\langle \sigma \rangle$ averaged cross section, $n \sim T^3 = \beta^{-3} \sim \lambda_{\text{th}}^{-3}$
- length scale over which macroscopic fluid fields vary L_{hydro} , $\partial_\mu \sim L_{\text{hydro}}^{-1}$

Note: since $\eta \sim (\langle \sigma \rangle \lambda_{\text{th}})^{-1} \implies$

$$\frac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \frac{1}{\langle \sigma \rangle n} \frac{1}{\lambda_{\text{th}}} \sim \frac{\lambda_{\text{th}}^3}{\langle \sigma \rangle \lambda_{\text{th}}} \sim \frac{\lambda_{\text{th}}^3}{\langle \sigma \rangle \lambda_{\text{th}}} \sim \frac{\eta}{s}$$

s entropy density, $s \sim n \sim T^3 = \beta^{-3} \sim \lambda_{\text{th}}^{-3}$

$\implies \frac{\eta}{s}$ solely determined by the 2 microscopic length scales!

Note: similar argument holds for $\frac{\zeta}{s}$, $\frac{\kappa}{\beta s}$

Results (II)

3 regimes:

- **dilute gas limit** $\frac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \frac{\eta}{s} \gg 1 \iff \langle \sigma \rangle \ll \lambda_{\text{th}}^2 \implies \text{weak-coupling limit}$

- **viscous fluids** $\frac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \frac{\eta}{s} \sim 1 \iff \langle \sigma \rangle \sim \lambda_{\text{th}}^2$

interactions happen on the scale $\lambda_{\text{th}} \implies \text{moderate coupling}$

- **ideal fluid limit** $\frac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \frac{\eta}{s} \ll 1 \iff \langle \sigma \rangle \gg \lambda_{\text{th}}^2 \implies \text{strong-coupling limit}$

gradient (derivative) expansion:

$$\ell_{\text{mfp}} \partial_{\mu} \sim \frac{\ell_{\text{mfp}}}{L_{\text{hydro}}} \equiv K \sim \delta \ll 1$$

K Knudsen number

\implies equivalent to $k \ell_{\text{mfp}} \ll 1$, k typical momentum scale

R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100

\implies separation of macroscopic fluid dynamics (large scale $\sim L_{\text{hydro}}$)
from microscopic particle dynamics (small scale $\sim \ell_{\text{mfp}}$)

Results (III)

Primary quantities: ϵ, p, n, s \iff **Dissipative quantities:** $\Pi, q^\mu, \pi^{\mu\nu}$

$$\text{If } K \sim \ell_{\text{mfp}} \partial_\mu \sim \delta \ll 1, \text{ then } \frac{\Pi}{\epsilon} \sim \frac{q^\mu}{\epsilon} \sim \frac{\pi^{\mu\nu}}{\epsilon} \sim \delta \ll 1$$

Dissipative quantities are small compared to **primary quantities**

\implies small deviations from local thermodynamical equilibrium!

Note: statement independent of value of $\frac{\zeta}{s}, \frac{\kappa}{\beta s}, \frac{\eta}{s}$!

Proof: Gibbs relation: $\epsilon + p = Ts + \mu n \implies \beta\epsilon \sim s$!

Estimate dissipative terms by their **Navier-Stokes values:**

$$\Pi \sim \Pi_{\text{NS}} = -\zeta \theta, \quad q^\mu \sim q_{\text{NS}}^\mu = \frac{\kappa}{\beta} \frac{n}{\beta(\epsilon + p)} \nabla^\mu \alpha, \quad \pi^{\mu\nu} \sim \pi_{\text{NS}}^{\mu\nu} = 2\eta \sigma^{\mu\nu}$$

$$\implies \frac{\Pi}{\epsilon} \sim -\frac{\zeta}{\beta\epsilon} \beta \theta \sim -\frac{\zeta}{s} \frac{\beta}{\lambda_{\text{th}}} \frac{\lambda_{\text{th}}}{\ell_{\text{mfp}}} \ell_{\text{mfp}} \theta \sim \ell_{\text{mfp}} \partial_\mu u^\mu \sim \delta,$$

$$\frac{q^\mu}{\epsilon} \sim \frac{\kappa}{\beta} \frac{1}{\beta\epsilon} \frac{n}{\beta(\epsilon + p)} \beta \nabla^\mu \alpha \sim \frac{\kappa}{\beta s} \frac{\beta}{\lambda_{\text{th}}} \frac{\lambda_{\text{th}}}{\ell_{\text{mfp}}} \ell_{\text{mfp}} \nabla^\mu \alpha \sim \ell_{\text{mfp}} \nabla^\mu \alpha \sim \delta,$$

$$\frac{\pi^{\mu\nu}}{\epsilon} \sim 2 \frac{\eta}{\beta\epsilon} \beta \sigma^{\mu\nu} \sim 2 \frac{\eta}{s} \frac{\beta}{\lambda_{\text{th}}} \frac{\lambda_{\text{th}}}{\ell_{\text{mfp}}} \ell_{\text{mfp}} \sigma^{\mu\nu} \sim \ell_{\text{mfp}} \nabla^{<\mu} u^{\nu>} \sim \delta, \quad \text{q.e.d.}$$

Results (IV)

IS equations:

$$\begin{aligned}
 \tau_{\Pi} \dot{\Pi} + \Pi &= \Pi_{\text{NS}} + \tau_{\Pi q} \mathbf{q} \cdot \dot{\mathbf{u}} - \ell_{\Pi q} \partial \cdot \mathbf{q} \\
 &\quad - \tau_{\Pi} \frac{\hat{\zeta}_1}{\zeta} \Pi^2 - \tau_{\Pi} \frac{\hat{\zeta}_2 \beta}{\kappa} \mathbf{q} \cdot \mathbf{q} - \tau_{\Pi} \frac{\hat{\zeta}_3}{2\eta} \pi^{\mu\nu} \pi_{\mu\nu} \\
 \tau_q \Delta^{\mu\nu} \dot{q}_\nu + q^\mu &= q_{\text{NS}}^\mu - \tau_{q\Pi} \Pi \dot{u}^\mu - \tau_{q\pi} \pi^{\mu\nu} \dot{u}_\nu \\
 &\quad + \ell_{q\Pi} \nabla^\mu \Pi - \ell_{q\pi} \Delta^{\mu\nu} \partial^\lambda \pi_{\nu\lambda} + \tau_q \omega^{\mu\nu} q_\nu \\
 &\quad - \tau_q \frac{\hat{\kappa}_1}{\zeta} q^\mu \Pi - \tau_q \frac{\hat{\kappa}_2}{2\eta} \pi^{\mu\nu} q_\nu \\
 \tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} &= \pi_{\text{NS}}^{\mu\nu} + 2\tau_{\pi q} q^{\langle\mu} \dot{u}^{\nu\rangle} + 2\ell_{\pi q} \nabla^{\langle\mu} q^{\nu\rangle} + 2\tau_\pi \pi_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda} \\
 &\quad - 2\tau_\pi \frac{\hat{\eta}_1}{2\eta} \pi_\lambda^{\langle\mu} \pi^{\nu\rangle\lambda} - 2\tau_\pi \frac{\hat{\eta}_2 \beta}{\kappa} q^{\langle\mu} q^{\nu\rangle} - 2\tau_\pi \frac{\hat{\eta}_3}{\zeta} \Pi \pi^{\mu\nu}
 \end{aligned}$$

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

A. Muronga, PRC 76 (2007) 014909 (and parts of $\hat{\zeta}_1$, $\hat{\kappa}_1$, $\hat{\eta}_3$)

B. Betz, D. Henkel, DHR, Prog. Part. Nucl. Phys. 62 (2009) 556

B. Betz, T. Koide, H. Niemi, DHR, in preparation

Results (V)

Remarks:

1. **Structure of second-order terms follows exclusively from Lorentz covariance**
2. **Coefficients can be computed from kinetic theory and Grad's 14-moment method** B. Betz, H. Niemi, T. Koide, DHR, in preparation
3. R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100: second-order fluid dynamics for **conformal** fluids (AdS/CFT correspondence)

⇒ second-order terms (in flat space):

$$4 \lambda_1 \sigma_\lambda^{<\mu} \sigma^{\nu>\lambda} - 2 \lambda_2 \sigma_\lambda^{<\mu} \omega^{\nu>\lambda} + \lambda_3 \omega_\lambda^{<\mu} \omega^{\nu>\lambda}$$

to 2nd order: may use $\pi^{\mu\nu} \simeq \pi_{\text{NS}}^{\mu\nu} = 2 \eta \sigma^{\mu\nu}$ to replace $\sigma^{\mu\nu}$ with $\pi^{\mu\nu}$

⇒ in kinetic theory: $\lambda_1 \equiv -\tau_\pi \eta \hat{\eta}_1$, $\lambda_2 \equiv -2\tau_\pi \eta$, $\lambda_3 \equiv 0$

Note: second-order terms from collision integral ⇒ $\hat{\eta}_1 \neq 1!$

cf. M.A. York, G.D. Moore, arXiv:0811.0729

Results (VI)

4. P. Romatschke, arXiv:0906.4787:

second-order fluid dynamics for **nonconformal** fluids without conserved charges

⇒ second-order terms (in flat space):

$$4\xi_1 \sigma^{\mu\nu} \sigma_{\mu\nu} + \xi_2 \theta^2 + \xi_3 \omega^{\mu\nu} \omega_{\mu\nu} + \xi_4 \nabla^\mu \ln s \nabla_\mu \ln s$$

to 2nd order: may use $\pi^{\mu\nu} \simeq 2\eta \sigma^{\mu\nu}$, $\Pi \simeq -\zeta \theta$

and 0th order acceleration eq. $\dot{u}^\mu \simeq \frac{\nabla^\mu p}{\epsilon + p} = c_s^2 \nabla^\mu \ln s$

⇒ in kinetic theory: $\xi_1 \equiv -\frac{1}{2} \tau_\Pi \eta \hat{\zeta}_3$, $\xi_2 \equiv -\tau_\Pi \zeta \hat{\zeta}_1$, $\xi_3 \equiv \xi_4 \equiv 0$

5. Coefficients $\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3, \hat{\kappa}_1, \hat{\kappa}_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3$ are (complicated) **dimensionless** functions of α, β

6. Viscosities and thermal conductivity ζ, η, κ , relaxation times $\tau_\Pi, \tau_q, \tau_\pi$, coefficients $\tau_{\Pi q}, \tau_{q\Pi}, \tau_{q\pi}, \tau_{\pi q}, \ell_{\Pi q}, \ell_{q\Pi}, \ell_{q\pi}, \ell_{\pi q}$ are (complicated) functions of α, β , divided by **tensor coefficients of second moment of collision integral**: $\sim \chi_i(\alpha, \beta) / \langle \sigma \rangle \rightarrow 0$ as cross section $\sigma \rightarrow \infty$ (“strong coupling limit”!)

⇒ $\Pi = q^\mu = \pi^{\mu\nu} \rightarrow 0$ **ideal fluid limit!**

Results (VII)

7. **IS** equations are **formally** independent of calculational frame (Eckart, Landau,...), **but ...**

8. **Values** of coefficients are **frame dependent!** We have analyzed:

(a) **Eckart (N)** or (net) charge frame:

$$\nu^\mu = 0, \quad \epsilon = \epsilon_0, \quad n = n_0$$

ϵ_0, n_0 : **energy density** and **charge density** in local thermodyn. equilibrium

(b) **Landau (E)** or energy frame:

$$q^\mu = 0, \quad \epsilon = \epsilon_0, \quad n = n_0$$

Note: in **IS** equations $q^\mu \equiv -\frac{\epsilon + p}{n} \nu^\mu$

(c) **Tsumura-Kunihiro-Ohnishi (TKO)** frame:

$$\nu^\mu = 0, \quad \epsilon = \epsilon_0 - 3\Pi, \quad n = n_0$$

We have checked agreement with the results of **IS** for most coefficients computed by **IS**...

9. R.h.s.: all terms except **NS** terms are of second order, $\sim \delta^2$

$\implies t < \tau_\Pi \sim \tau_q \sim \tau_\pi$: dissipative terms relax towards their **NS** values,

$t > \tau_\Pi \sim \tau_q \sim \tau_\pi$: last terms on r.h.s. and **NS** terms on l.h.s. largely cancel, second-order terms govern evolution!

Conclusions and open problems

1. Derived **Israel-Stewart (IS)** equations from **kinetic theory** via **Grad's 14-moment method** \implies **new second-order terms!**
2. **Results consistent with**
R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100
M.A. York, G.D. Moore, arXiv:0811.0729
3. **Coefficients of terms in IS equations are frame dependent**
 \implies **have (not yet completely) been computed in various frames**
(Eckart, Landau, TKO)
4. **Generalization to a system of various particle species**
(done: quarks, antiquarks, gluons), **various conserved charges**
cf. M. Prakash, M. Prakash, R. Venugopalan, G. Welke, Phys. Rept. 227 (1993) 321
G. Denicol, DHR, in preparation
5. **Numerical implementation**
E. Molnar, H. Niemi, DHR, in preparation