# *Viscosities in the hadron gas*

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# Outline

- Quick review of the kinetic theory approach vs the diagramatic method for calculating transport coefficients
- A diagramatic calculation of the shear viscosity in the meson gas in ChPT: unitarity, KSS bound, comparison with other results for the hadron gas, ...
- A diagramatic calculation of the bulk viscosity in the meson gas in ChPT: trace anomaly, sum rules, comparison with other results, ...
- Conclusions

- In presence of viscosities, the energy-momentum tensor is modified. To first order in gradients,

fluid velocity

$$T_{ij} = P\delta_{ij} + \eta \underbrace{\left(\partial_i U_j + \partial_j U_i + \frac{2}{3}\delta_{ij}\nabla\cdot\boldsymbol{U}\right)}_{\text{thermodynamic force}} - \zeta \underbrace{\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{U}}_{\text{thermodynamic force}} .$$

(local rest frame)

pressure

shear viscosity

bulk viscosity

In kinetic theory: $$T^{\mu\nu}(x) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3 p^0}\, p^\mu p^\nu\, f(x,p)$$

out-of-equilibrium distribution function

- The Uehling-Uhlenbeck transport equation: $$p^\mu \partial_\mu f = C[f]$$

Eg., collision term for $1\,2 \to 1'\,2'$ (bosons):

$$C[f_1] = \frac{1}{2(2\pi)^3}\int \frac{\mathrm{d}^3\boldsymbol{p}_2}{p_2^0}\frac{\mathrm{d}^3\boldsymbol{p}_1'}{p_1'^0}\frac{\mathrm{d}^3\boldsymbol{p}_2'}{p_2'^0}\,\left[f_1' f_2'(1+f_1)(1+f_2) - f_1 f_2 (1+f_1')(1+f_2')\right]$$
$$\times\, \delta^{(4)}(p_1+p_2-p_1'-p_2')\; |\langle p_2', p_1'|\hat{T}|p_1,p_2\rangle|^2 \;.$$

- Consider a small deviation from equilibrium: $f(x,p) = f_{\rm eq}(x,p) + \delta f_{\rm out}(x,p)$

$$\delta f_{\rm out}(x,p) \equiv f_{\rm eq}(x,p)[1+f_{\rm eq}(x,p)]\phi(x,p)$$

By linearizing the transport equation with respect to $\phi$:

$$p^\mu \partial_\mu f_{\rm eq}|_{\rm lin} = \beta p^0 [q_\zeta(|\boldsymbol{p}|)\boldsymbol{\nabla}\cdot\boldsymbol{U} + q_\eta(|\boldsymbol{p}|)\hat{p}_i\hat{p}_j \overset{\circ}{\overline{\partial_i U_j}}] f_{\rm eq}(1+f_{\rm eq}) \ , \quad \Longleftarrow \quad f_{\rm eq}(x,p) = \frac{1}{e^{\beta p_\mu U^\mu}-1}$$

$$C[f_1]_{\rm lin} = \frac{1}{2(2\pi)^3} f_{1,\rm eq} \int \frac{d^3\boldsymbol{p}_2}{p_2^0}\frac{d^3\boldsymbol{p}'_1}{{p'}_1^0}\frac{d^3\boldsymbol{p}'_2}{{p'}_2^0}\ f_{2,\rm eq}(1+f'_{1,\rm eq})(1+f'_{2,\rm eq})[\phi'_1+\phi'_2-\phi_1-\phi_2]$$
$$\times\, \delta^{(4)}(p_1+p_2-p'_1-p'_2)\ |\langle p'_2,p'_1|\hat{T}|p_1,p_2\rangle|^2 \equiv f_{1,\rm eq}\,\mathcal{C}[\phi]$$

with $\overset{\circ}{\overline{\partial_i U_j}} \equiv \partial_i U_j + \partial_j U_i + \frac{2}{3}\delta_{ij}\nabla\cdot\boldsymbol{U}$ .

Then $\phi$ must be of the form: $\phi = A(|\boldsymbol{p}|)\underbrace{\boldsymbol{\nabla}\cdot\boldsymbol{U}} + B(|\boldsymbol{p}|)\hat{p}_i\hat{p}_j\underbrace{\overset{\circ}{\overline{\partial_i U_j}}}$

thermodynamic force associated to the bulk viscosity

thermodynamic force associated to the shear viscosity

$$\delta T^{\mu\nu}(x) \equiv \int \frac{d^3\boldsymbol{p}}{p^0}\ p^\mu p^\nu\ f_{\rm eq}(x,p)[1+f_{\rm eq}(x,p)]\phi(x,p) \quad \Longrightarrow$$

expressions for the shear and bulk viscosities

# The kinetic theory approach to calculate transport coefficients 

Then we can write the transport equation for each type of deviation from equilibrium symbolically as: *Arnold, Moore & Jaffe*, JHEP 11, 001 (2000) *Arnold, Dogan & Moore*, PRD 74, 085021 (2006)

some set of indices

$$\mathcal{S}^a = \mathcal{C}[\chi^a] \ , \qquad \text{and} \qquad \eta, \zeta \propto \langle \mathcal{S}^a | \chi^a \rangle \ ,$$

source

corresponding deviation from equilibrium

where
$$\mathcal{S}_\eta^{ij} \equiv -T q_\eta(|\boldsymbol{p}|) \overset{\circ}{\overline{\hat{p}^i \hat{p}^j}} f_{\mathrm{eq}}(1 + f_{\mathrm{eq}}) \ , \quad \chi_\eta^{ij} \equiv \overset{\circ}{\overline{\hat{p}^i \hat{p}^j}} B(|\boldsymbol{p}|)$$

$$\mathcal{S}_\zeta \equiv -T q_\zeta(|\boldsymbol{p}|) f_{\mathrm{eq}}(1 + f_{\mathrm{eq}}) \ , \quad \chi_\zeta \equiv A(|\boldsymbol{p}|)$$

$$\langle f | g \rangle \equiv \beta^3 \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \ f(p) g(p)$$

Finally,
$$\eta = \frac{2}{15} \langle \mathcal{S}_\eta | \hat{\mathcal{C}}^{-1} | \mathcal{S}_\eta \rangle \ , \quad \zeta = \langle \mathcal{S}_\zeta | \hat{\mathcal{C}}^{-1} | \mathcal{S}_\zeta \rangle \ .$$

# Diagramatic method for calculating transport coefficients

- Linear Response Theory (LRT): consider a small perturbation, $H(t) = H_0 + V(t)$

time-independent hamiltonian

time-dependent perturbation

$\hat{H}(t)$ eigenstate in the Schrödinger image

$$\Rightarrow \qquad \delta\langle\hat{O}(t)\rangle \equiv \langle\Psi(t)|\hat{O}(t)|\Psi(t)\rangle - \langle\Psi^{(0)}(t)|\hat{O}(t)|\Psi^{(0)}(t)\rangle$$

$$= -\mathrm{i}\int_0^\infty \mathrm{d}t' \ \langle\Psi^{(0)}(0)|\theta(t-t')[\hat{O}_{\mathrm{H}}(t), \hat{V}_{\mathrm{H}}(t')]|\Psi^{(0)}(0)\rangle \ .$$

operators in the Heisenberg image

$\hat{H}_0$ eigenstate in the Schrödinger image

- Applied to viscosities:

$$\eta = \frac{1}{20}\lim_{q^0\to 0^+}\lim_{|\boldsymbol{q}|\to 0^+}\frac{\partial\rho_\eta(q^0,\boldsymbol{q})}{\partial q^0} \ , \quad \zeta = \frac{1}{2}\lim_{q^0\to 0^+}\lim_{|\boldsymbol{q}|\to 0^+}\frac{\partial\rho_\zeta(q^0,\boldsymbol{q})}{\partial q^0} \ ,$$

with

$$\rho_\eta(q^0,\boldsymbol{q}) = 2\,\mathrm{Im}\,\mathrm{i}\int \mathrm{d}^4x \ \mathrm{e}^{\mathrm{i}q\cdot x}\theta(t)\langle[\hat{\pi}_{ij}(x), \hat{\pi}^{ij}(0)]\rangle \ , \quad \rho_\zeta(q^0,\boldsymbol{q}) = 2\,\mathrm{Im}\,\mathrm{i}\int \mathrm{d}^4x \ \mathrm{e}^{\mathrm{i}q\cdot x}\theta(t)\langle[\hat{\mathcal{P}}(x), \hat{\mathcal{P}}(0)]\rangle \ .$$

where $\quad \pi_{ij} \equiv T_{ij} - g_{ij}T^l_{\ l}/3 \ , \quad \mathcal{P} \equiv \overbrace{-T^l_{\ l}/3}^{\text{pressure}} - v_s^2 T_{00} - \mu N^0 \ .$

pressure

energy density

some conserved charge

$g_{\mu\nu} = \mathrm{diag}\,(+,-,-,-)$

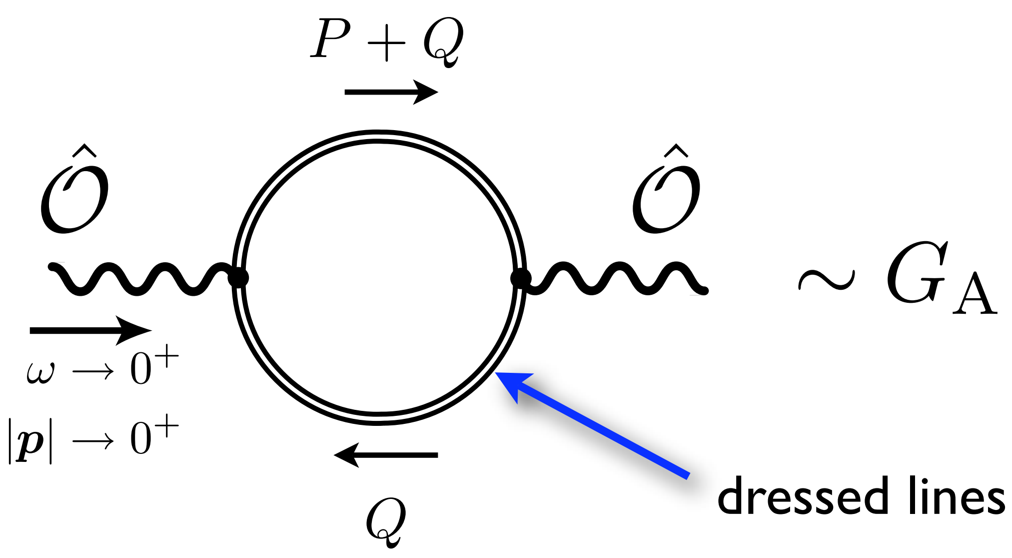
speed of sound in the fluid

# Diagramatic method for calculating transport coefficients



- Consider for instance $\lambda\phi^4$: to one-loop order,

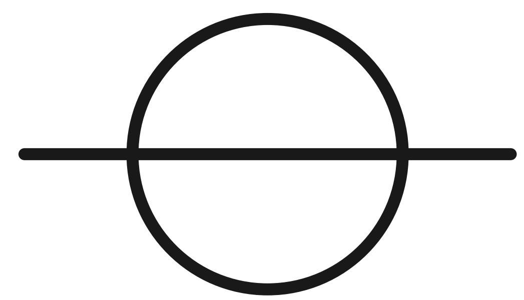
*Jeon*, PRD 52, 3591 (1995)



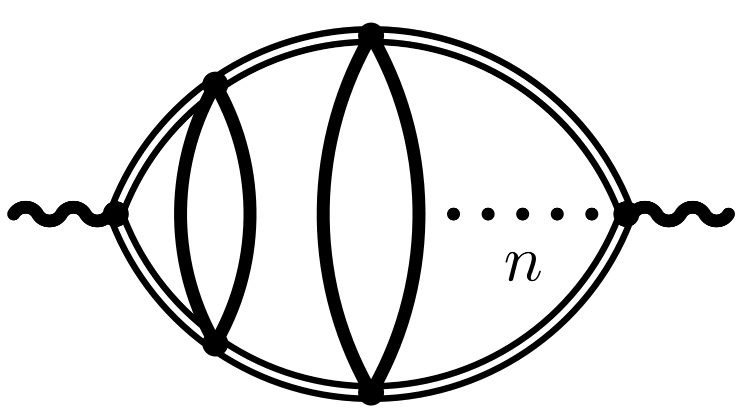

$$\sim G_{\mathrm{A}}(q^0,\boldsymbol{q})G_{\mathrm{R}}(q^0,\boldsymbol{q}) \simeq \frac{\pi}{4E_{\boldsymbol{q}}^2\Gamma_{\boldsymbol{q}}}\left[\delta(q^0-E_{\boldsymbol{q}})+\delta(q^0+E_{\boldsymbol{q}})\right]$$

if $\Gamma_{\boldsymbol{q}}$ is small

particle width

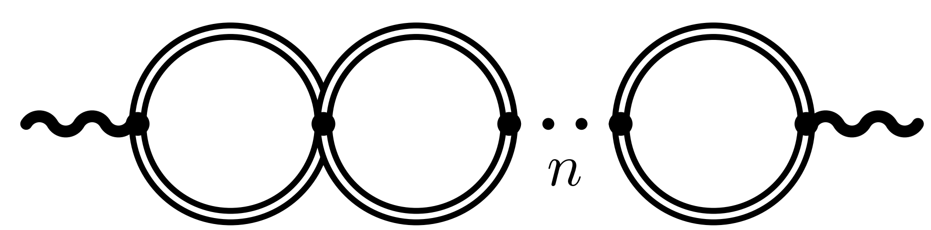
★ Particle width: $\Gamma \sim \mathrm{Im}$  $\sim \lambda^2$

- Therefore, in $\lambda\phi^4$ a resummation is necessary:




$= \mathcal{O}(1/\lambda^2)$

*(ladder diagram)*




$= \mathcal{O}(1/\lambda^{n+1})$

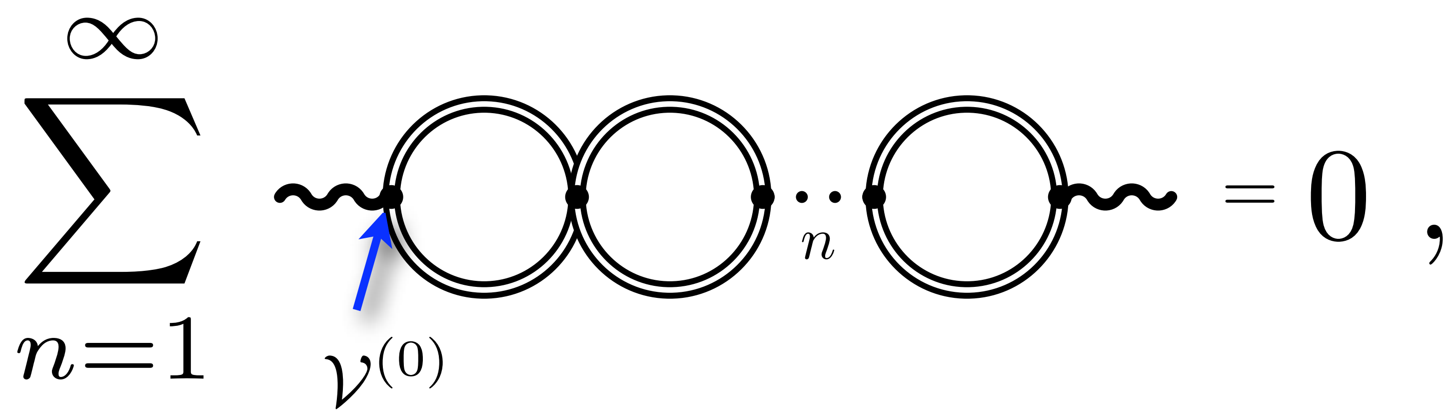
*(bubble diagram)*

# Shear viscosity in $\lambda\phi^4$

*Jeon*, PRD 52, 3591 (1995)

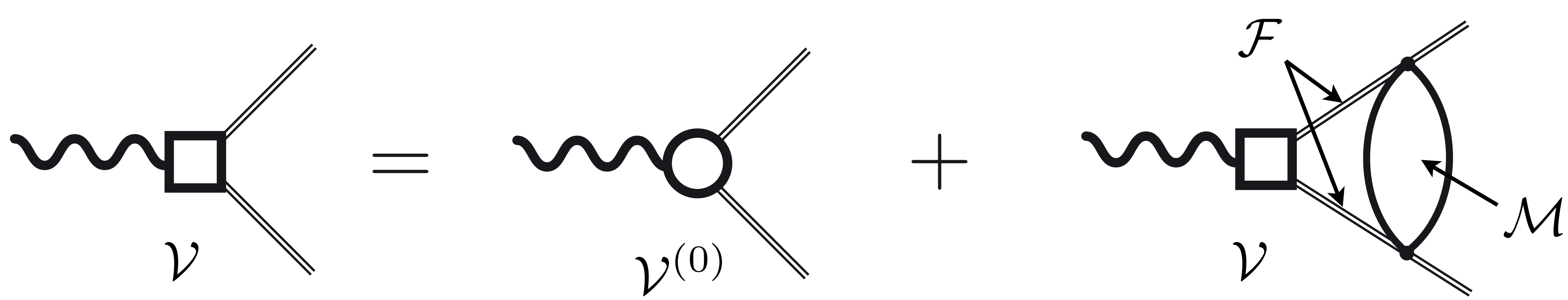
*Jeon & Yaffe*, PRD 53, 5799 (1996)

- Bubble diagrams can be easily resummated:




because of rotational invariance ($\mathcal{V}^{(0)}_{ij} = \partial_i\phi\partial_j\phi + \frac{1}{3}\delta_{ij}\partial_k\phi\partial^k\phi$).

- The resummation of ladder diagrams instead implies to solve an integral equation:




$\Rightarrow$ $|\mathcal{V}\rangle = |\mathcal{V}^{(0)}\rangle + \hat{\mathcal{K}}|\mathcal{V}\rangle$ , $\quad \mathcal{K} \equiv \mathcal{M}\mathcal{F}$ .

$$\eta = \frac{\beta}{10}\lim_{\omega\to 0^+}\lim_{|\boldsymbol{p}|\to 0^+}\langle\mathcal{V}^{(0)}|\hat{\mathcal{F}}|\mathcal{V}\rangle[1+\mathcal{O}(\lambda)] \ .$$

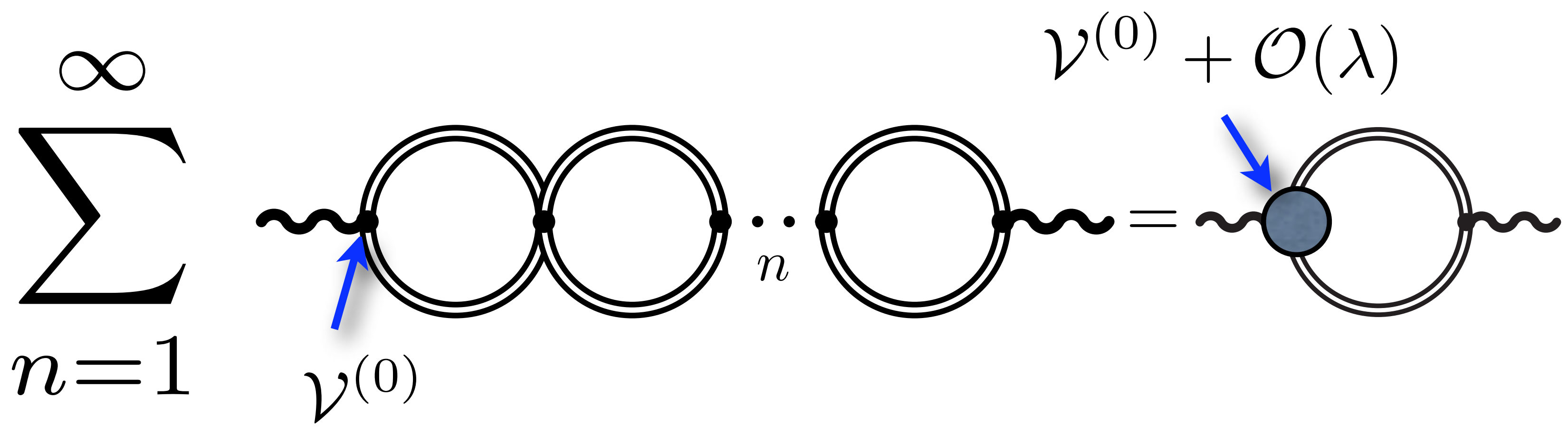
# Bulk viscosity in $\lambda\phi^4$



*Jeon*, PRD 52, 3591 (1995)

*Jeon & Yaffe*, PRD 53, 5799 (1996)

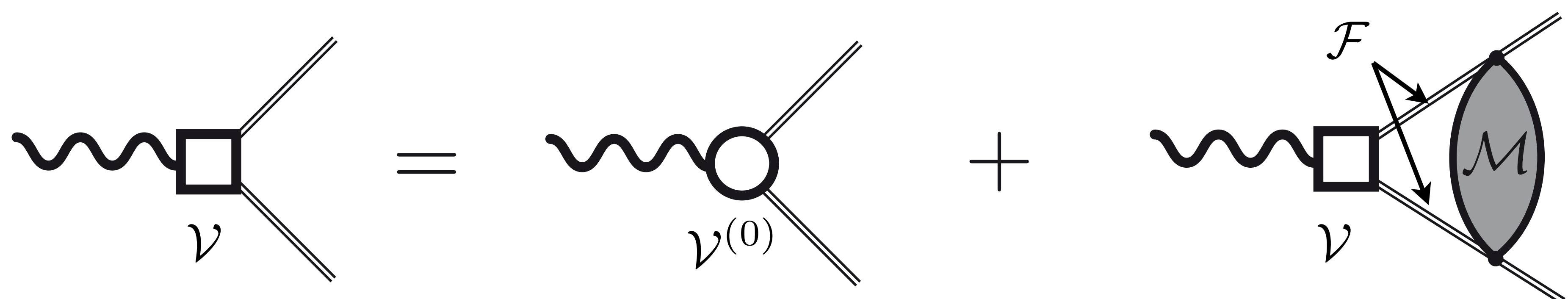
- For $\zeta$, bubble diagrams cannot be neglected:

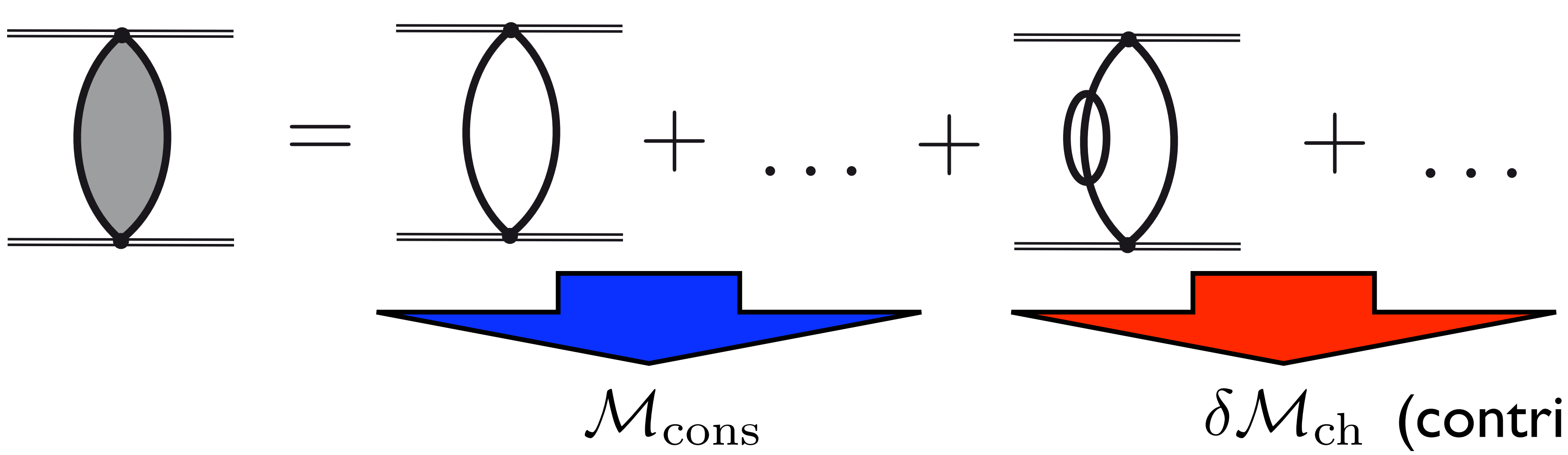



, and $\mathcal{V}^{(0)} \sim \mathcal{O}(\lambda)$

★ Because the real part of a bubble does not contain pinching poles.

- In this case, the resummation of ladder diagrams involves more complicated rungs:




(contribution from number-changing processes)

$$\zeta = \beta \lim_{\omega\to 0^+} \lim_{|\boldsymbol{p}|\to 0^+} \langle \mathcal{V}^{(0)} | \hat{\mathcal{F}} | \mathcal{V} \rangle [1 + \mathcal{O}(\lambda)] \ .$$

# Chiral Perturbation Theory (ChPT)

- We're interested in the (non-perturbative) low-energy regime of QCD, i.e. $E \lesssim 1\ \mathrm{GeV}$ and $T \lesssim 200\ \mathrm{MeV}$. There, chiral symmetry is spontaneously broken:

$$\chi \equiv \mathrm{SU}(3)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{R}} \equiv \mathrm{SU}(3)_{\mathrm{V}} \times \mathrm{SU}(3)_{\mathrm{A}} \longrightarrow \mathrm{SU}(3)_{\mathrm{V}}\ .$$

- In that regime, the degrees of freedom are the corresponding Goldstone bosons: pions, kaons and etas.

- Chiral symmetry acts non-linearly on the Goldstone bosons: $U(x) \stackrel{\chi}{\mapsto} RU(x)L^{\dagger}$

$\in \mathrm{SU}(3)_{\mathrm{R}}$ $\in \mathrm{SU}(3)_{\mathrm{L}}$

Gell-Mann matrices

$$\text{with} \quad U(x) \equiv \exp\left(\mathrm{i}\frac{\phi(x)}{F_0}\right)\ , \quad \text{and} \quad \phi(x) = \sum_{a=1}^{8} \lambda_a \phi_a(x)$$

Goldstone bosons

$$\Rightarrow \quad [Q_a^{\mathrm{V}}, \phi_b] = \mathrm{i} f_{abc}\phi_c\ , \quad [Q_a^{\mathrm{A}}, \phi_b] = g_{ab}(\phi)\ .$$

a non-linear function

- **ChPT lagrangian:** The most general expansion in terms of derivatives of the field $U(x)$ and masses that fulfills all the symmetries of QCD:

$$\mathcal{L}_{\mathrm{ChPT}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots \qquad \text{(infinite \# of terms)}$$

# ChPT: lagrangians

- Leading order:

$$\mathcal{L}_2 = \frac{F_0^2}{4}\,\mathrm{Tr}\{(\nabla_\mu U)(\nabla^\mu U)^\dagger\} + \frac{F_0^2}{4}\,\mathrm{Tr}\{\chi U^\dagger + U\chi^\dagger\}\ .$$

- Next-to-leading order:

$$\begin{aligned}
\mathcal{L}_4 = {} & L_1\left(\mathrm{Tr}\{(\nabla_\mu U)(\nabla^\mu U)^\dagger\}\right)^2 + L_2\,\mathrm{Tr}\{(\nabla_\mu U)(\nabla_\nu U)^\dagger\}\,\mathrm{Tr}\{(\nabla^\mu U)(\nabla^\nu U)^\dagger\} \\
& + L_3\,\mathrm{Tr}\{(\nabla_\mu U)(\nabla^\mu U)^\dagger(\nabla_\nu U)(\nabla^\nu U)^\dagger\} + L_4\,\mathrm{Tr}\{(\nabla_\mu U)(\nabla^\mu U)^\dagger\}\,\mathrm{Tr}\{\chi U^\dagger + U\chi^\dagger\} \\
& + L_5\,\mathrm{Tr}\{(\nabla_\mu U)(\nabla^\mu U)^\dagger(\chi U^\dagger + U\chi^\dagger)\} + L_6\left(\mathrm{Tr}\{\chi U^\dagger + U\chi^\dagger\}\right)^2 \\
& + L_7\left(\mathrm{Tr}\{\chi U^\dagger - U\chi^\dagger\}\right)^2 + L_8\,\mathrm{Tr}\{U\chi^\dagger U\chi^\dagger + \chi U^\dagger \chi U^\dagger\} \\
& - \mathrm{i}L_9\,\mathrm{Tr}\{f^{\mathrm{R}}_{\mu\nu}(\nabla^\mu U)(\nabla^\nu U)^\dagger + f^{\mathrm{L}}_{\mu\nu}(\nabla^\mu U)^\dagger(\nabla^\nu U)\} + L_{10}\,\mathrm{Tr}\{U f^{\mathrm{L}}_{\mu\nu} U^\dagger f^{\mu\nu}_{\mathrm{R}}\} \\
& + H_1\,\mathrm{Tr}\{f^{\mathrm{R}}_{\mu\nu} f^{\mu\nu}_{\mathrm{R}} + f^{\mathrm{L}}_{\mu\nu} f^{\mu\nu}_{\mathrm{L}}\} + H_2\,\mathrm{Tr}\{\chi\chi^\dagger\}\ .
\end{aligned}$$

The constants $F_0, B_0, L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9, L_{10}, H_1, H_2$ are energy- and temperture-independent, and are determined experimentally.

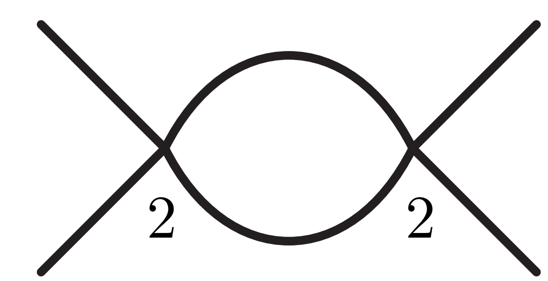
- Dimension, $D$, of a Feynman diagram:

$$\text{Re-scaling:} \begin{cases} p_i \mapsto t p_i \\ m_\mathrm{q} \mapsto t^2 m_\mathrm{q} \end{cases} \quad \Rightarrow \quad \mathcal{M}(tp_i, t^2 m_\mathrm{q}) = t^D \mathcal{M}(p_i, m_\mathrm{q}) \ .$$

amplitude of the diagram

- Weinberg's theorem:

$$\boxed{D = 2 + \sum_n N_n (n-2) + 2L \ .}$$

Eg., 

$= \mathcal{O}(p^4)$

number of loops

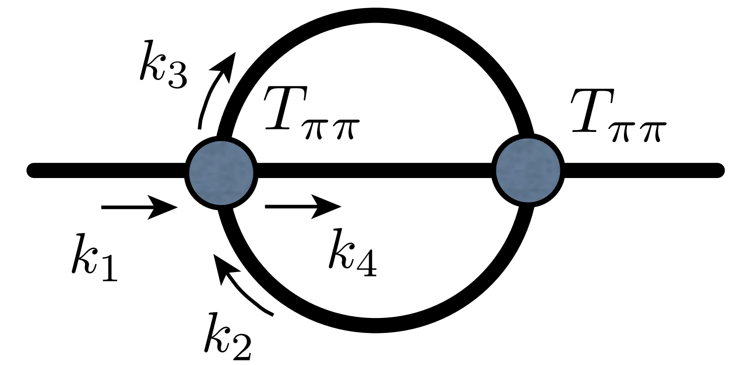
$p = E, |\boldsymbol{p}|, T, M$

number of vertices from $\mathcal{L}_n$

Perturbation theory with respect to the scales: $\Lambda_\chi \sim 1\ \mathrm{GeV}$ (for momenta), $\Lambda_T \sim 200\ \mathrm{MeV}$ (for temperatures).

# Thermal width in ChPT

- Pion thermal width: $\Gamma \sim \mathrm{Im}$




Dilute gas approximation: $$\Gamma(k_1) = \frac{1}{2}\int \frac{\mathrm{d}^3\boldsymbol{k}_2}{(2\pi)^3}\ \mathrm{e}^{-\beta E_2}\sigma_{\pi\pi} v_{\mathrm{rel}}(1-\boldsymbol{v}_1\cdot\boldsymbol{v}_2)$$

Scattering cross section: $$\sigma_{\pi\pi}(s) \simeq \frac{32\pi}{3s}\left[|t_{00}(s)|^2 + 9|t_{11}(s)|^2 + 5|t_{20}(s)|^2\right] \ .$$

here we can introduce the effect of resonances and medium evolution thereof

- ChPT violates the unitarity condition for high $p$: $\mathrm{S}^\dagger \mathrm{S} = 1 \Rightarrow \mathrm{Im}\, t_{IJ}(s) = \sigma(s)|t_{IJ}(s)|^2$, with $\sigma(s) \equiv \sqrt{1-4M_\pi^2/s}$.

  Because partial waves are essentially polynomials in $p$: $t_{IJ}(s) = t_{IJ}^{(1)}(s) + t_{IJ}^{(2)}(s) + \mathcal{O}(s^3)$ .

- The Inverse Amplitude Method (IAM): *Gomez Nicola & Pelaez,* PRD 65, 054009 (2002).

$$t_{IJ}(s) \simeq \frac{t_{IJ}^{(1)}(s)}{1 - t_{IJ}^{(2)}(s;T)/t_{IJ}^{(1)}(s)} \ .$$

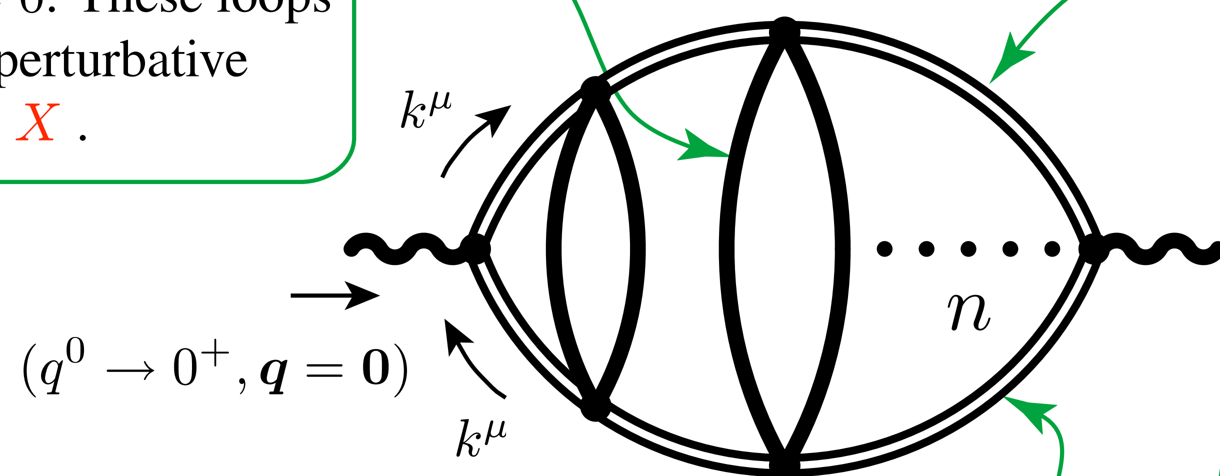
It verifies the unitarity condition exactly and reproduces resonant states.

# Diagramatic analysis in ChPT

*Gomez Nicola & DFF,* PRD 73, 045025 (2006).

## Ladder diagrams:

lines with $\Gamma = 0$. These loops (rungs) give a perturbative contribution $\sim X$ .




$$= \begin{cases} \mathcal{O}(X^n Y)\ , & \text{for } T \ll M_\pi \\ \mathcal{O}(X^n Y^{n+1})\ , & \text{for } T \simeq M_\pi \end{cases}$$

each pair of lines with $\Gamma \neq 0$ and equal momentum give a pinching pole contribution $\sim Y$

If $T \gtrsim M_\pi$, $X \sim 1$, derivative vertices start to dominate $\Rightarrow$ a large number of diagrams become important.

If we only consider *constant* vertices:

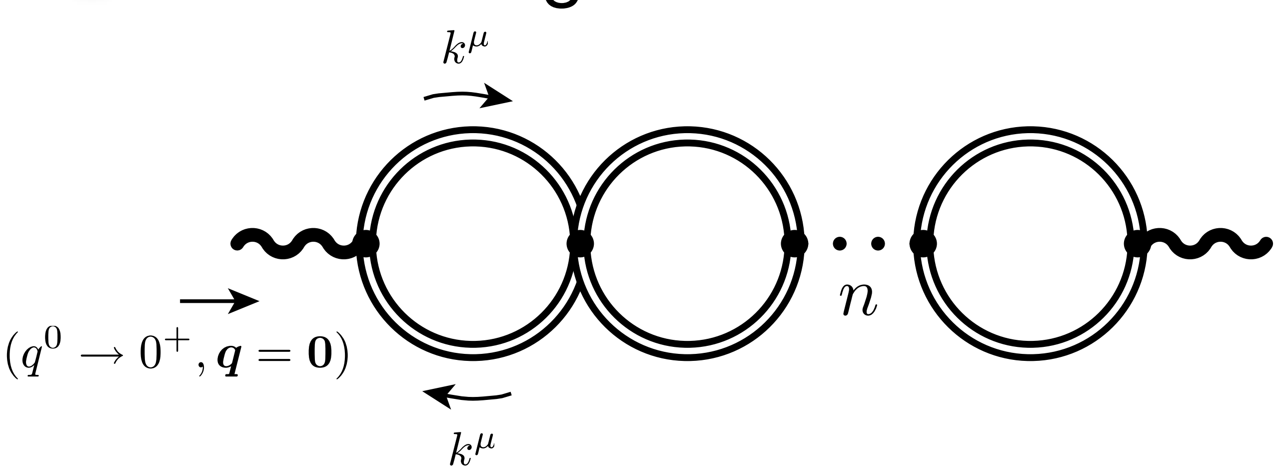
$$T \ll M_\pi\ , \quad Y \sim \sqrt{\frac{M_\pi}{T}}\ , \quad X \sim \frac{1}{Y}\left(\frac{M_\pi}{4\pi F_\pi}\right)^2 .$$

$$T \simeq M_\pi\ , \quad Y \sim 1\ , \quad X \lesssim \left(\frac{M_\pi}{4\pi F_\pi}\right)^2 .$$

pion decay constant

Weinberg's theorem does not provide the (non-perturbative) right order for TC at low $T$: $\mathcal{O}(p^{2n}) \gg \mathcal{O}(p^{4n})$.

## Bubble diagrams:




$$= \mathcal{O}(B^{(0)} X^{n-1})$$

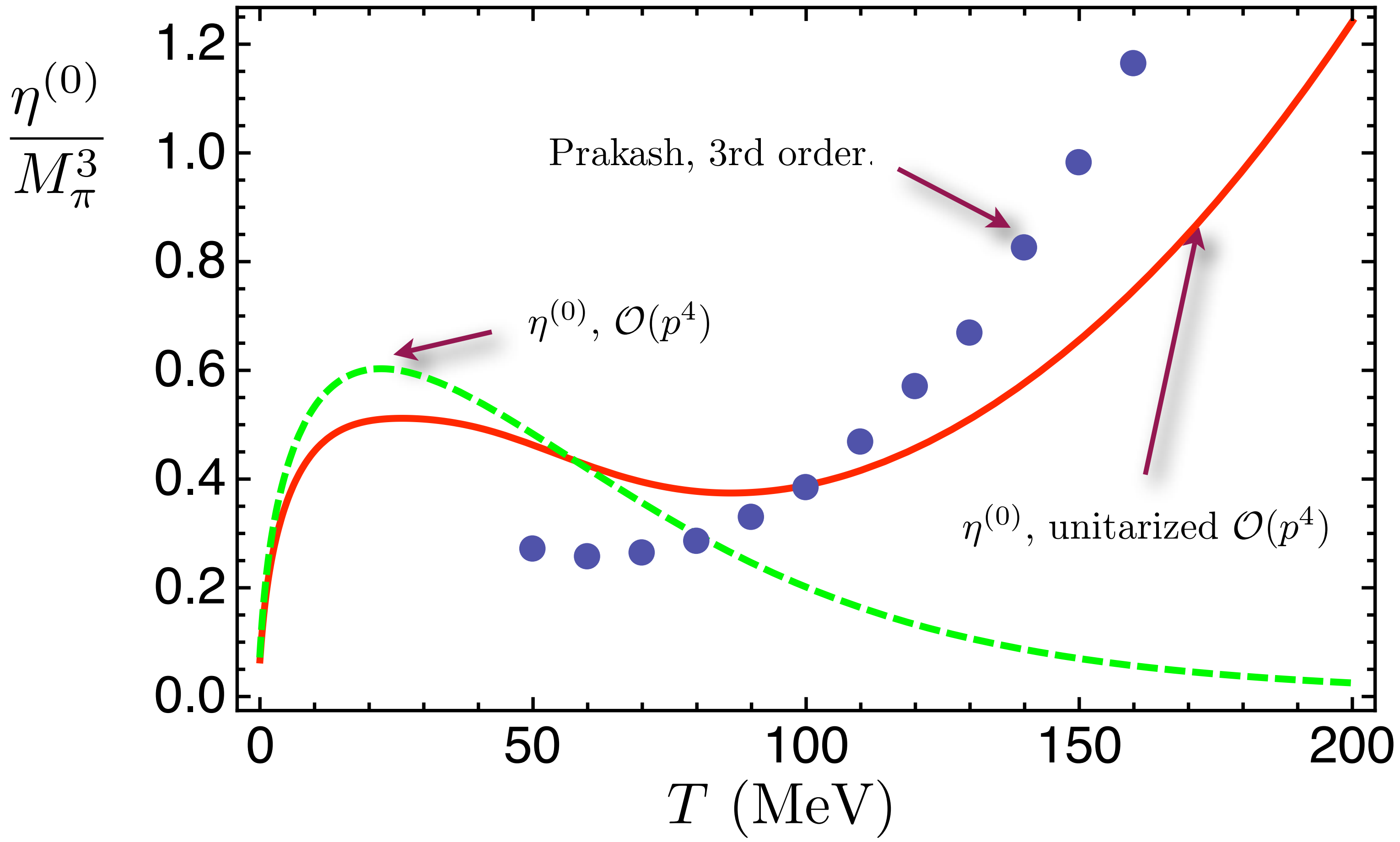
contribution from a simple bubble $\sim Y$

This simple counting allows us to quickly obtain the functional form of TC at low $T$.

# Shear viscosity of a pion gas

- Results:

*Gomez Nicola & DFF,* EPJC 62, 37 (2009).

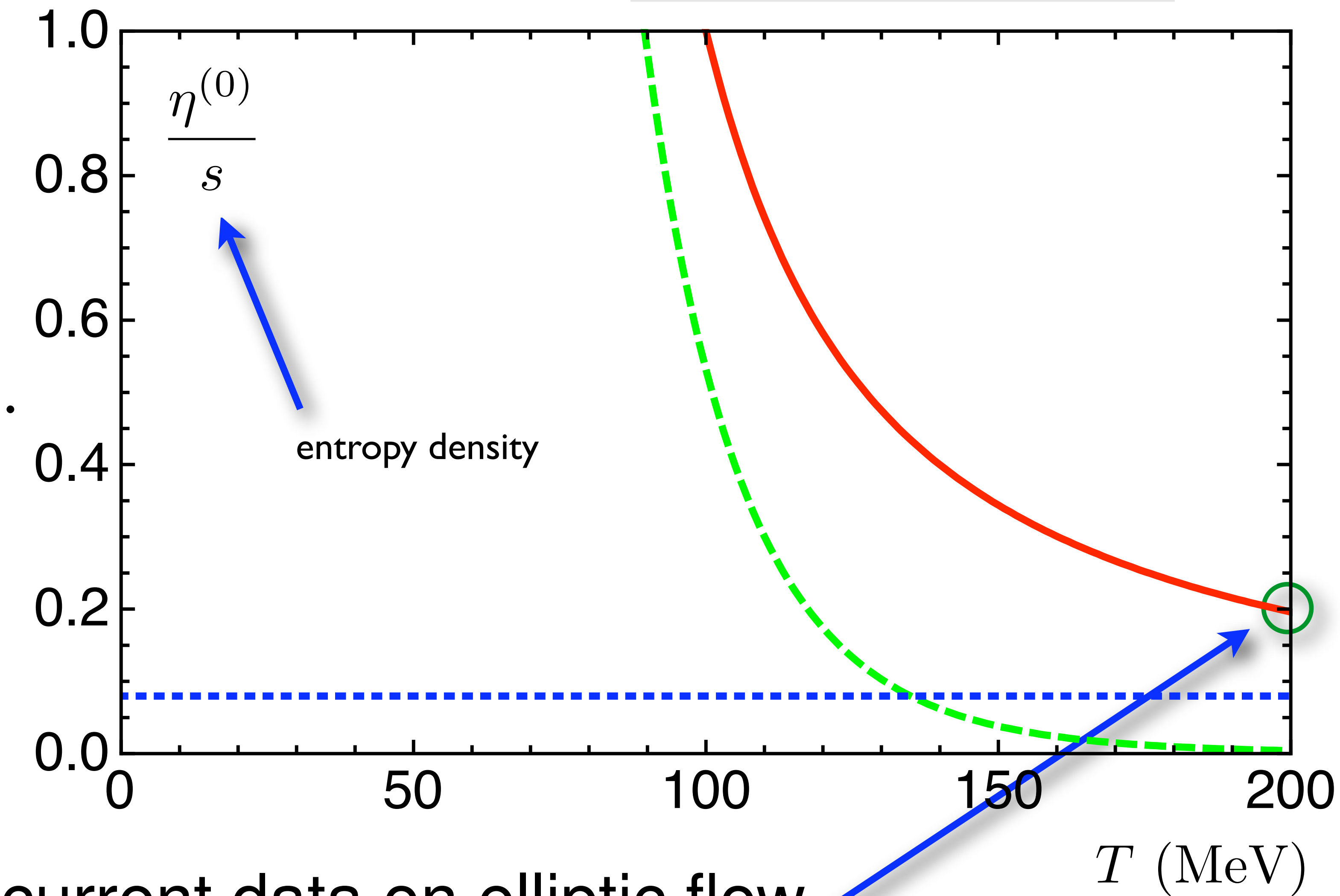



By KT: $\eta \sim M_\pi v n l$, but $l \sim \dfrac{1}{\sigma_{\pi\pi} n}$.

Then for $T \ll M_\pi$, $\eta, \zeta \sim \sqrt{T}$. ✔

- AdS/CFT bound:

*Kovtun, Son & Starinets,* PRL 94, 111601 (2005)




- Sound attenuation length: $\omega = v_\mathrm{s} k + \dfrac{1}{2}\mathrm{i}\varGamma_\mathrm{s} k^2$ .

without $\zeta$,

$$\varGamma_\mathrm{s} \simeq \frac{4\eta}{3sT} \,, \quad \longrightarrow \quad \varGamma_\mathrm{s}(T = 180 \text{ MeV}) \simeq 1.1 \text{ fm}.$$

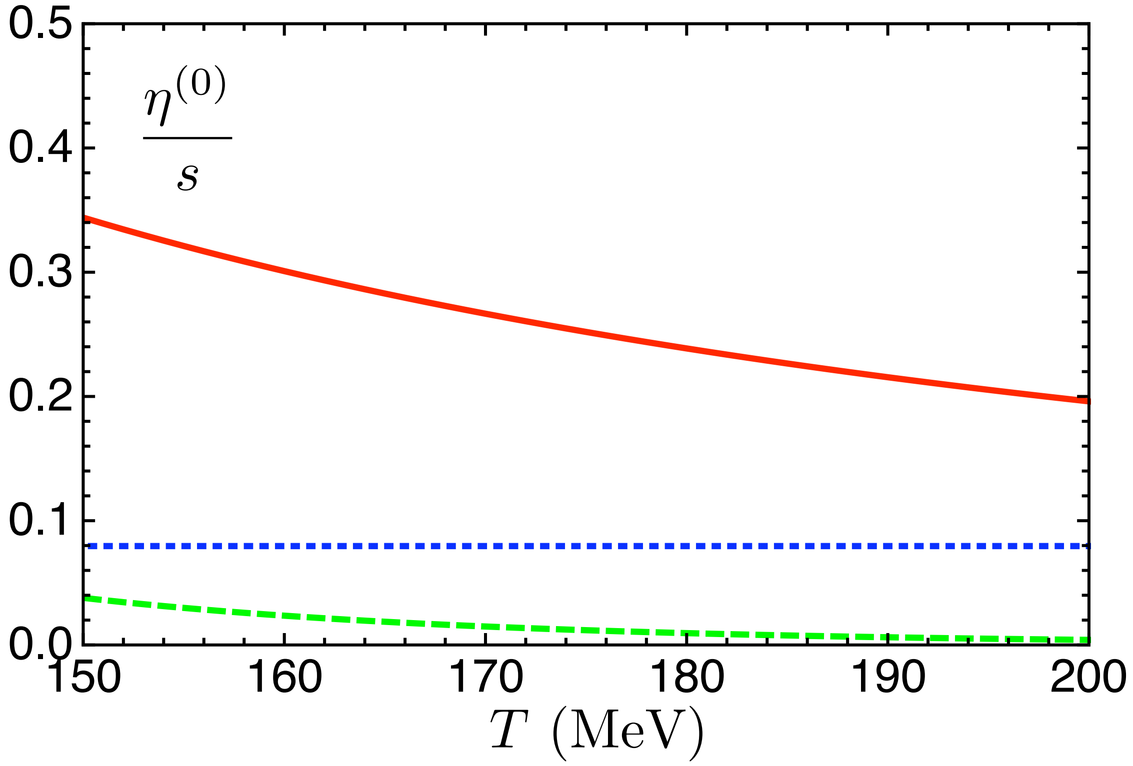
*Teaney,* PRC 68, 034913 (2003)

- A value $\eta/s < 0.24$ is necessary to explain the current data on elliptic flow.

*Romatschke & Romatschke,* PRL 99, 172301 (2007)

# The value of $\eta/s$ near the phase transition

- A minimum near $T_\mathrm{c}$ for a pion gas:




By KT: $\eta \sim mvnl \sim \epsilon\tau$, and $s \sim n$.

$$\Rightarrow \frac{\eta}{s} \sim E\,\tau \gtrsim 1 \text{ (uncertainty principle)}$$

$$\tau \sim \frac{1}{\Gamma} \Rightarrow \frac{\eta}{s} \text{ increases at high } T$$

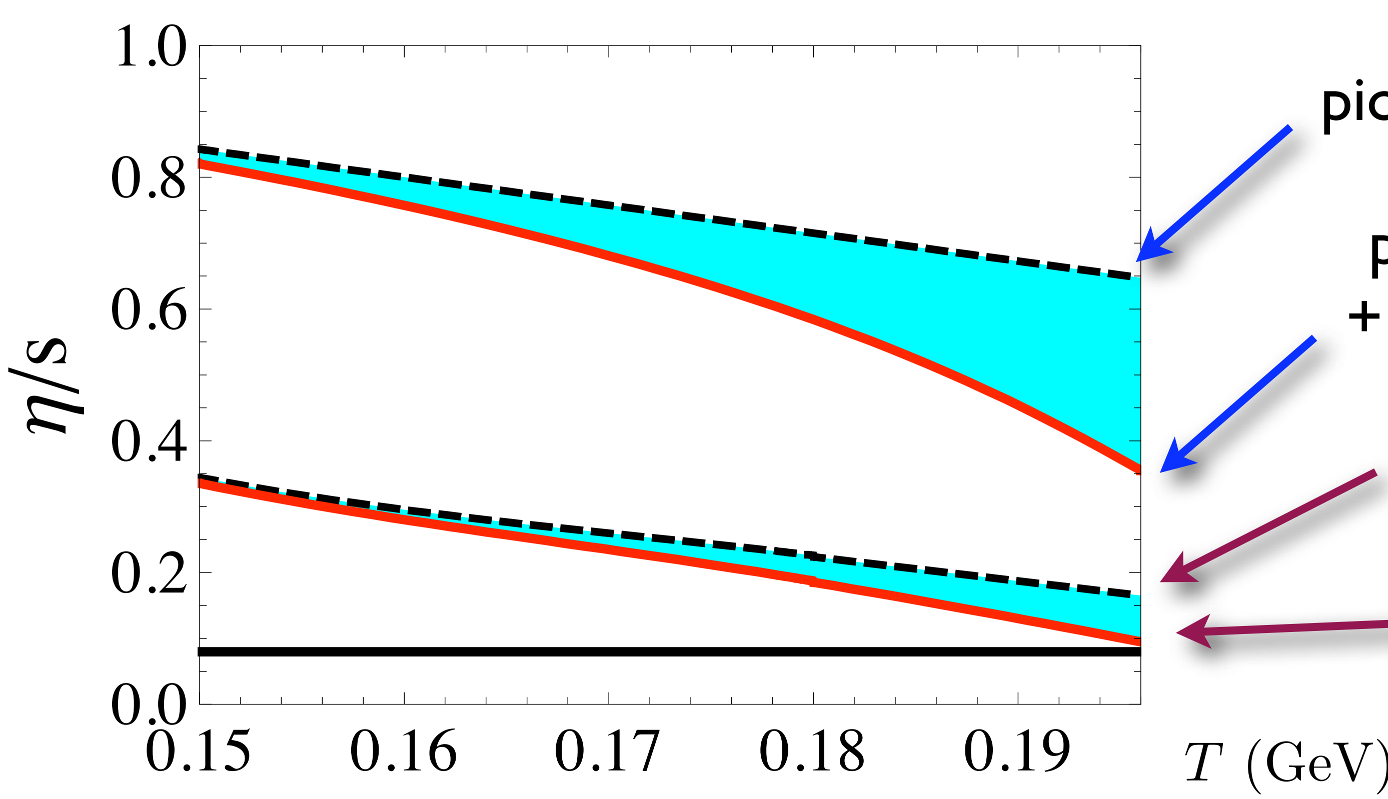
*Dobado & Llanes-Estrada,* EPJ C49, 1011 (2007)

Large $N_\mathrm{c}$: $$\zeta/s = \begin{cases} \mathcal{O}(N_\mathrm{c}^2)\ , & T \ll M_\pi \\ \mathcal{O}(1)\ , & T \to \infty \end{cases}$$

*Arnold, Moore, & Yaffe,* JHEP 0011, 001 (2000)

- Full hadron resonance gas:

*Noronha-Hostler, Noronha, & Greiner,* arXiv: 0811.1571




$$\eta \sim \underbrace{\alpha}_{\mathcal{O}(1)} \sum_i n_i \langle p\rangle_i \lambda_i$$

# Trace anomaly, sum rules, and bulk viscosity



- QCD trace anomaly: $\partial_\nu J^\nu_{\mathrm{dil}} = T^\mu{}_\mu = \dfrac{\beta(g)}{2g} G^a_{\mu\nu} G^{\mu\nu}_a + (1+\gamma(g))\bar{q}Mq$

  related to bulk viscosity: $\zeta = \dfrac{1}{9} \lim_{\omega\to 0^+} \dfrac{1}{\omega} \int_0^\infty \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, \mathrm{e}^{\mathrm{i}\omega t} \langle [\hat{T}^\mu{}_\mu(x), \hat{T}^\nu{}_\nu(0)] \rangle = \dfrac{\pi}{9} \lim_{\omega\to 0^+} \dfrac{\rho_{\theta\theta}(\omega,\mathbf{0})}{\omega}$

- Sum rule: $\displaystyle\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{\rho_{\theta\theta}(\omega,0)}{\omega} = -\left(4 - T\frac{\partial}{\partial T}\right)\langle\theta\rangle_T = T^5 \frac{\partial}{\partial T} \frac{(\epsilon - 3P)^*}{T^4} + 16|\epsilon_{\mathrm{v}}|$

  $\langle\cdot\rangle^* \equiv \langle\cdot\rangle_T - \langle\cdot\rangle_0$

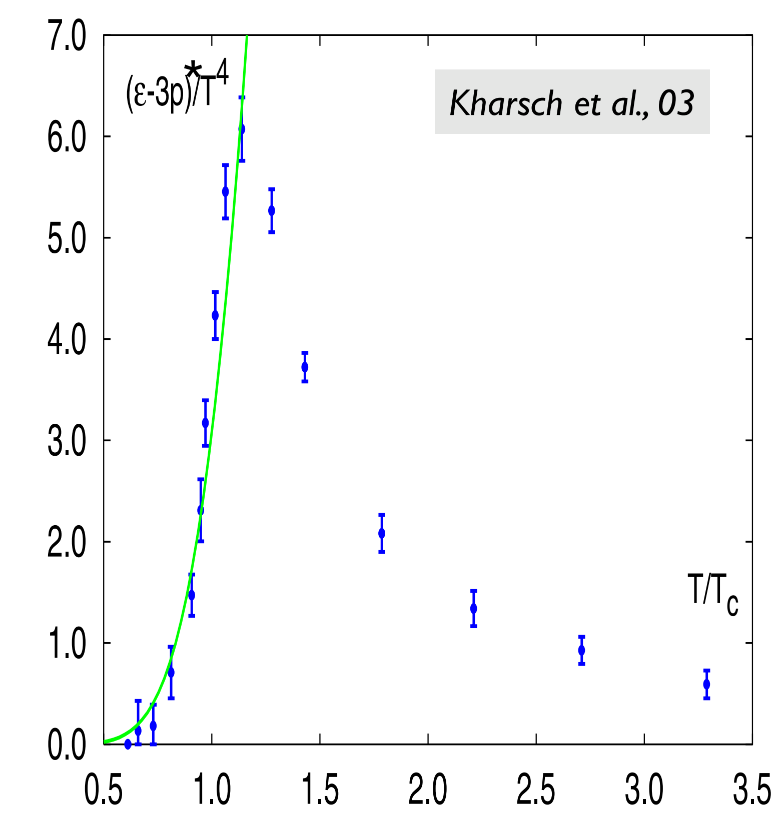
  *Kharzeev & Tuchin*, JHEP 0809:093,2008    *Kharsch, Kharzeev & Tuchin*, PLB 663, 217 (2008)

  $\langle\theta\rangle_T \equiv \langle T^\mu_\mu\rangle_T = \epsilon - 3P$

- Ansatz: $\dfrac{\rho_{\theta\theta}(\omega,\mathbf{0})}{\omega} = \dfrac{9\zeta}{\pi} \dfrac{\omega_0^2}{\omega_0^2 + \omega^2}\,, \quad \omega_0 \sim 1\ \mathrm{GeV}$

vacuum energy density

$$\Rightarrow \quad \zeta(T) = \frac{1}{9\omega_0(T)} \left[ T^5 \frac{\partial}{\partial T} \frac{\langle\theta\rangle_T - \langle\theta\rangle_0}{T^4} + 16|\epsilon_{\mathrm{v}}| \right] .$$




An increase of $\zeta$ near $T_c$?

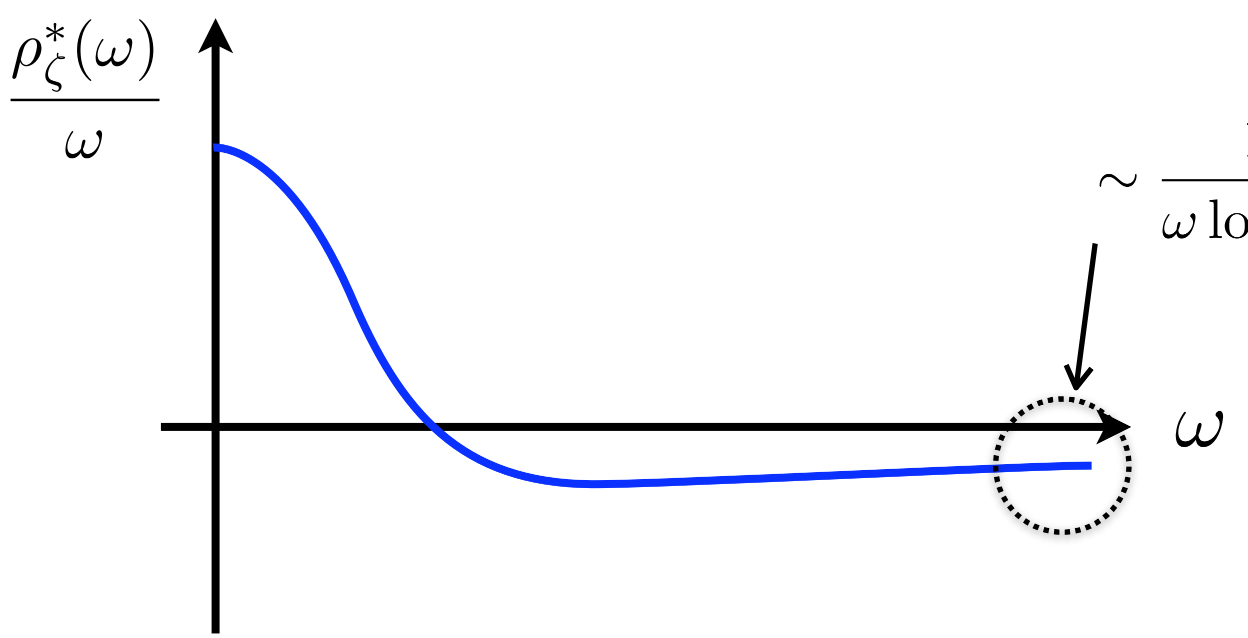
# Trace anomaly, sum rules, and bulk viscosity

- There is a recent modification of the sum rule (corresponding to exchanging the external frequency and momentum limits): *Romatschke & Son*, arXiv:0903.3946

$$3(\epsilon + P)(1 - 3c_\mathrm{s}^2) - 4(\epsilon - 3P) = \frac{2}{\pi}\int \frac{\mathrm{d}\omega}{\omega}\left[\rho_\zeta(\omega) - \rho_\zeta^{T=0}(\omega)\right] .$$

- An ansatz for $\rho_\zeta(\omega) - \rho_\zeta^{T=0}(\omega)$ near $\omega = 0$ might miss important information from the high-$\omega$ region:

*Caron-Huot*, PRD 79, 125009 (2009)




- Even in $\lambda\phi^4$, the correlation between $\zeta$ and $T^\mu_\mu$ is not direct:

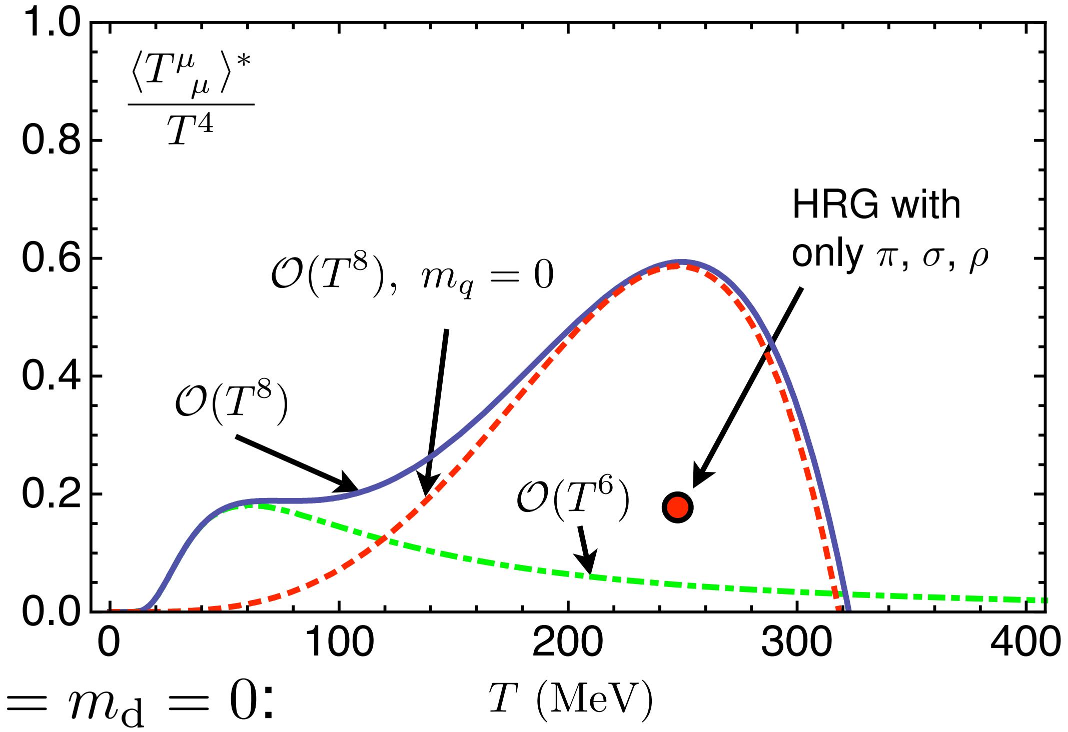
$$T \ll m : \begin{cases} T^\mu_\mu \sim T^{3/2} m^{5/2} \mathrm{e}^{-m/T} \\ \zeta \sim \mathrm{e}^{2m/T} m^6/\lambda^4 T^3 \end{cases} , \quad m \equiv 0 : \begin{cases} T^\mu_\mu \sim \beta(\lambda) T^4 \\ \zeta \sim \lambda T^3 \log^2 \lambda \end{cases}$$

*Jeon & Yaffe*, PRD 53, 5799 (1996)

# Trace anomaly of a gas of pions

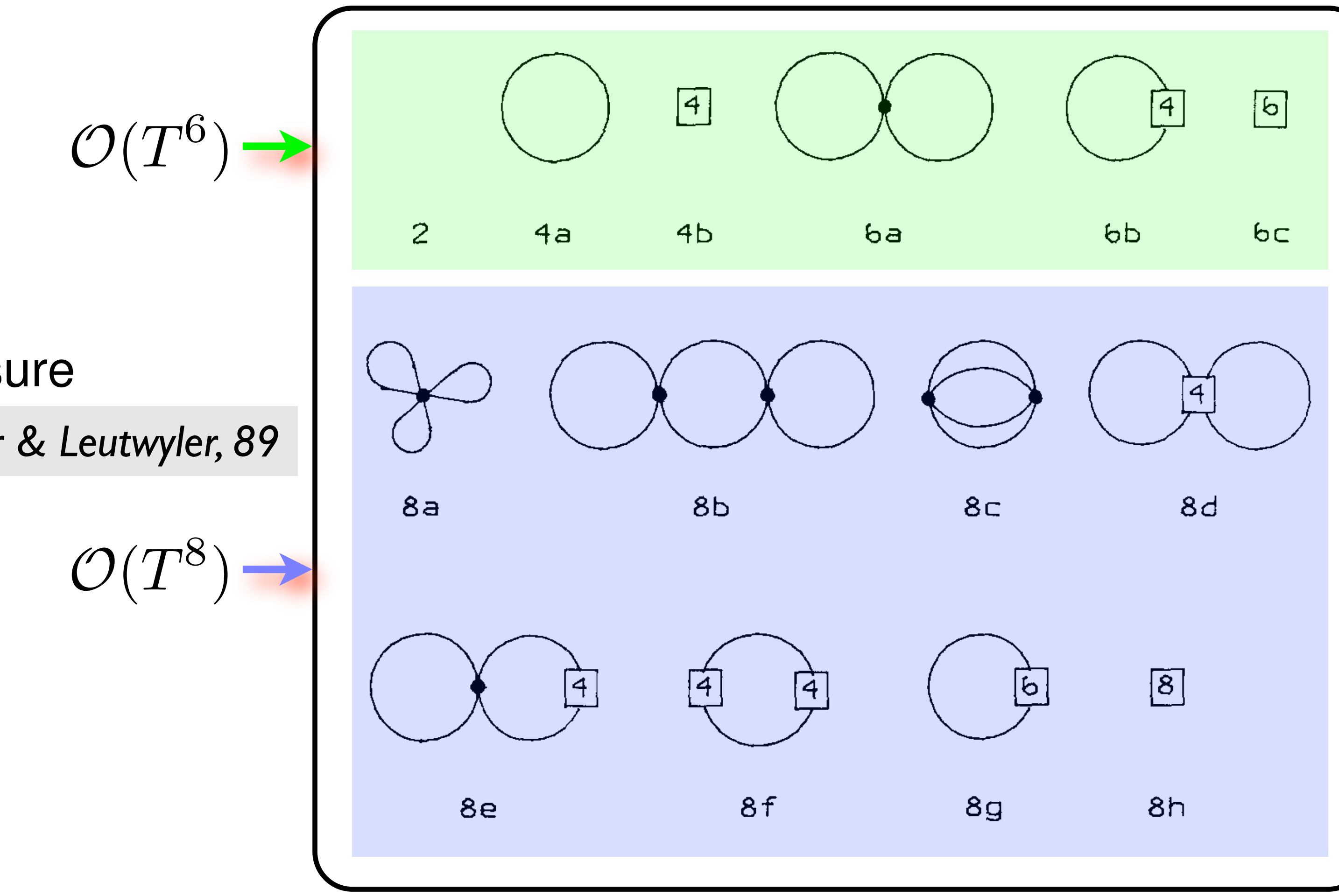


- From the ChPT pressure: $\langle T^{\mu}{}_{\mu} \rangle_T = T^5 \dfrac{\mathrm{d}}{\mathrm{d}T}\left(\dfrac{P}{T^4}\right)$




Pressure

*Gerber & Leutwyler, 89*




For $m_\mathrm{u} = m_\mathrm{d} = 0$:

$$\langle T^{\mu}{}_{\mu} \rangle^* = \frac{\pi^2}{270}\frac{T^8}{F_\pi^4}\left(\ln\frac{\Lambda_\mathrm{p}}{T} - \frac{1}{4}\right) \quad , \qquad \Lambda_\mathrm{p} \sim 400 \text{ MeV} \; .$$

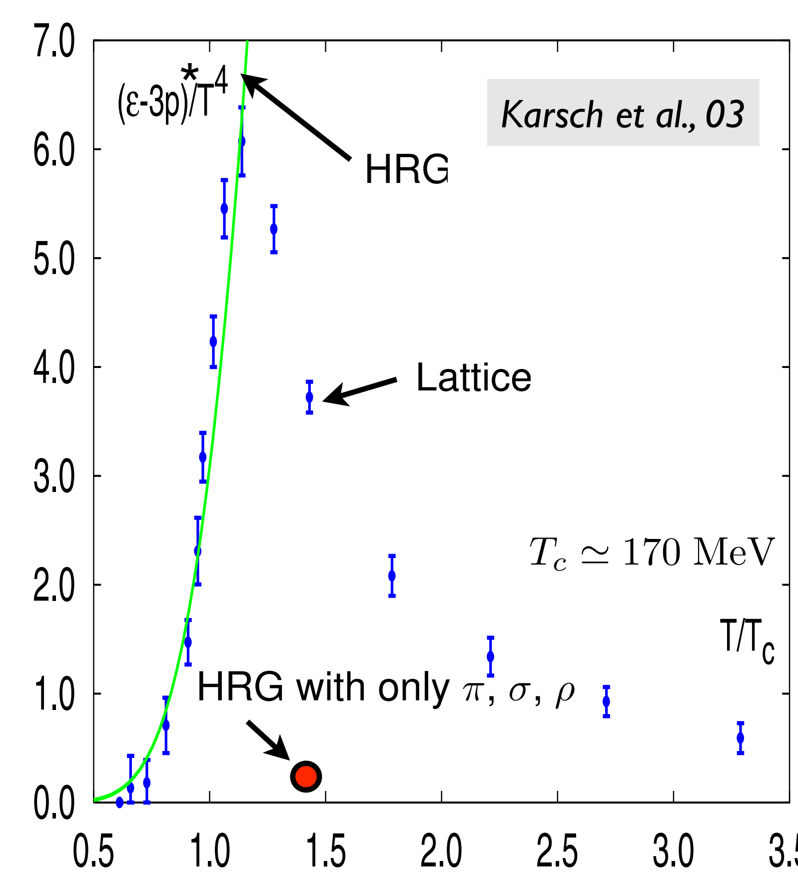
- Hadron Resonance Gas vs Lattice (2+1 q's):

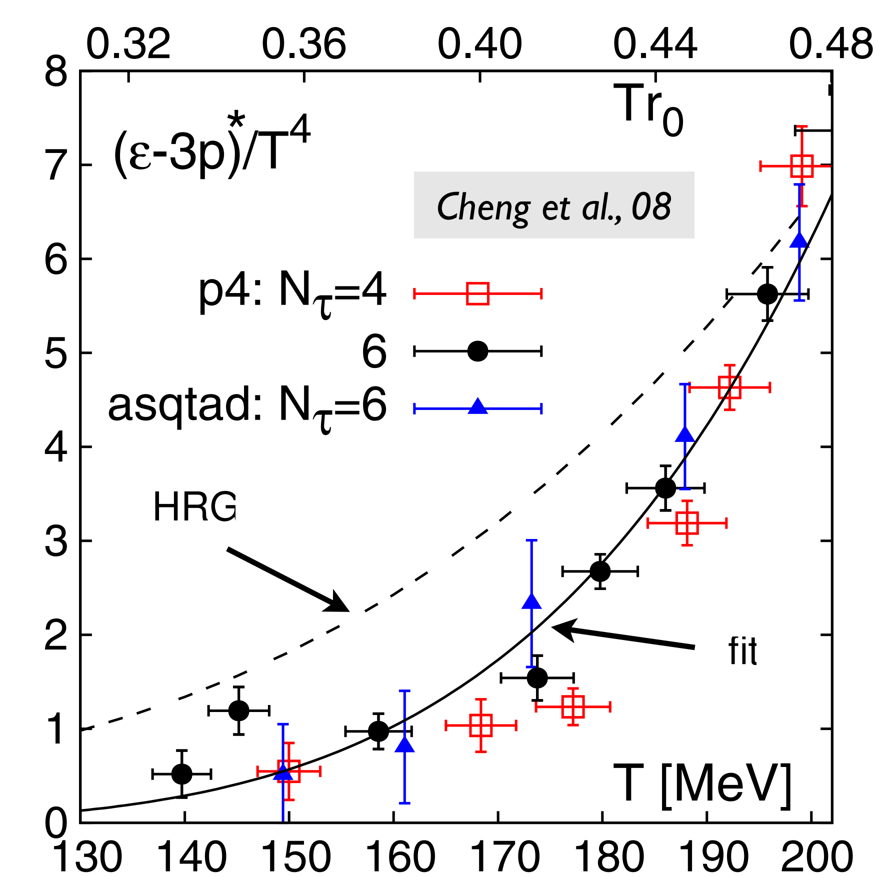
HRG approximation: all the resonances in the PDB up to $2$ GeV are included, 1026 in total, introduced as free states.

*Karsch et al., 03*

$$\Delta \equiv \frac{\epsilon - 3P}{T^4} = \sum_{i=1}^{1026} \frac{\epsilon_i - 3P_i}{T^4}$$

$$\stackrel{*}{=} \sum_{i=1}^{1026} \frac{g_i}{2\pi^2} \sum_{k=1}^{\infty} (-\eta)^{k+1} \frac{(\beta m_i)^3}{k} K_1(k\beta m_i)$$



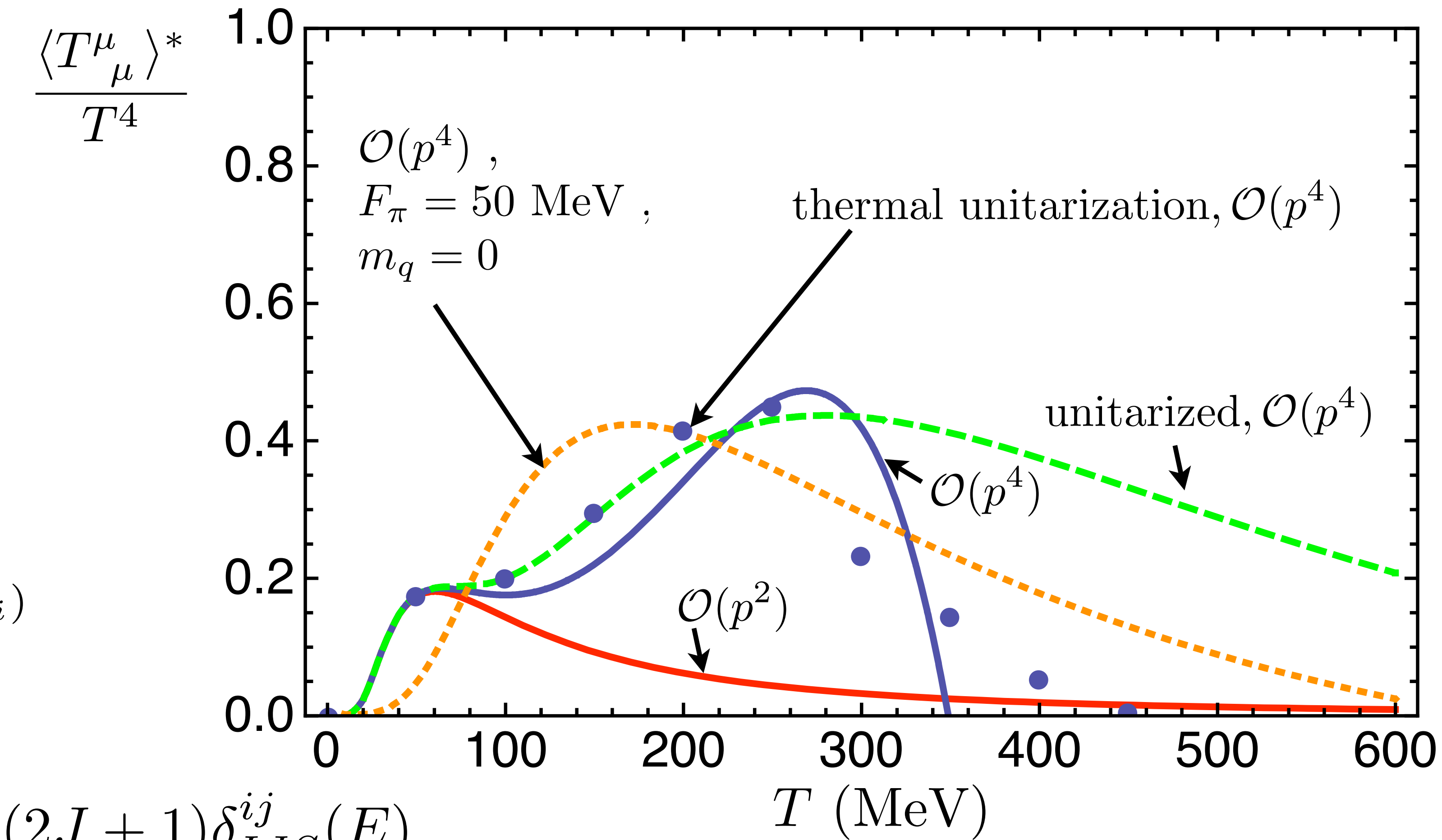
# The role of in-medium resonances

- Trace anomaly in the Virial Gas Approximation (dilute gas):

$$\beta P = \sum_i \left( B_i^{(1)} \xi_i + B_i^{(2)} \xi_i^2 + \sum_{j \geq i} B_{\rm int} \xi_i \xi_j + \dots \right)$$

$$\xi_i \equiv \mathrm{e}^{\beta(\mu_i - m_i)} \ , \quad B_i^{(n)} = \frac{g_i \eta_i^{n+1}}{2\pi^2 n} \int_0^\infty \mathrm{d}p \ p^2 \mathrm{e}^{-n\beta(E_i - m_i)}$$

$$B_{ij}^{int} = \frac{\mathrm{e}^{\beta(m_i+m_j)}}{2\pi^3} \int_{m_i+m_j}^{\infty} \mathrm{d}E \ E^2 K_1(\beta E) \sum_{I,J,S} (2I+1)(2J+1)\delta_{IJS}^{ij}(E)$$

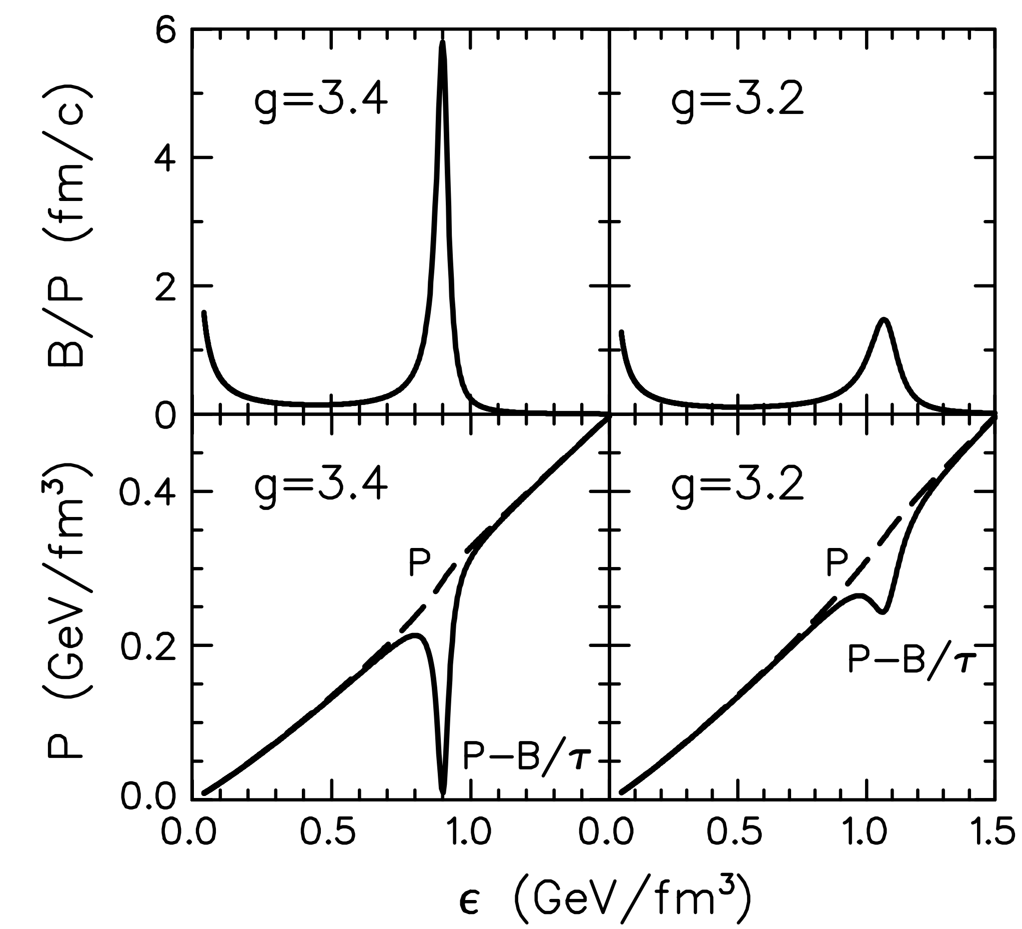



Therefore, we would not expect a large effect on $\zeta$ due to in-medium modification of the $\sigma$ and $\rho$ resonances.

- Bulk viscosity in the Linear Sigma Model:

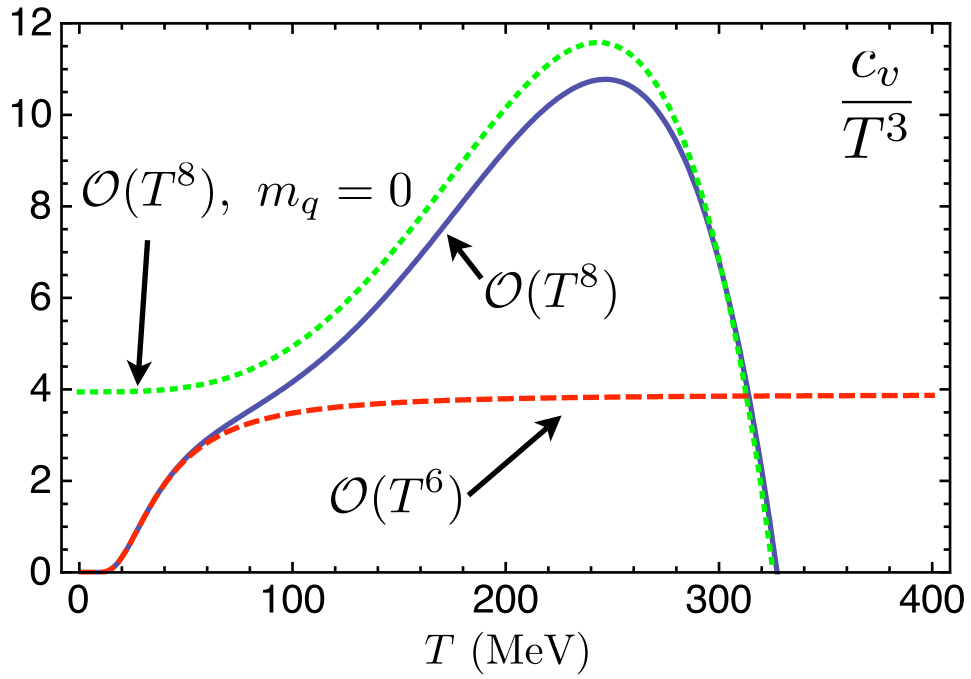
*Paech & Pratt*, PRC 74, 014901 (2006)

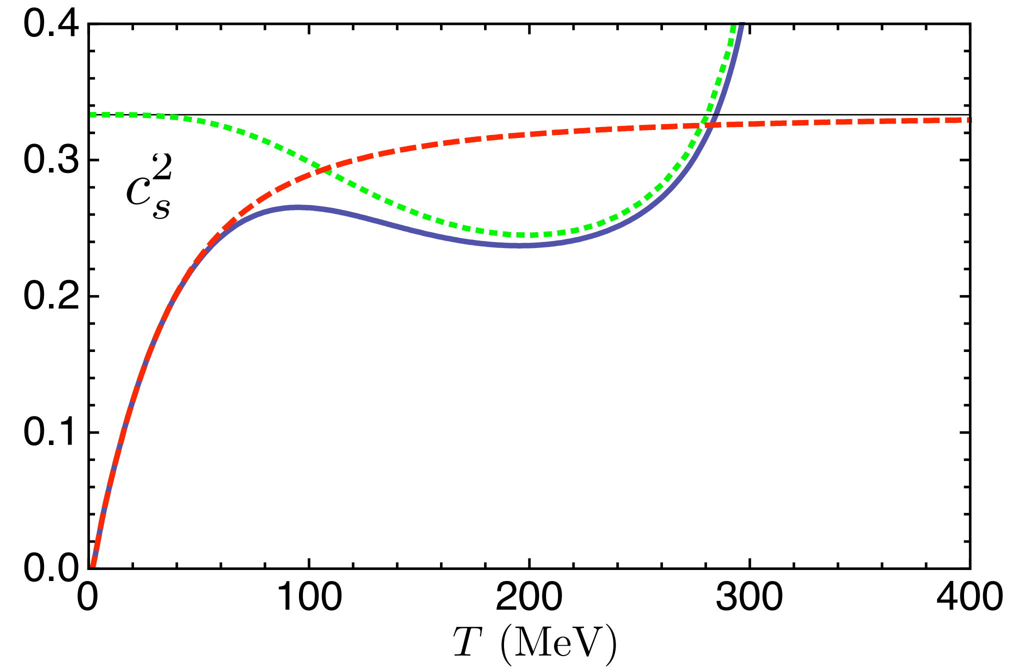
$$\zeta \propto \frac{\Gamma_\sigma}{m_\sigma^2}$$



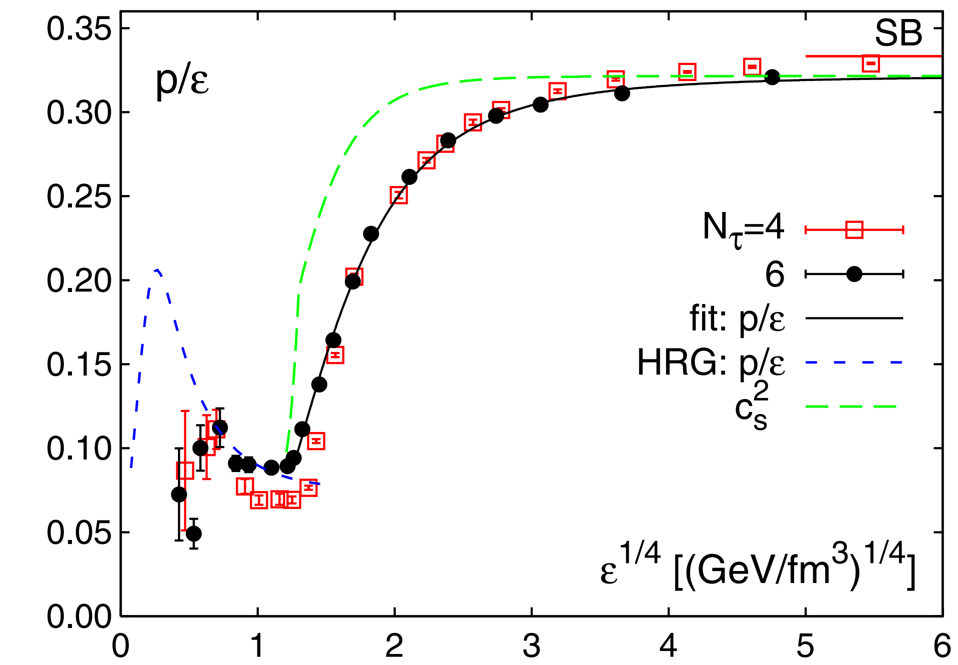
# Bulk viscosity of a gas of pions

- Heat capacity and speed of sound (ChPT):

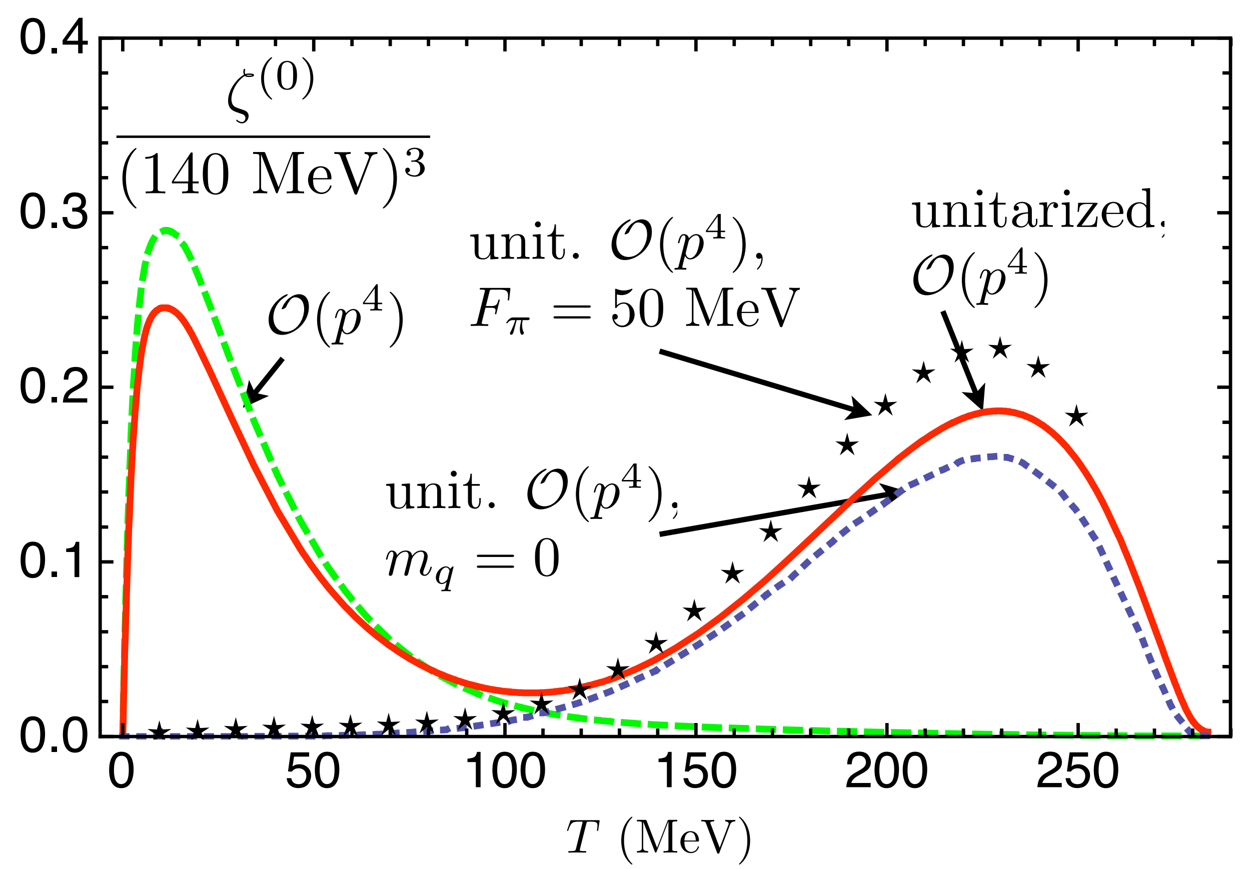



- Lattice (2+1 flavors): *Cheng et al., 08*




- Bulk viscosity (including only $2 \to 2$ processes): *Gomez Nicola & DFF,* PRL 102, 121601(2009)




$$\zeta^{(0)} = \int_0^\infty \mathrm{d}p\, \frac{3p^2(p^2/3 - c_s^2 E_p^2)^2}{4\pi^2 T E_p^2 \Gamma_p}\, n_B(E_p)[1 + n_B(E_p)]$$

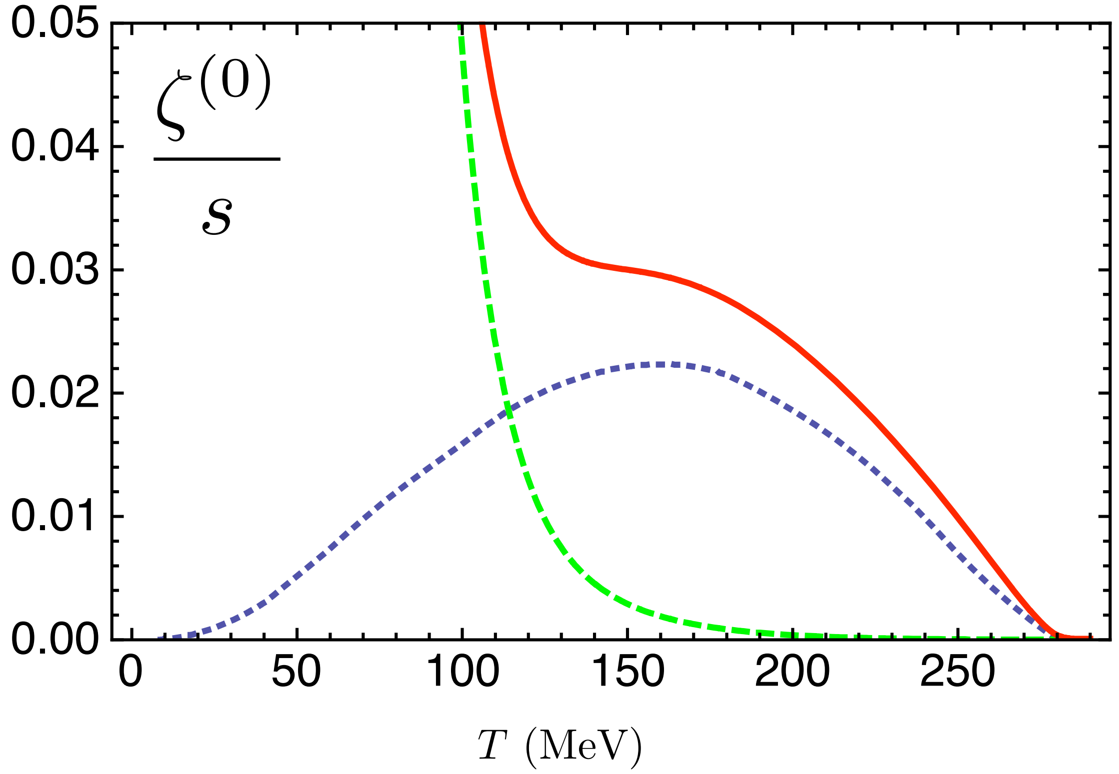
$$T \ll M_\pi :\ \zeta^{(0)} \simeq 0.36\, \eta^{(0)}$$

$$T \simeq M_\pi :\ \zeta^{(0)} \sim 10^{-1} \eta^{(0)}$$

$$T \gg M_\pi :\ \zeta^{(0)} \sim \left(\frac{1}{3} - v_s^2\right)^2 \eta^{(0)}$$

# Bulk viscosity of a gas of pions

- The $\zeta/s$ quotient near $T_c$ and the speed of sound: By KT: $\zeta \sim mvnl\left(\frac{1}{3}-v_s^2\right)^2$

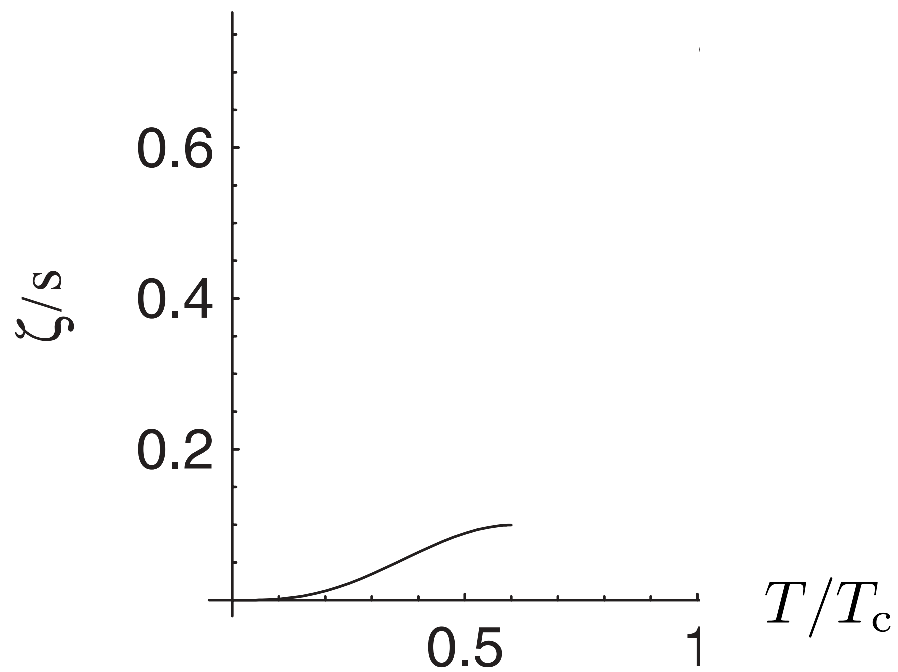



According to this, for the full hadron resonance gas near $T_c$:

$$\frac{\zeta}{s}(T=T_c) \simeq A\left(\frac{1}{3}-c_s^2\right)^2 \simeq 0.3 \gtrsim \frac{\eta}{s}(T_c)$$

approximately independent of the number of degrees of freedom

- $\zeta/s$ for the massless pion gas in KT: *Chen & Wang*, PRC 79, 044913 (2009)




$$T_{\mu\nu} = g_\pi \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{f_{eq}}{E_p}\left[p_\mu p_\nu(1+g_1)+\frac{g_2 g_{\mu\nu}}{\beta^2}+\frac{g_3 U_\mu U_\nu}{\beta^2}\right]$$

$$\frac{\zeta}{s}(T=T_c) \gtrsim 3 \text{ ?}$$

Even a bigger effect might come from vertex corrections.

# Conclusions

- The ChPT diagramatic method presented allows to easily obtain the functional form of transport coefficients at low $T$, including the in-medium evolution of resonances.
- The method can be extended to include other degrees of freedom: kaons, etas, baryons, and the corresponding resonances.
- Resonances make the quotient $\eta/s$ for a pion gas fulfill the KSS bound and reach a minimum near $T_c$.
- There are several indications that there is a maximum of the bulk viscosity near $T_c$ driven by the maximum of the trace anomaly.
- Some estimations suggest that $\zeta/s$ might be larger that $\eta/s$ near $T_c$.
- Several effects contribute to a large bulk viscosity: small speed of sound, vertex corrections, and resonances.

# Backup slides

# Equivalence between the diagramatic and the KT approaches



*Jeon*, PRD 52, 3591 (1995) *Jeon & Yaffe*, PRD 53, 5799 (1996)

- Consider for instance $\lambda\phi^4$. For $T \gg m$, apparently the KT treatment is not applicable:

$$l_{\rm free} \sim \frac{1}{T} \lesssim l_{\rm Compton}(T=0)$$

- However, for a weakly coupled theory, at an arbitrary temperature there is an effective KT description:

$$l_{\rm free} \sim \frac{1}{\lambda^2 T} > l_{\rm Compton}(T) \sim \frac{1}{\sqrt{\lambda}T}$$

★ Essentially, one identifies $A$ and $B$ in the KT description with the effective vertices of the diagramatic analysis, and the rung with the collision operator $\hat{\mathcal{C}}$.

★ In the dispersion relation of the effective quanta enters the thermal mass instead of the vacuum mass.

★ Scattering amplitudes are evaluated using thermal propagators.

★ $T^{\mu\nu}(x) \equiv T^{\mu\nu}_{\rm eq} - \int \frac{{\rm d}^3\boldsymbol{p}}{(2\pi)^3 E_p} \left( p^\mu p^\nu - U^\mu U^\nu T^2 \frac{\partial^2 m_{\rm th}}{\partial T^2} \right) f_{\rm eq}(1+f_{\rm eq})\phi \, .$

# Zero modes and particle-number changing processes



*Jeon*, PRD 52, 3591 (1995) *Jeon & Yaffe*, PRD 53, 5799 (1996) *Arnold, Dogan & Moore*, PRD 74, 085021 (2006)

- In order to calculate a transport coefficient, we need to invert the collision operator: $\eta, \zeta \propto \langle \mathcal{S}|\hat{\mathcal{C}}^{-1}|\mathcal{S}\rangle$.

- $\hat{\mathcal{C}}$ has one exact zero mode corresponding to energy conservation, $|E_0\rangle$, and an approximate one, $|N_0\rangle$, corresponding to the particle-number conserving terms in $\hat{\mathcal{C}}$. This is not important for $\eta$ (because $\langle E_0, N_0|\mathcal{S}_\eta\rangle = 0$), but it is for $\zeta$:

  ★ $|E_0\rangle$ is not problematic, we simply consider the vector space orthogonal to it (since $|E_0\rangle$ is not actually a departure from equilibrium).

  ★ Since $\hat{\mathcal{C}}$ is hermitian, let's consider an orthonormal basis of eigen-states:

$$|\chi\rangle = \sum_n \chi_n |f_n\rangle, \text{ with } |f_0\rangle \equiv |N_0\rangle$$

$$\zeta \propto \langle \mathcal{S}_\zeta|\chi\rangle = \langle \mathcal{S}_\zeta|\hat{\mathcal{C}}^{-1}|\mathcal{S}_\zeta\rangle = \sum_n S_n^\zeta C_n^{-1} S_n^\zeta = \sum_{n\neq 0} S_n^\zeta \frac{1}{C_n} S_n^\zeta + S_0^\zeta \frac{1}{\delta C_0(\mathrm{n-changing})} S_0^\zeta\,,$$

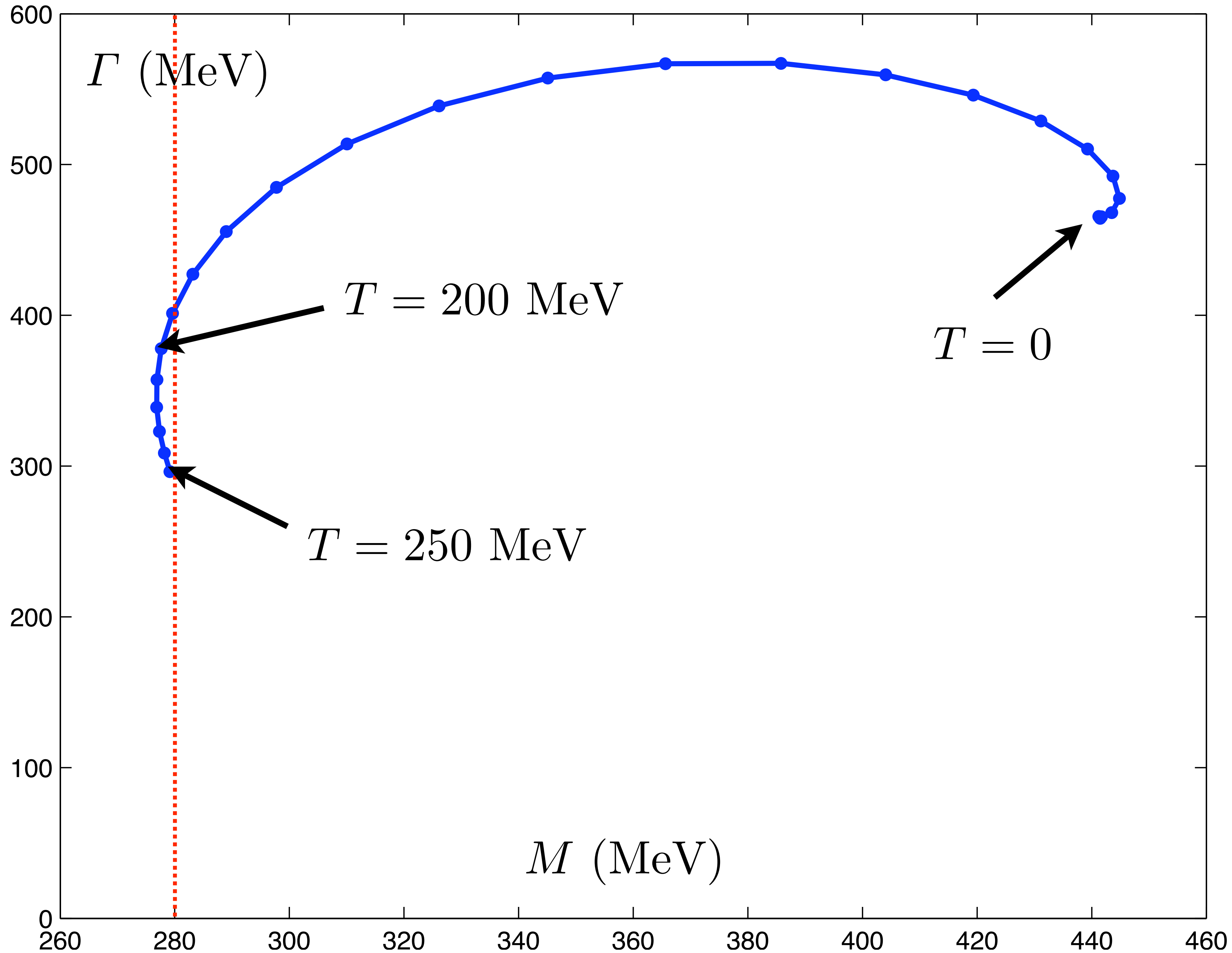
with $C_n = C_n(\mathrm{cons}) + \delta C_n(\mathrm{n-changing})$.

It dominates in QCD at high $T$.

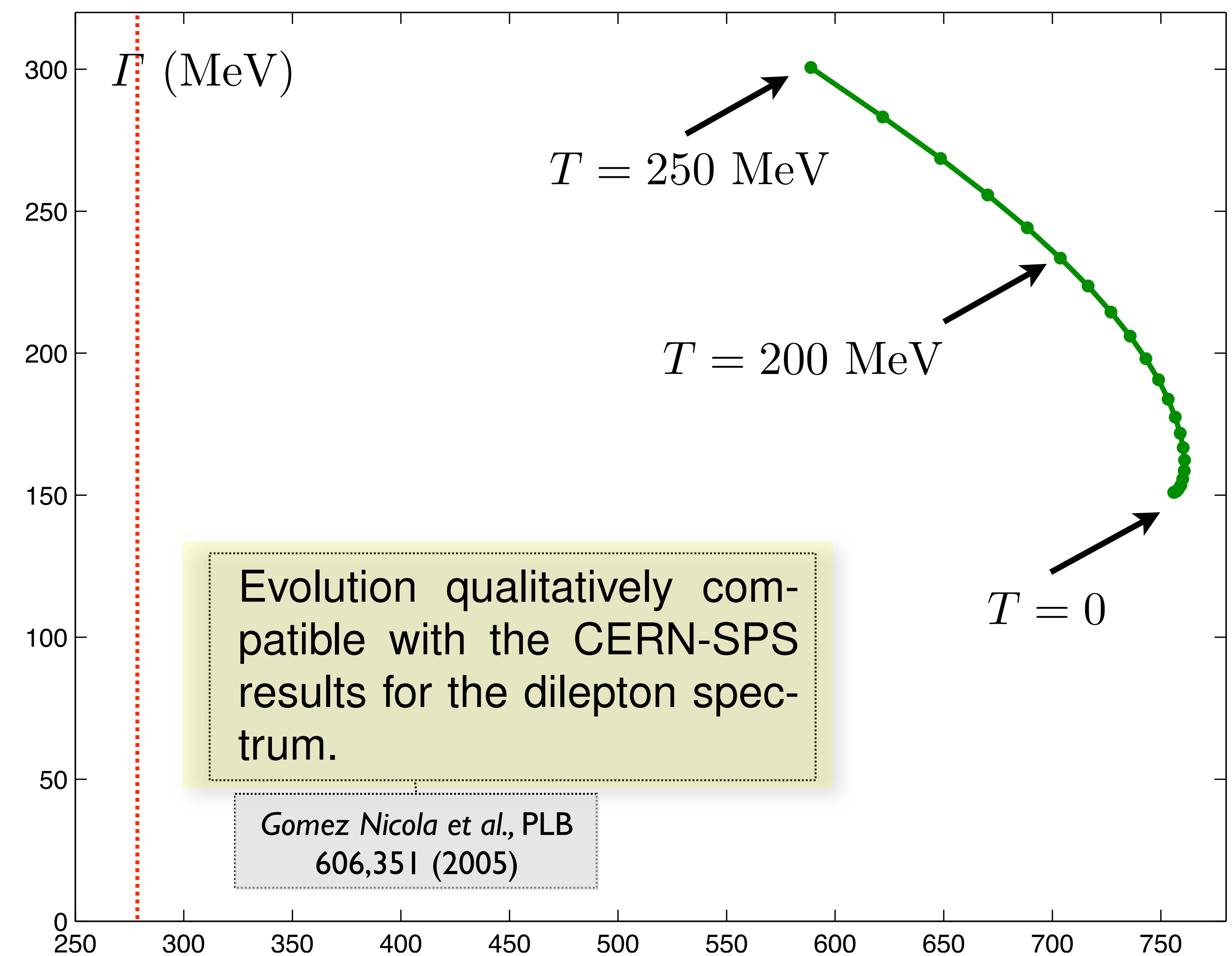
It dominates in $\lambda\phi^4$ at any $T$.

# Thermal evolution of the resonances

- $\sigma$ resonance (channel $IJ = 00$):




- $\rho$ resonance (channel $IJ = 11$):




Evolution qualitatively compatible with the CERN-SPS results for the dilepton spectrum.

*Gomez Nicola et al.*, PLB 606,351 (2005)

*Cabrera, Gomez Nicola, & DFF*, EPJC 61 879 (2009).

*Herruzo, Gomez Nicola, & DFF*, PRD 76, 085020 (2007).

more details on this in Angel Gomez Nicola's talk tomorrow

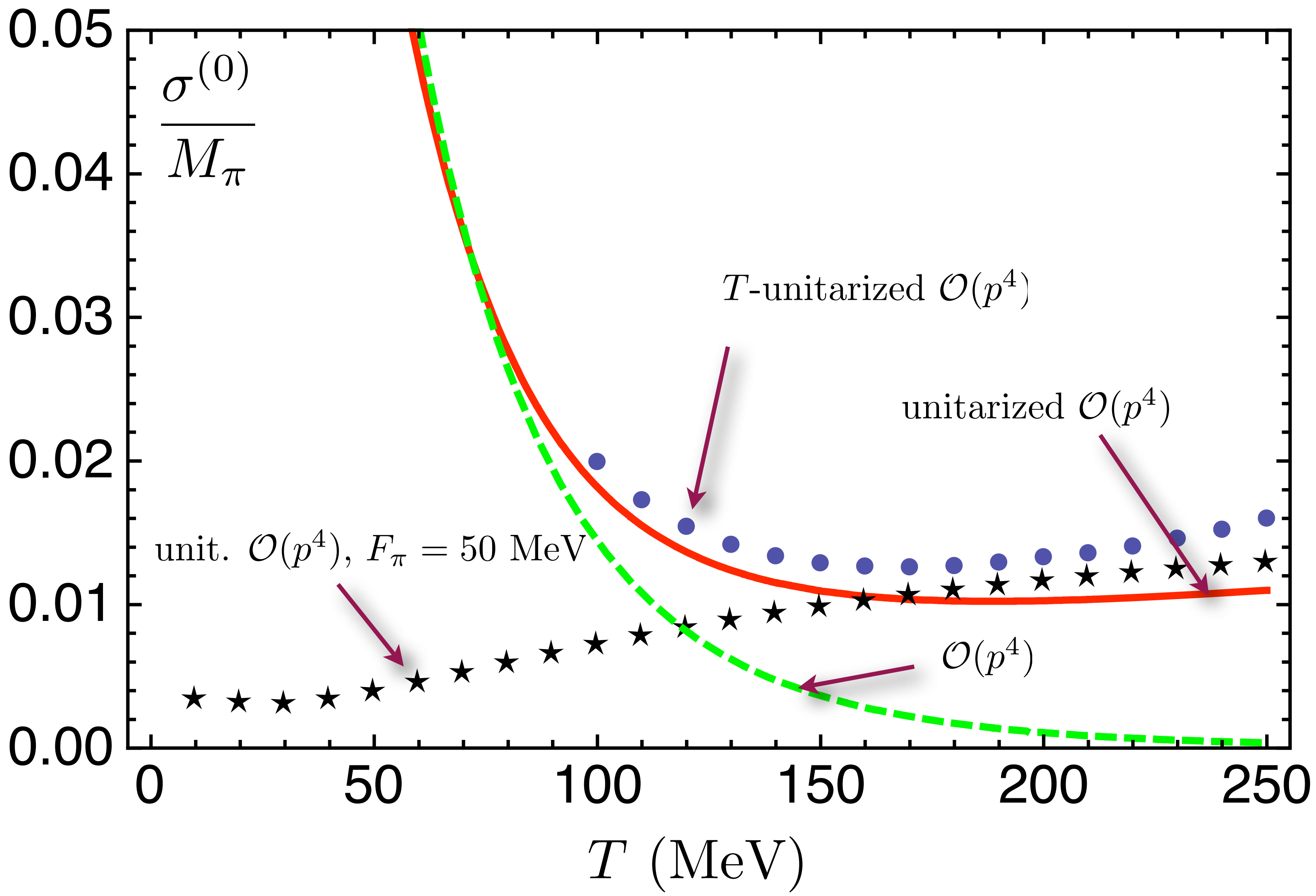
# Electrical conductivity of a pion gas

- Definition: $j^i = \sigma E^i_{\text{ext}}$
- Kubo's formula:

electric current

$$\sigma = -\frac{1}{6}\lim_{q^0\to 0^+}\lim_{|\boldsymbol{q}|\to 0^+}\frac{\partial\rho_\sigma(q^0,\boldsymbol{q})}{\partial q^0}\,, \quad \rho_\sigma(q^0,\boldsymbol{q}) = 2\,\mathrm{Im}\,\mathrm{i}\int \mathrm{d}^4x\; \mathrm{e}^{\mathrm{i}q\cdot x}\theta(t)\langle[\hat{J}_i(x),\hat{J}^i(0)]\rangle\;.$$

- Results:




According to kinetic theory: $\sigma \sim \frac{e^2 n_{\text{ch}}\tau}{M_\pi}$, but $\tau \sim 1/\Gamma$, and $\Gamma \sim n v \sigma_{\pi\pi}$.

For $T \ll M_\pi$, $n \sim (M_\pi T)^{3/2}\mathrm{e}^{-M_\pi/T}$, $v \sim \sqrt{T/M_\pi}$, and $\sigma_{\pi\pi}$ is a constant, $\Rightarrow$ $\sigma \sim 1/\sqrt{T}$. ✔

$$T \ll M_\pi: \quad \sigma^{(0)} \simeq 15\frac{e^2 F_\pi^4}{T^{1/2}M_\pi^{5/2}}$$

# Thermal conductivity of a gas of pions

- Definition: $T^{i0} - hN^i = \kappa \frac{T^2}{h} \partial_i \left(\frac{\mu}{T}\right)$

some conserved current

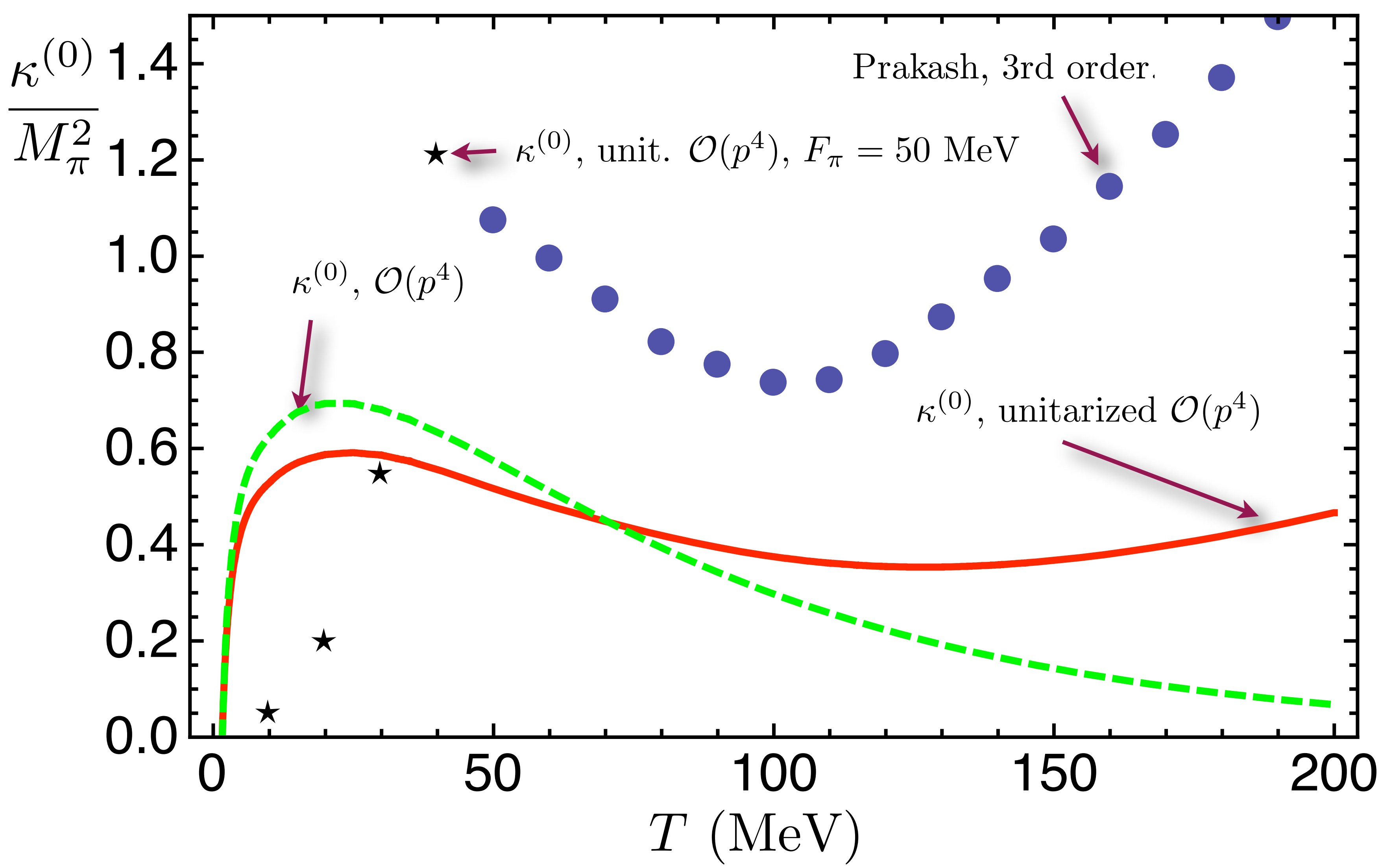
enthalpy per particle

- Kubo's formula:

$\equiv T^{i0} - hN^i$

$$\kappa = -\frac{\beta}{6} \lim_{q^0 \to 0^+} \lim_{|\boldsymbol{q}| \to 0^+} \frac{\partial \rho_\kappa(q^0, \boldsymbol{q})}{\partial q^0} , \quad \rho_\kappa(q^0, \boldsymbol{q}) = 2\,\mathrm{Im}\,\mathrm{i} \int \mathrm{d}^4x \; \mathrm{e}^{\mathrm{i}q\cdot x} \theta(t) \langle [\hat{\mathcal{T}}_i(x), \hat{\mathcal{T}}^i(0)] \rangle \; .$$

- Results:




From KT: $\kappa \sim T^{-1}(\bar{e} - h)lv$.

For $T \ll M_\pi$, $\bar{e} \sim M_\pi$, $h \sim 5T/2 + M_\pi$,
$\Rightarrow \kappa \sim T^{1/2}$. ✔

$$T \ll M_\pi : \quad \kappa^{(0)} \simeq 63 \, \frac{T^{1/2} F_\pi^4}{M_\pi^{5/2}}$$