Complex Actions and Stochastic Quantisation

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1. Complex Langevin method and real time evolution
    - Results for a scalar oscillator, SU(2) gauge theory
    - Connection with Schwinger Dyson equations
    - Optimization methods: reweighting, gauge fixing

2. Non-zero chemical potential
    - Toy modells of QCD, QCD in heavy quark aprroximation
    - Bose Gas at zero temperature

Non-zero chemical potential

Euclidean gauge theory with fermions: $Z = \int dU \exp(-S_E) \, det(M)$

For nonzero chemical potential, the fermion determinant is complex

Sign problem ——► Naïve Monte-Carlo breaks down

Methods going around the problem work for $\mu = \mu_B/3 < T$

Multi parameter reweighting

Fodor, Katz '02

Analytic continuation of results obtained at imaginary μ

Lombardo '00; de Forcrand, Philipsen '02

Taylor expansion in $(\mu/T)^2$

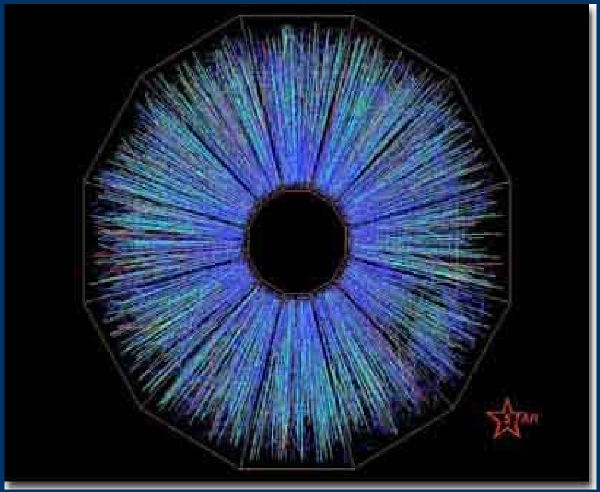
de Forcrand et al. '99; Hart, Laine, Philipsen '00; Gavai and Gupta '03;

Stochastic quantisation

Aarts and Stamatescu '08

Aarts '08

Real-time evolution



High occupation numbers
At $n = O(1/\alpha)$ all diagrams become large
preventing perturbative treatment

On the lattice: mainly equilibrium methods so far, static quantities
with few exceptions

Late times approaching thermal equilibrium:
Classical approximation breaks down

Direct Method: Schödinger equation for the wave function: $\Psi[A_\mu^a(x)]$
Impossible!

Formulation with non-equilibrium generator function $Z[J] = \int D\Phi\, e^{i\int_C L(\Phi,J)\,dt}$

averages with complex weight is needed! e^{iS_M}
Importance sampling doesn't work

Stochastic Quantization

Parisi, Wu (1981)

Weighted, normalized average: $$\langle O \rangle = \frac{\int e^{-S(x)} O(x) dx}{\int e^{-S(x)} dx}$$

Stochastic process for x: $$\frac{dx}{d\tau} = -\frac{\partial S}{\partial x} + \eta(\tau)$$

Gaussian noise $\langle \eta(\tau) \rangle = 0 \quad \langle \eta(\tau)\eta(\tau') \rangle = 2\delta(\tau - \tau')$

Averages are calculated along the trajectories:

$$\langle O \rangle = \frac{1}{T} \int_0^T O(x(\tau)) d\tau$$

Fokker-Planck equation for the probability distribution of P(x):

$$\frac{\partial P}{\partial \tau} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x} + P \frac{\partial S}{\partial x}\right) = -H_{FP} P$$

Real action $\rightarrow$ positive eigenvalues

for real action the Langevin method is convergent

Real-time evolution

$$\langle O(t)\rangle = \langle i|U(0,t)OU(t,0)|i\rangle$$ Schwinger-Keldysh contour

Nonequilibrium generating functional $$Z[J] = \int D\Phi\, e^{i\int_c L(\Phi,J)dt}$$

Real time= Langevin method with complex action! $$\frac{d\phi}{d\tau} = i\frac{\partial S}{\partial \phi} + \eta(\tau)$$

Klauder '83, Parisi '83, Hueffel, Rumpf '83,
Okano, Schuelke, Zeng '91, ...
applied to nonequilibrium: Berges, Stamatescu '05, ...

5D classical langevin system → 4D quantum averages

<u>The field is complexified</u>

real scalar → complex scalar

link variables: SU(2) → SL(2,C)
compact → non-compact

Is it still the same theory?

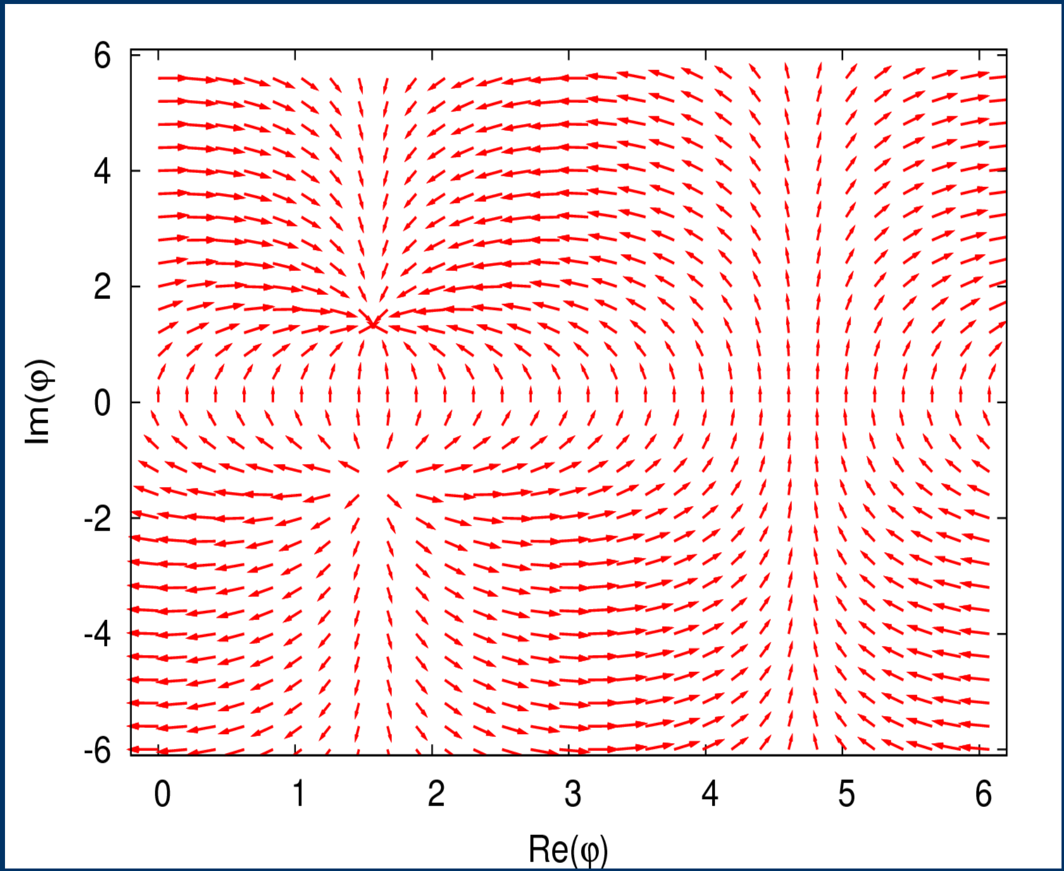
Yes: real (SU(2)) averages
Schwinger-Dyson equations fulfilled

No general proof of convergence

But Schwhinger-Dyson eqs. are fulfilled

Runaway trajectories present

Typical drift structure



Noise is real "horizontal"

Runaway if field stays at $\frac{3}{2}\pi$

In continuum probabilty of a runaway=0

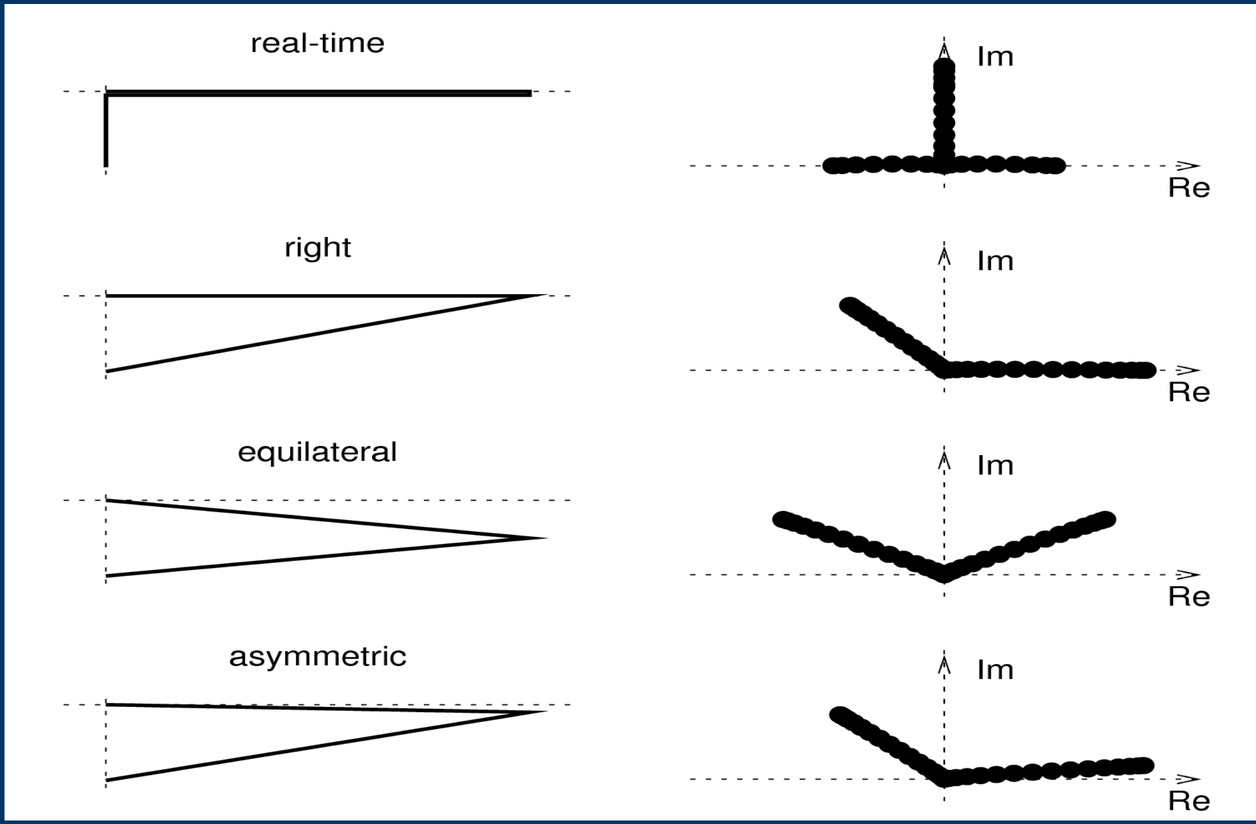
Discretised: getting far away

Numerical problem
drift proportional to field

Solution: small stepsize
restarting with different seed

Action of the langevin method= path integral on a complex time-contour

downwards sloped countour: regulator



Studied models:

-- quantum oscillator (0+1)D scalar field theory

-- SU(2) pure gauge (3+1)D field theory with Wilson action

J. Berges, Sz. Borsányi, D. Sexty, I.-O. Stamatescu PRD75 045007

-- few variable toy models: U(1) one-plaquette models

SU(2) one-plaquette model

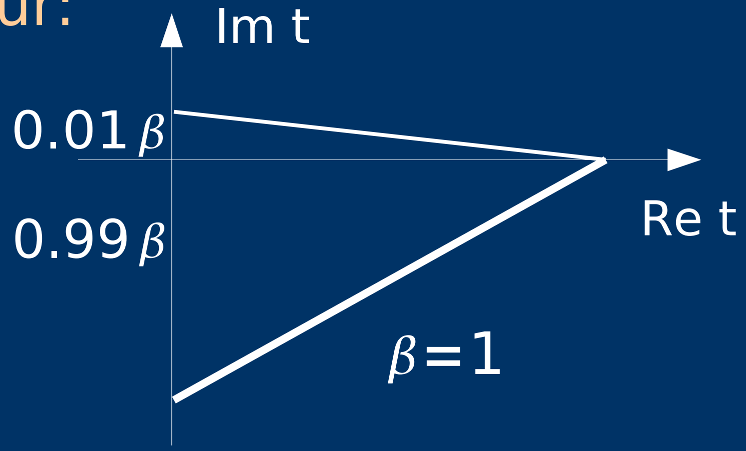
J. Berges, D. Sexty NPB799 306

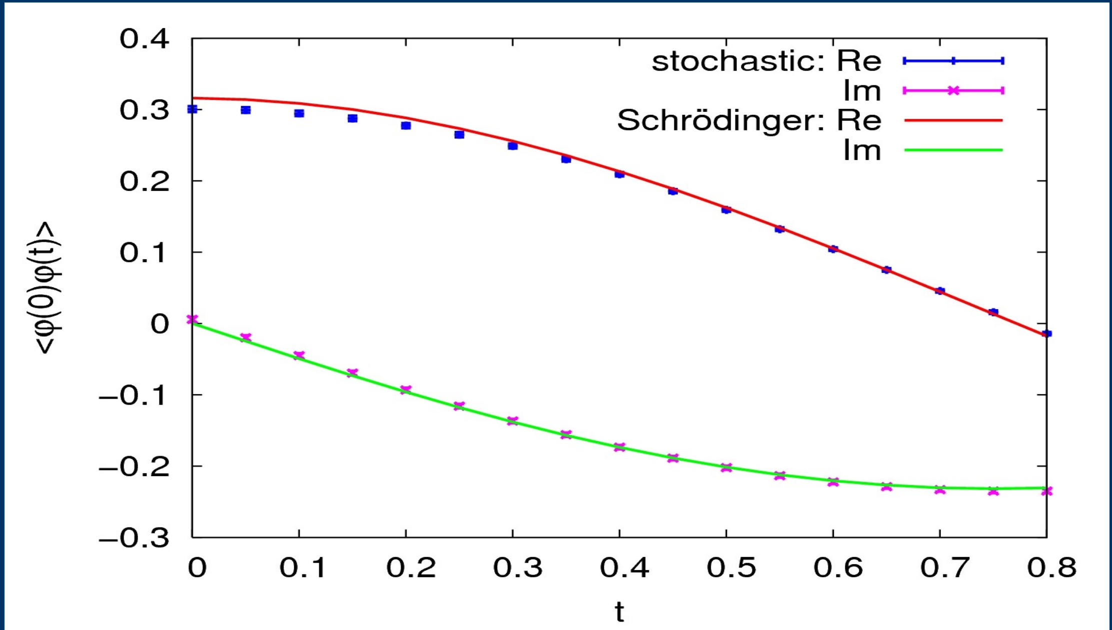
Real-time two point function of quantum oscillator

Thermal equilibrium:
periodic boundary cond.

Imaginary extent gives $\beta = \frac{1}{T}$

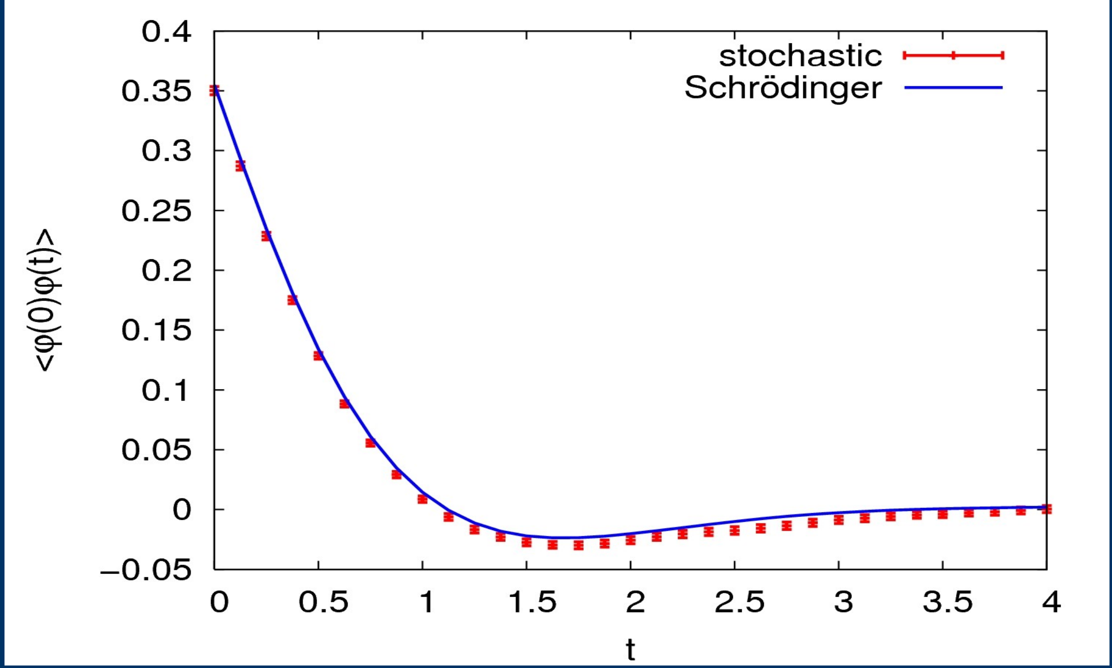
Asymmetric contour:





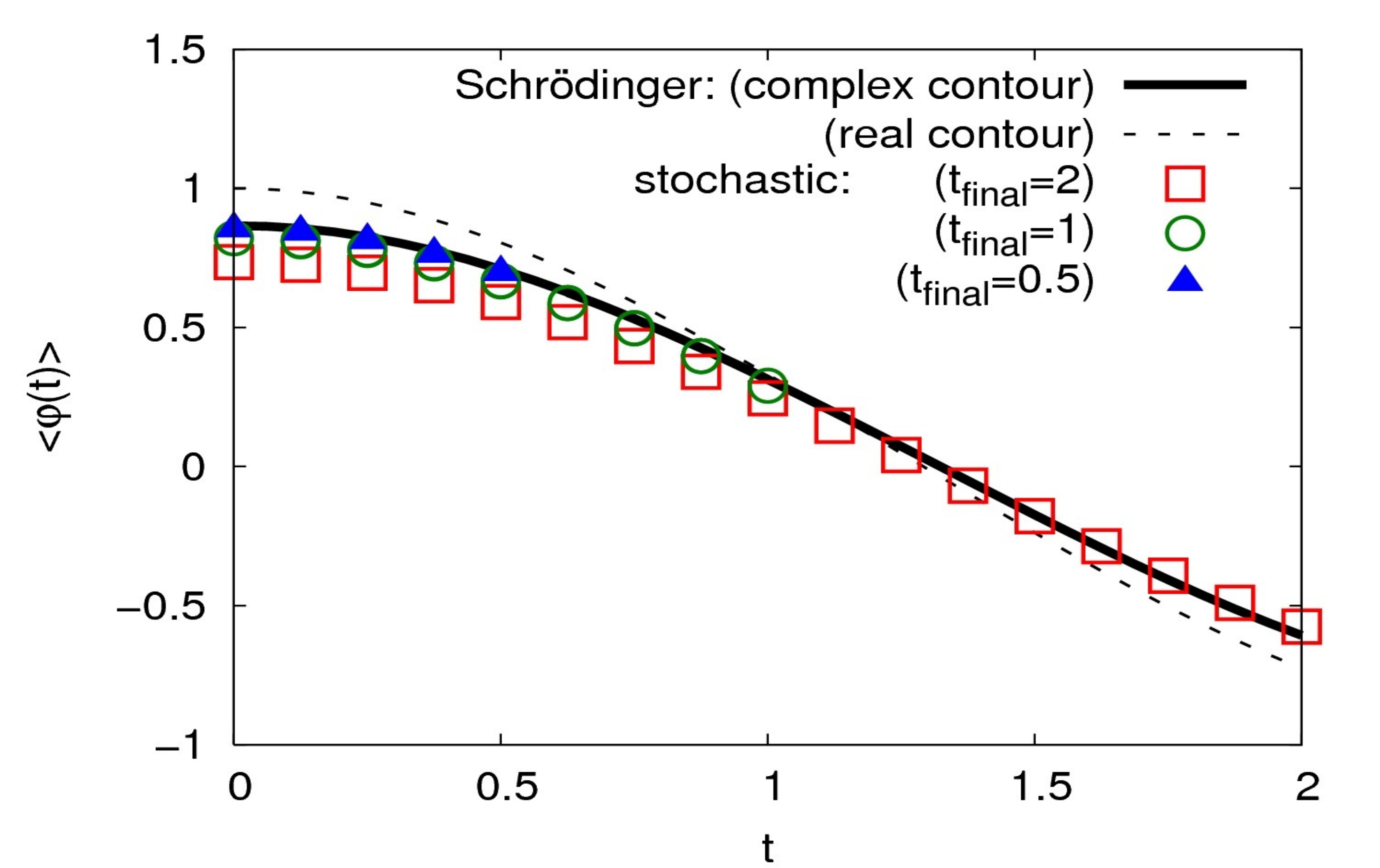
Smaller temperature
longer contour

$\beta = \Lambda$



Reproduces the
Schrodinger equation result.

Non-equilibrium time evolution



Gaussian initial density matrix

Contour with 5% slope

Bigger real time extent worse agreement

Schwinger Dyson equations for lattice gauge theory

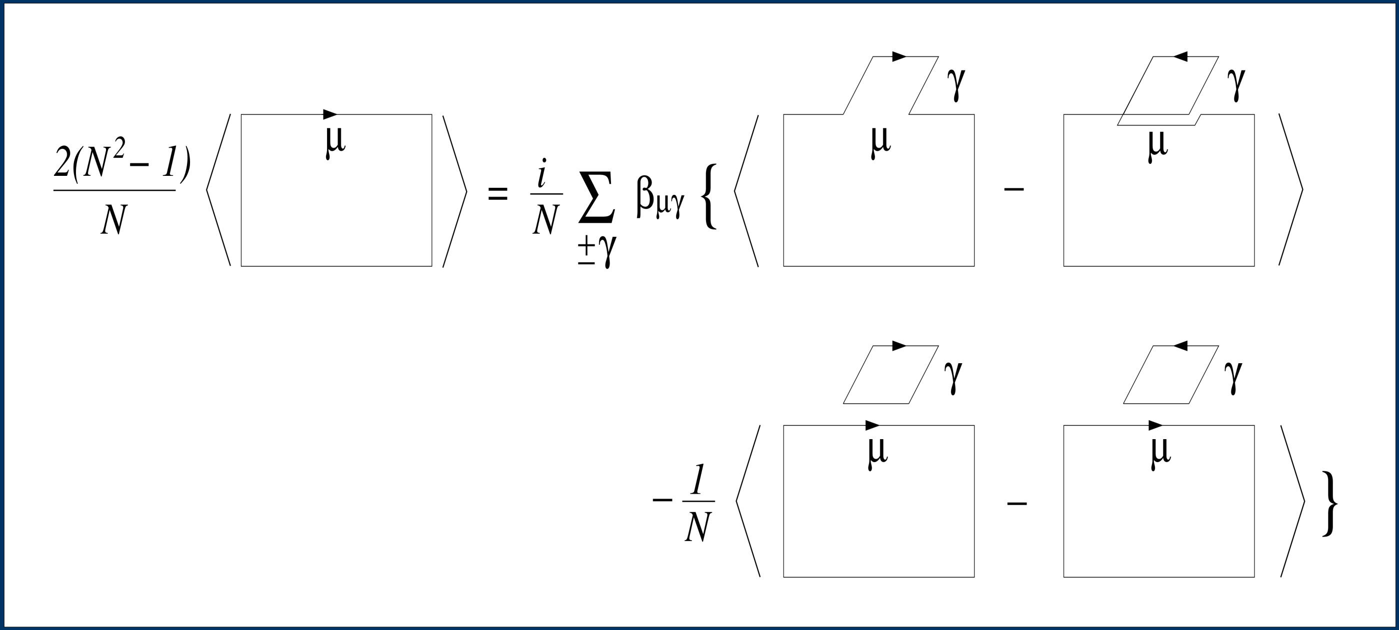
Langevin-time equilibrium reached:

$$\langle U_{x\mu a}(\tau+d\tau)U^{-1}_{x\mu a}(\tau)\rangle=1 \quad \Rightarrow \quad \langle D_{x\mu a}S\rangle=0$$ First Schwinger Dyson equation

Plaquette average is Langevin time independent

$$\langle U_{x,\mu\nu}(\tau+d\tau)\rangle=\langle U_{x,\mu\nu}(\tau)\rangle \quad \Rightarrow$$ Schwinger Dyson equation for plaquette average

can also be derived using the properties of Haar integration in the original integration over group space

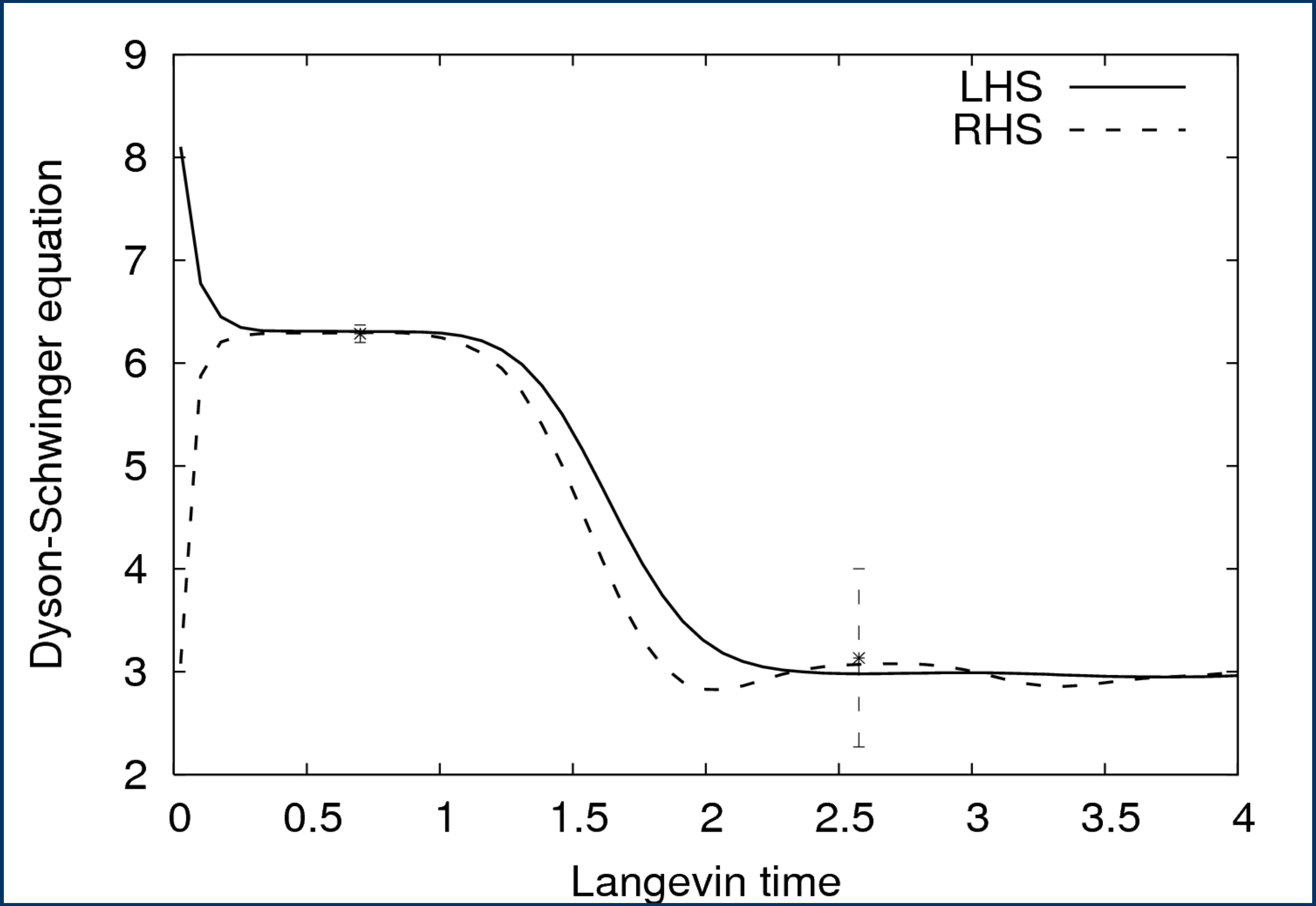


This method gives solutions of SD equations (all of them!)

(loophole: one might get unphysical solution)

SU(2) field theory

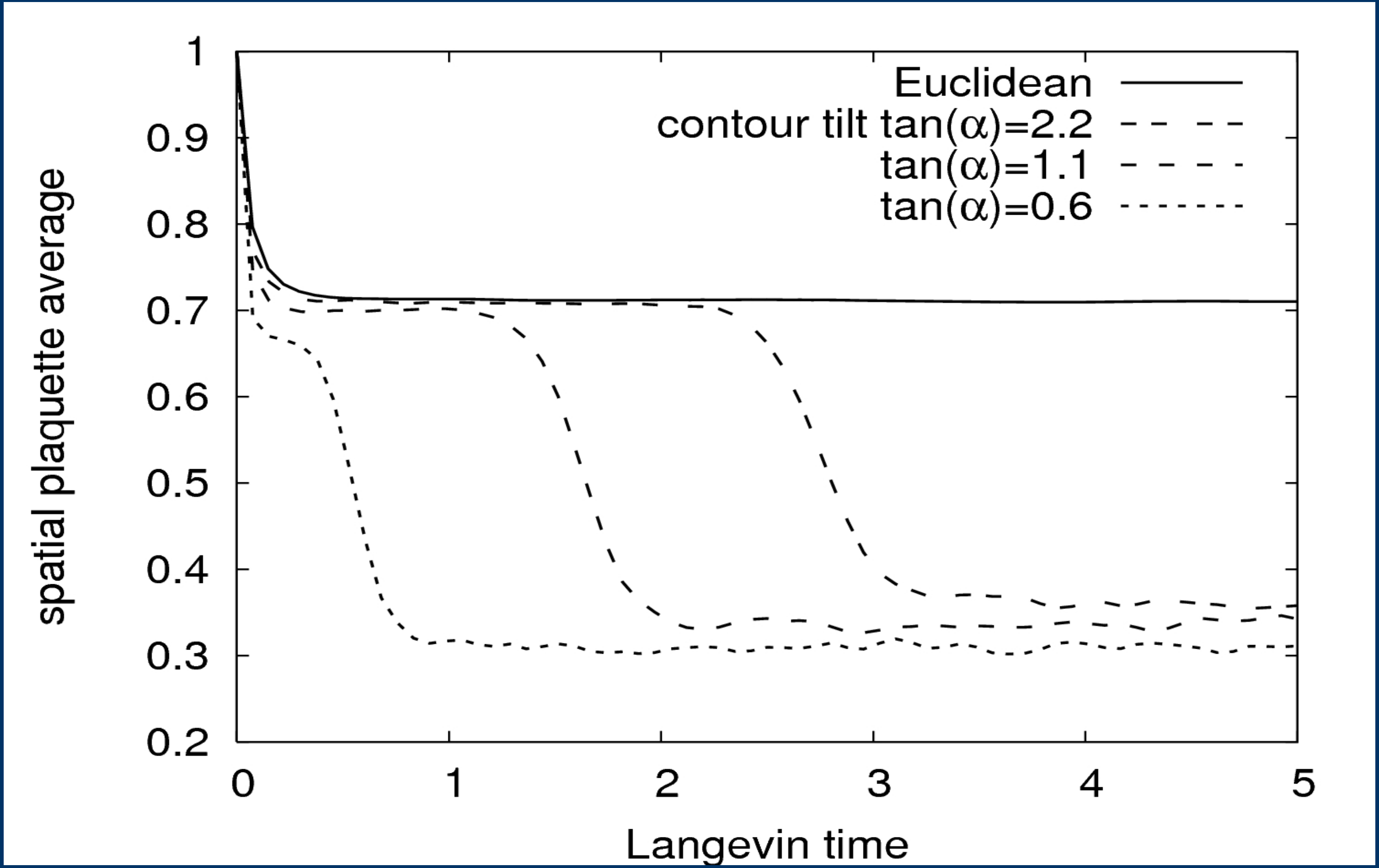
Numerical check of the Schwinger-Dyson equation



SD equations are fulfilled in both regions

SU(2) gauge theory without gaugefixing

without gauge fixing, non-physical fixpoint is always present



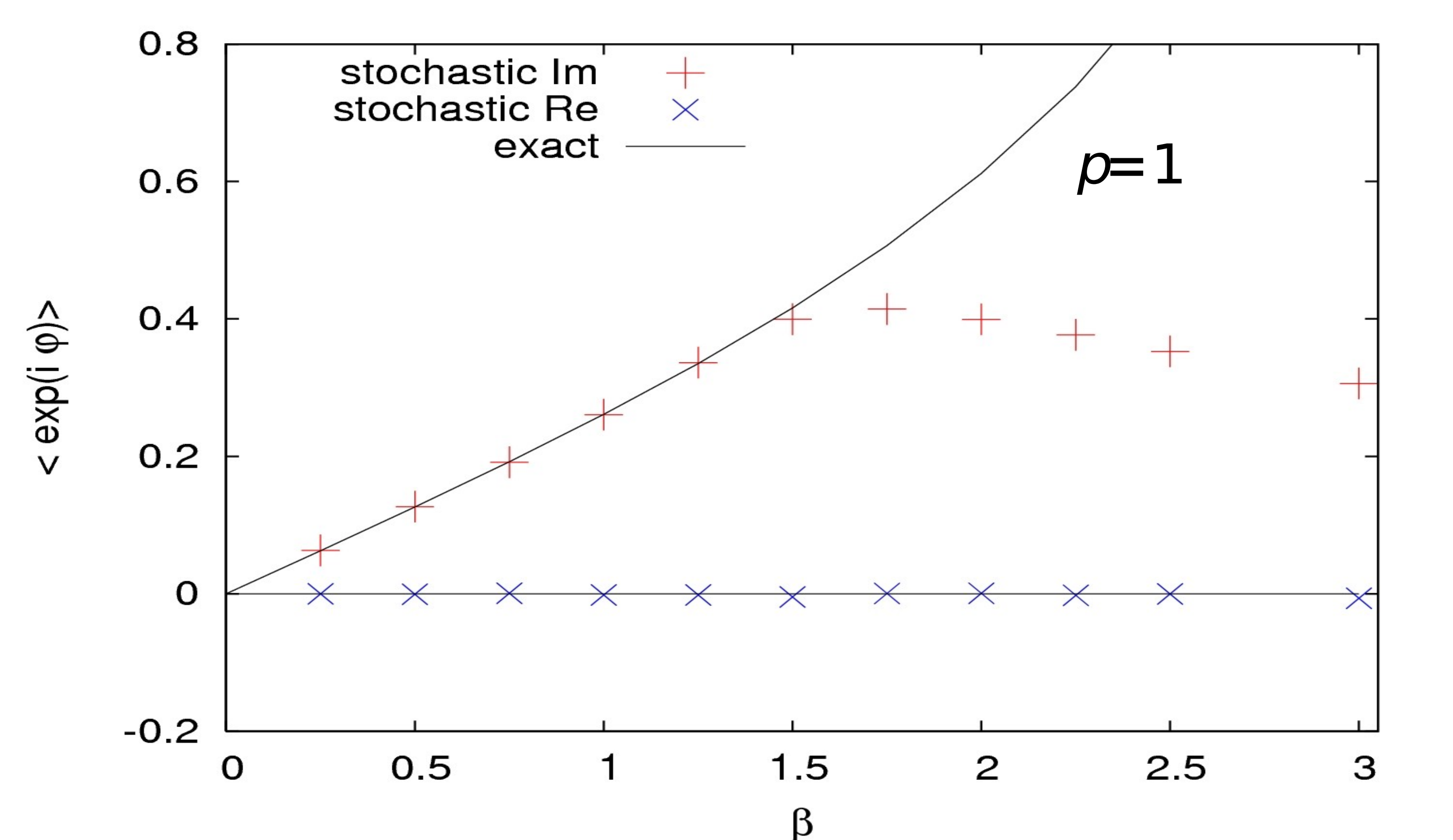
How to stabilize the first (physical) result?

U(1) One plaquette model

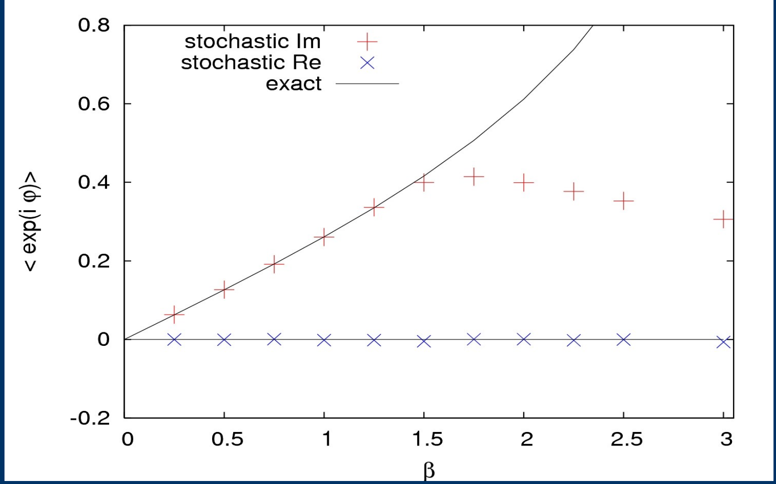
Using the action $S_p = i\beta\cos(\varphi) + ip\varphi$

Langevin equation: $\frac{d\varphi}{d\tau} = -i\beta\sin\varphi + ip + \eta(\tau)$

Correct results obtained for $\langle \exp(i\varphi) \rangle$ in the region: $\beta \leq p$



Flowchart: normalized drift vectors on the complex plane

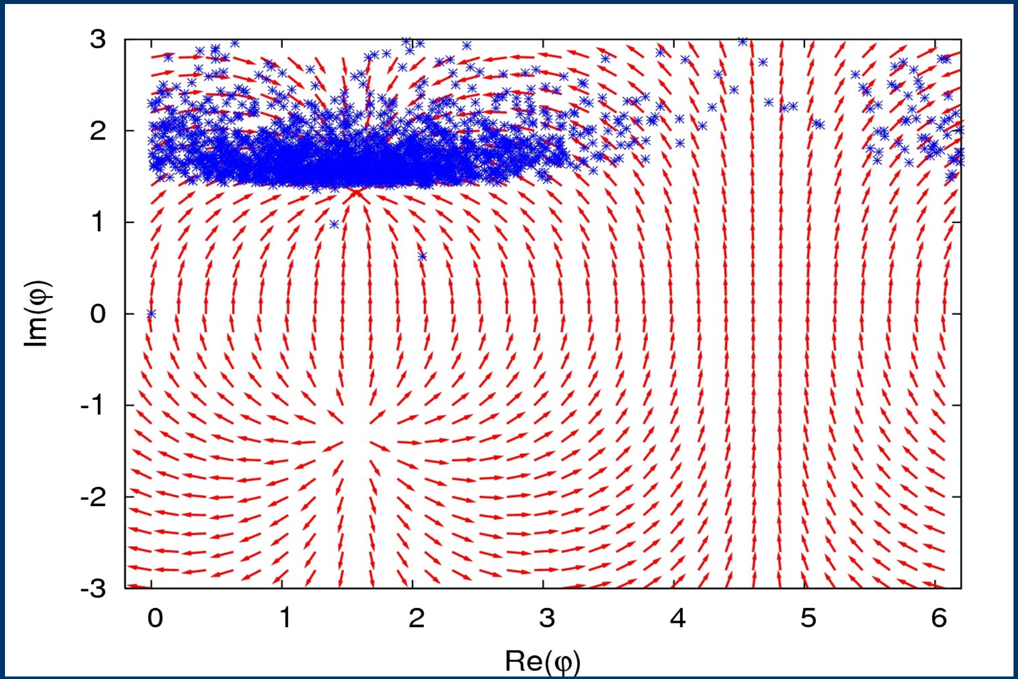


shows fixedpoint (zero drift term) structure on the complex φ plane

Attractive fixedpoint present

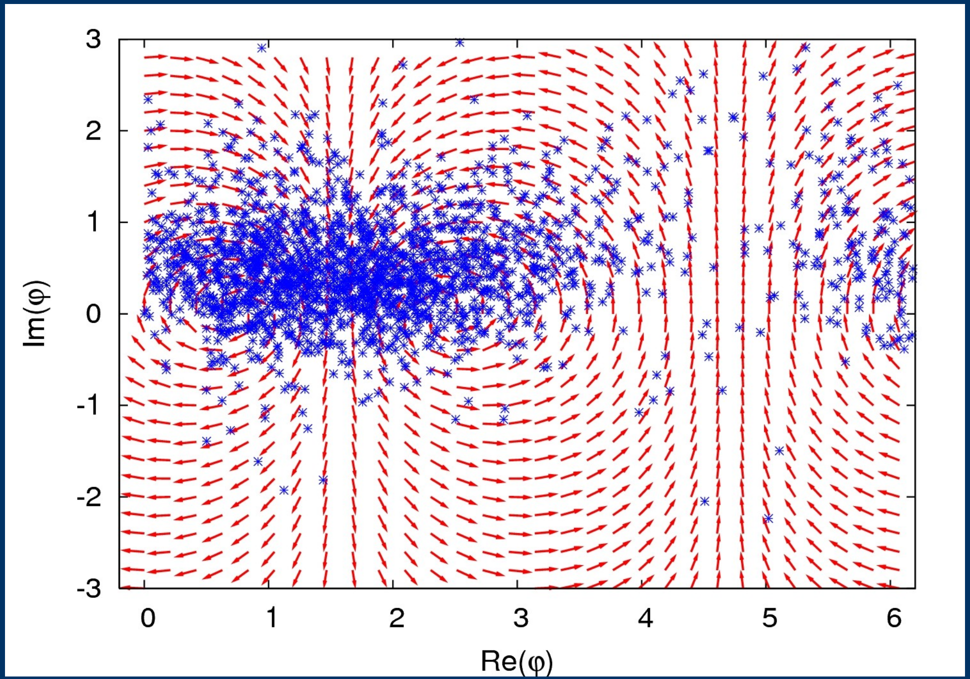
smaller distribution
correct results

$\beta=0.5, p=1$



No attractive fixedpoint present
(only indifferent)
larger distribution
incorrect results

$\beta=1.5, p=1$



Gaugefixing in SU(2) one plaquette model

SU(2) one plaquette model: $S = i\beta \, Tr\, U \qquad U \in SU(2)$

“gauge” symmetry: $U \rightarrow W U W^{-1}$ complexified theory: $U, W \in SL(2, \mathbb{C})$

exact averages by numerical integration: $\langle f(U) \rangle = \frac{1}{Z} \int_0^{2\pi} d\varphi \int d\Omega \sin^2 \frac{\varphi}{2} \, e^{i\beta \cos\frac{\varphi}{2}} f(U(\varphi, \hat{n}))$

Langevin updating $U' = \exp\left(i \lambda_a (\epsilon \, i D_a S[U] + \sqrt{\epsilon}\, \eta_a)\right) U$

parametrized with Pauli matrices

$$U = \exp\left(i \frac{\varphi \hat{n} \hat{\sigma}}{2}\right) = \left(\cos\frac{\varphi}{2}\right) \mathbf{1} + i \left(\sin\frac{\varphi}{2}\right) \hat{n} \hat{\sigma}$$

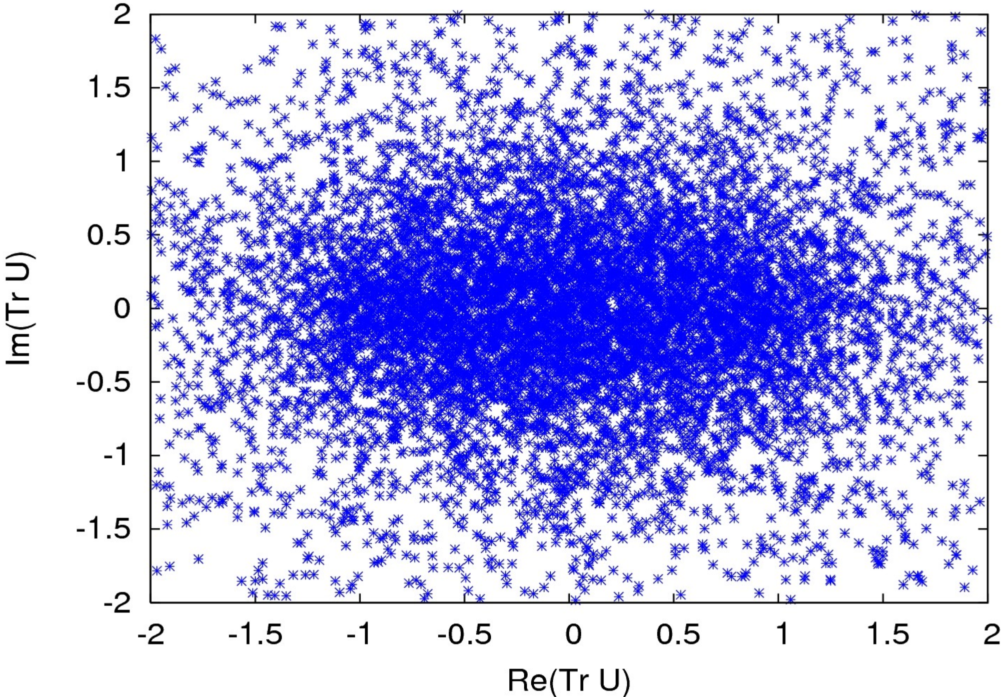
$$U = a \mathbf{1} + i b_i \sigma_i \qquad a^2 + b_i b_i = 1$$

After each Langevin timestep: fix gauge condition

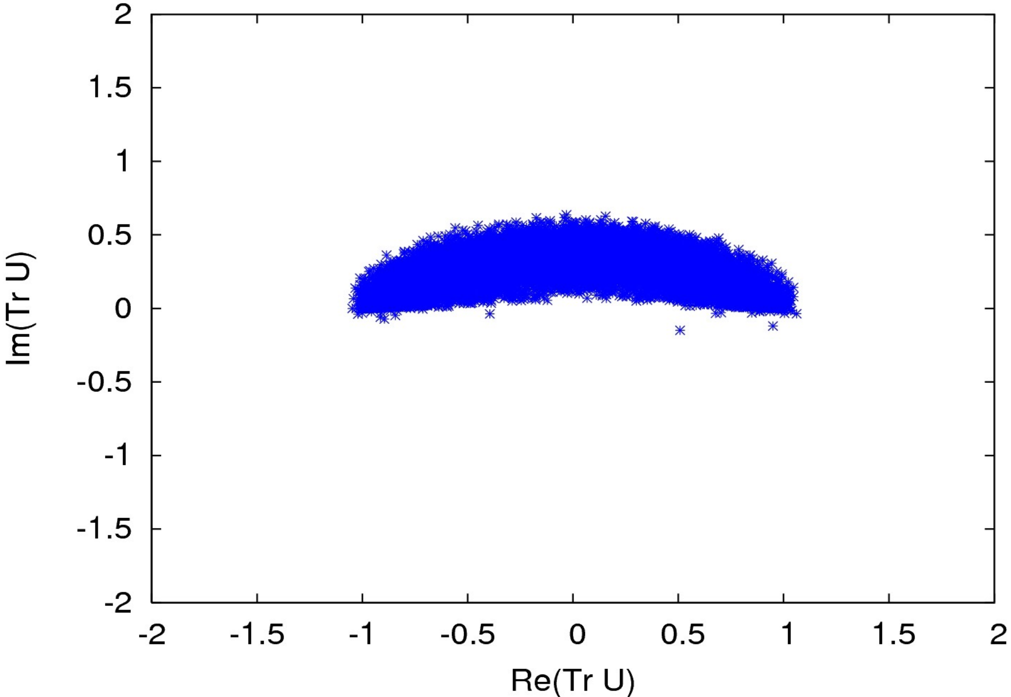
$U = a \mathbf{1} + i \sqrt{1 - a^2}\, \sigma_3 \qquad b_i = (0, 0, \sqrt{1 - a^2})$

SU(2) one-plaquette model

Distributions of Tr(U) on the complex plane



Without gaugefixing



With gaugefixing

Exact result from integration: $\langle Tr\, U \rangle = i$ ٠.٢٦١١

From simulation:

$$(-0.02 \pm 0.02) + i(-0.01 \pm 0.02)$$

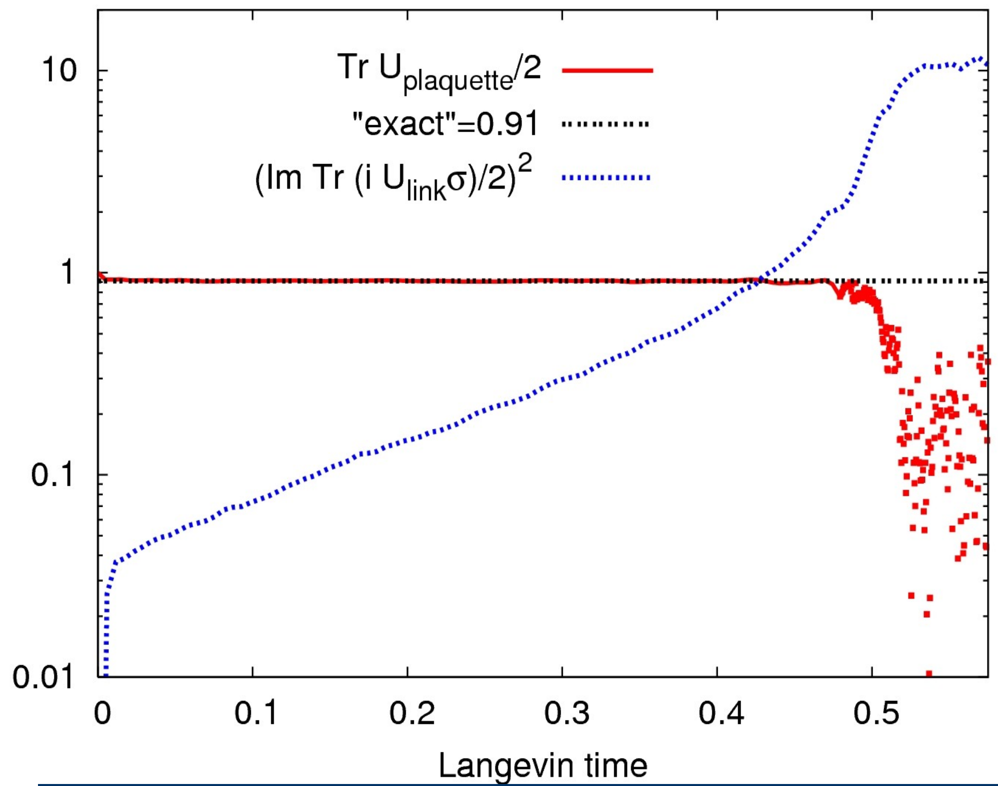
$$(-0.004 \pm 0.006) + i(0.260 \pm 0.001)$$

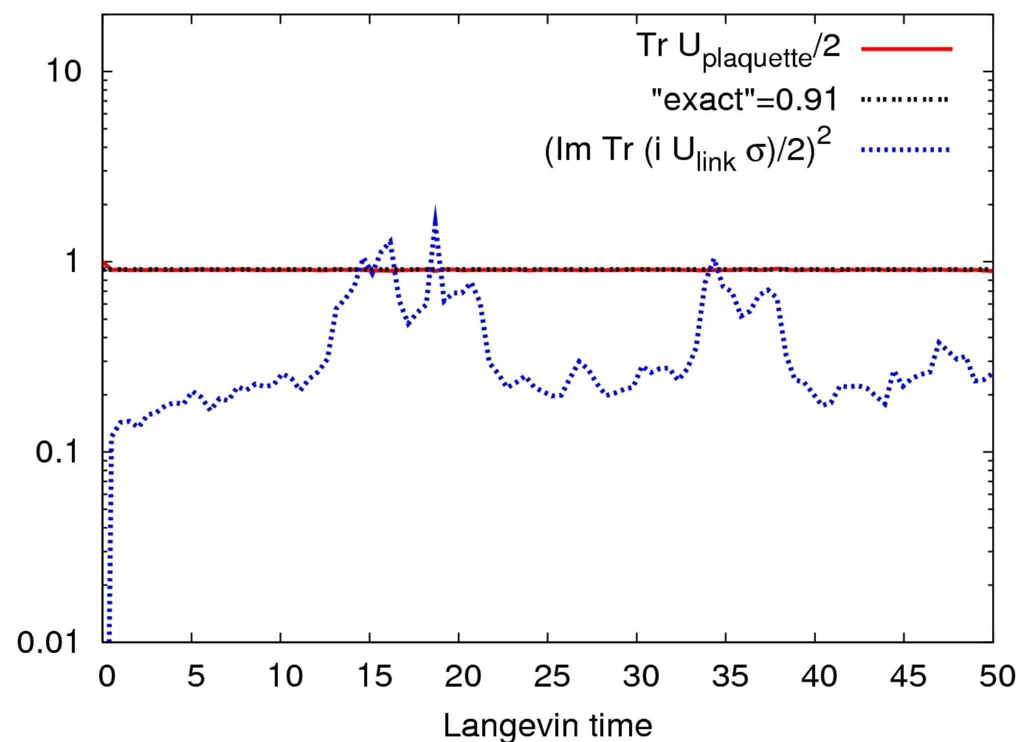
With gauge fixing, all averages are correctly reproduced

SU(2) field theory

$(Im\,Tr\,U)^2$ measures size of distribution

Without gauge fixing
non physical fixed point





Gauge fixing
small lattice coupling → large β

Correct result stabilizes

However:

Lattice coupling $g = 0.5$

Scaling region $g \geq 1$

Applications to non-zero chemical potential

Investigated models:

U(1) and SU(3) one plaquette models with fermion determinant
QCD with fermions in heavy quark approximation (only time hopping)

Aarts and Stamatescu '08

Bose Gas at finite chemical potential

Aarts '08

Method: $Z = \int DU\, e^{-S_B} \det M \quad \rightarrow \quad Z = \int DU\, e^{-S_B + \ln \det M}$

Langevin equation for the gauge fields from the complex action: $S_{eff} = -S_B + \ln \det M$

Encouraging results so far

still needs to be done: full QCD

or as first step: Thirring model

Simple model of QCD with finite chemical potential

Euclidean U(1) One plaquette model with "fermion determinant"

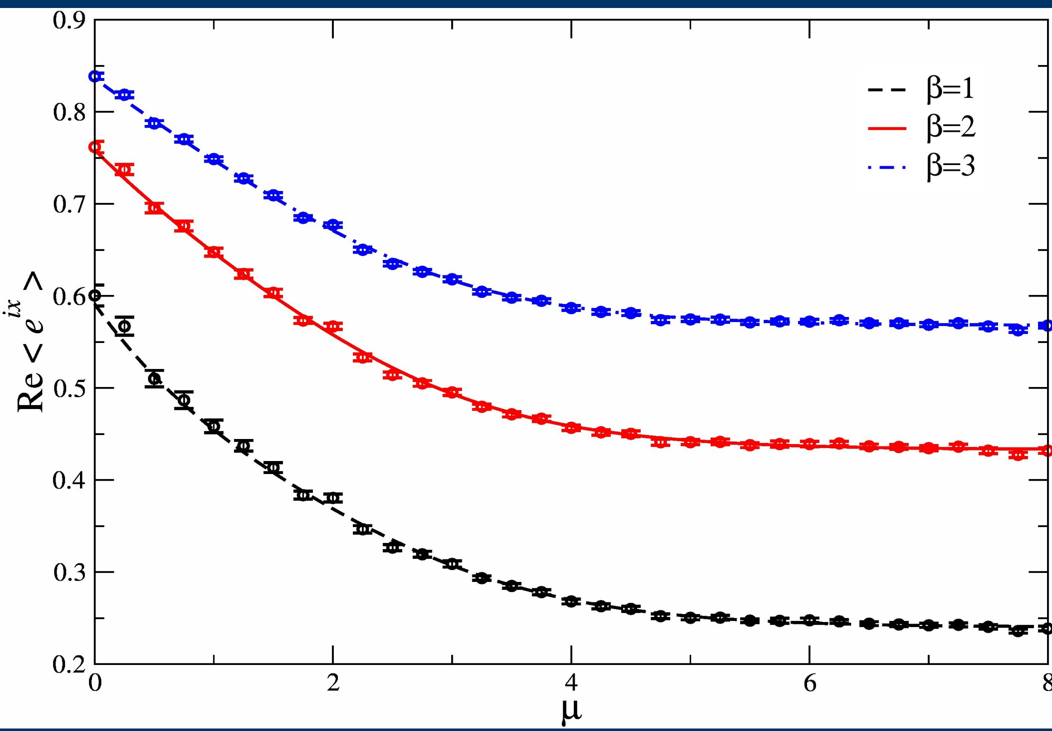
$$Z=\int_0^{2\pi} dx\, e^{-S_B} \det M \qquad S_B=-\frac{\beta}{2}(U+U^{-1})=-\beta\cos(x) \qquad U=e^{ix}$$

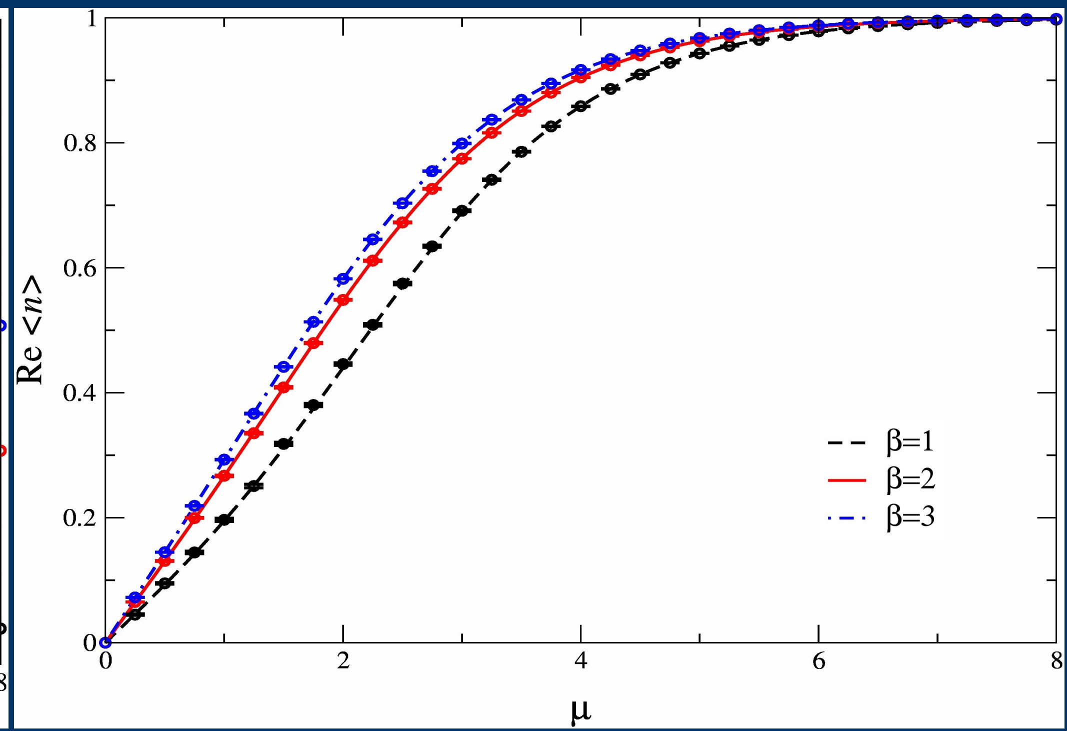
$$\det M=1+\frac{1}{2}\kappa(e^{\mu}U+e^{-\mu}U^{-1})=1+\kappa\cos(x-i\mu)$$

Similar to QCD fermion determinant:

$$\det M(\mu)=[\det M(-\mu)]^* \qquad \det M(i\mu_I) \quad \text{is real}$$

Exact averages calculated by numerical integration





Fixedpoint structure

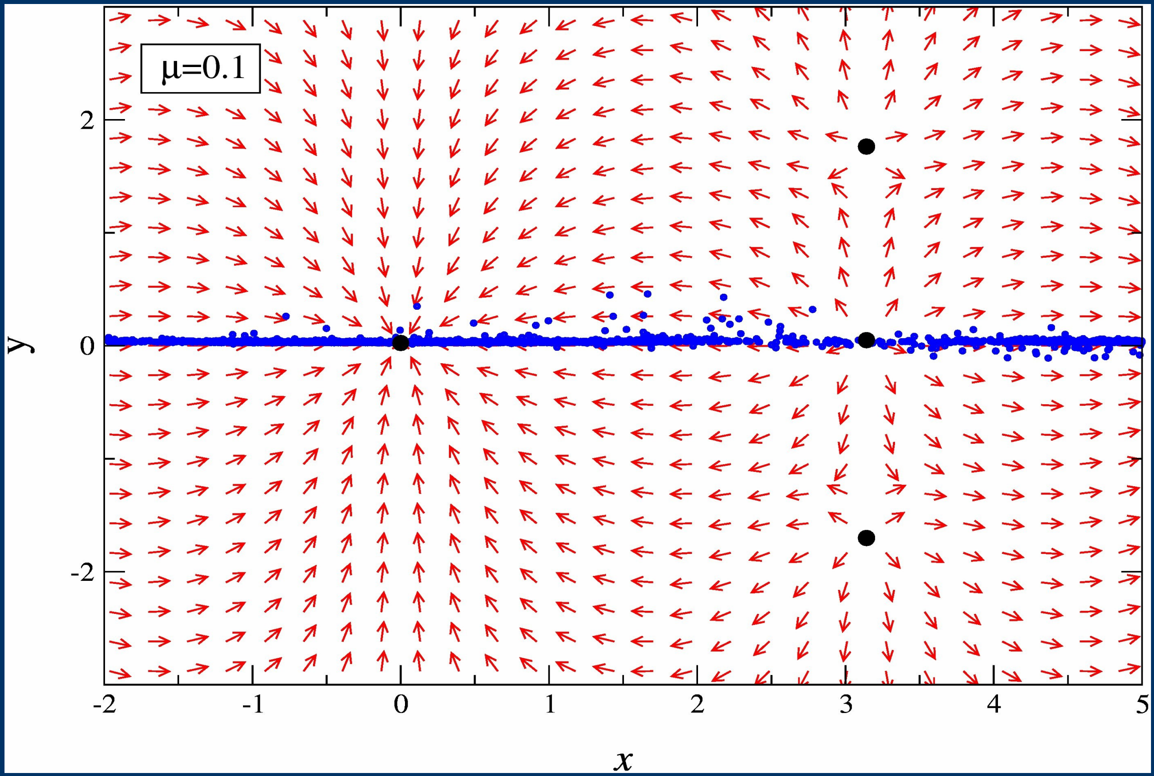
Distribution centered
around attractive fixedpoints
of the flow

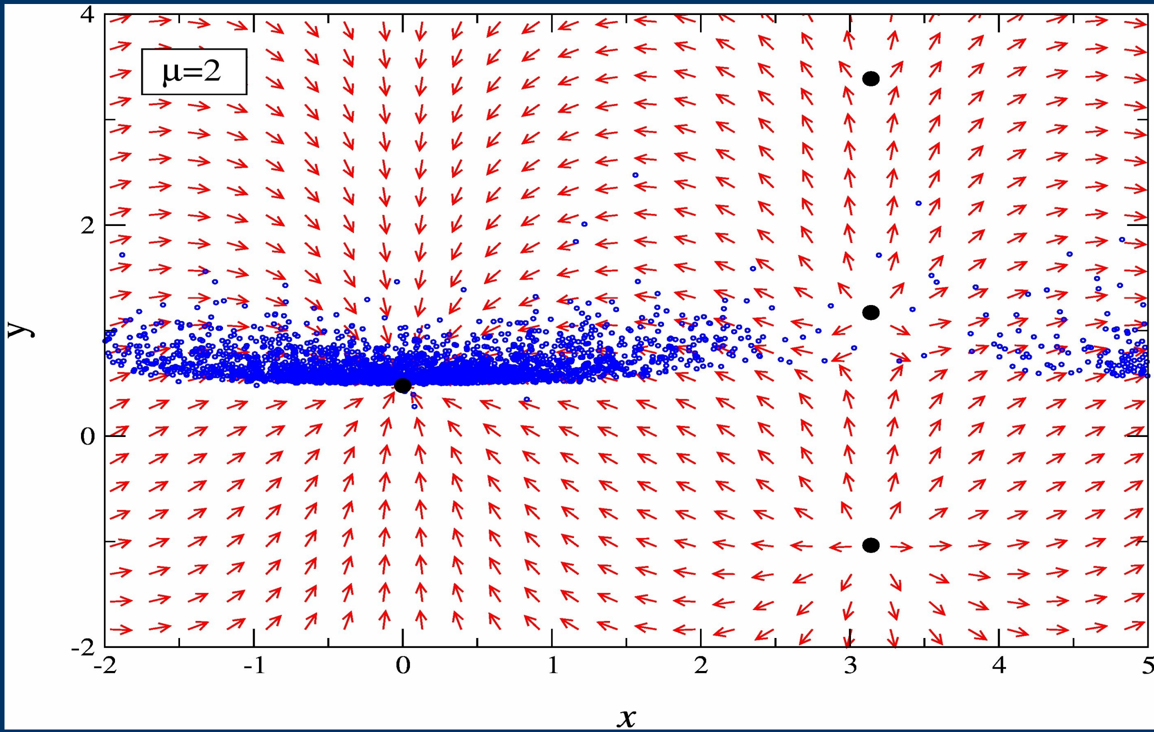
μ grows

↓

Fixed points move
no change in analytical structure

No breakdown,
Langevin works for high μ

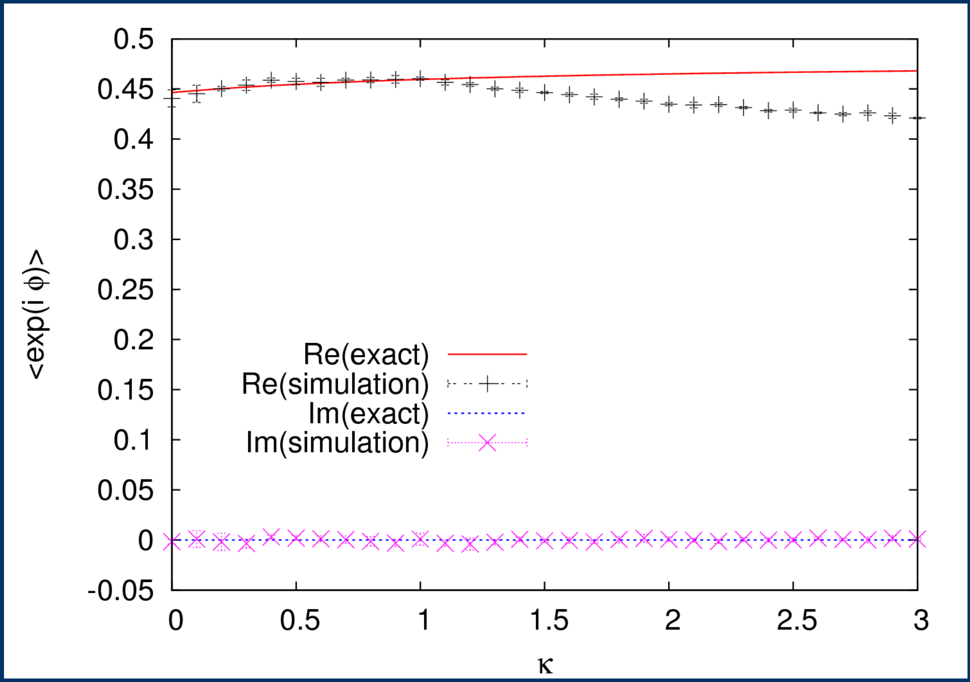


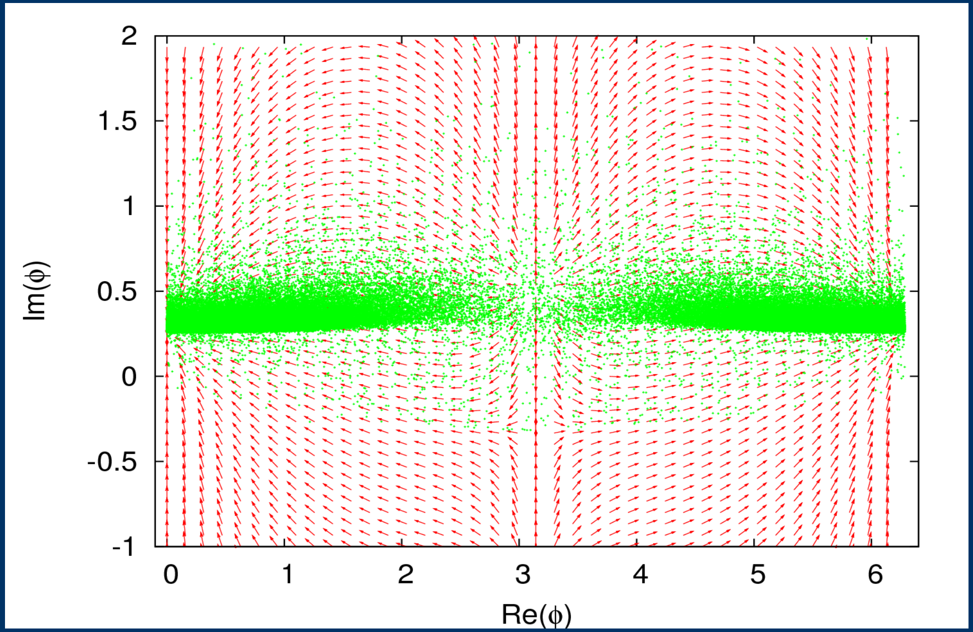


Breakdown at high kappa

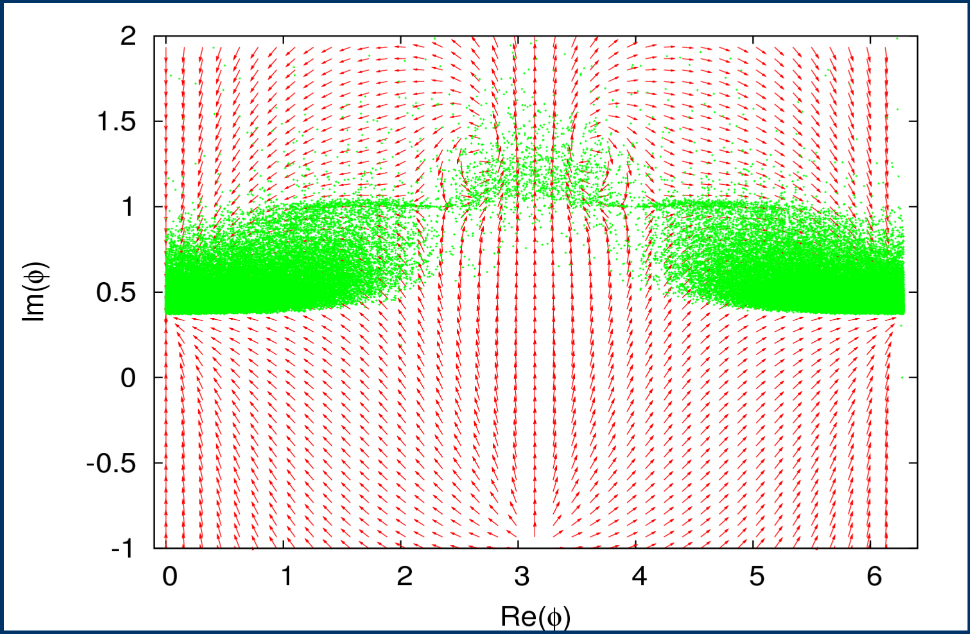
At $\kappa=1$ the fixedpoint structure changes

A new attractive fixedpoint sucks away part of the distribution





$\kappa=0.5$



$\kappa=1.0$

Bose Gas at zero temperature

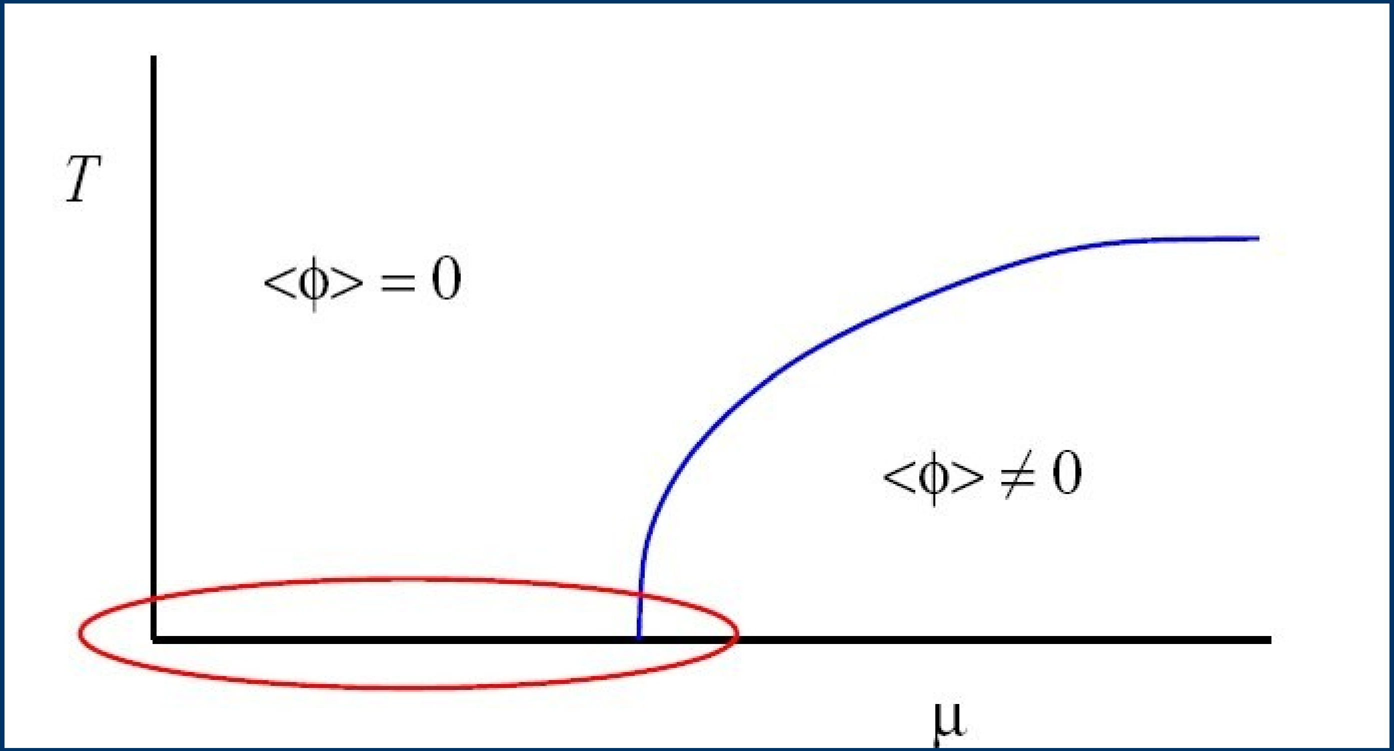
Silver Blaze problem:

No dependence on chemical potential for $\mu < m$

$$S = |\partial_\nu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu(\bar{\phi}\partial_4\phi - \partial_4\bar{\phi}\phi) + \lambda|\phi|^4$$

Complexification with Langevin method:

Complex scalar field → Two real fields → Two complex fields

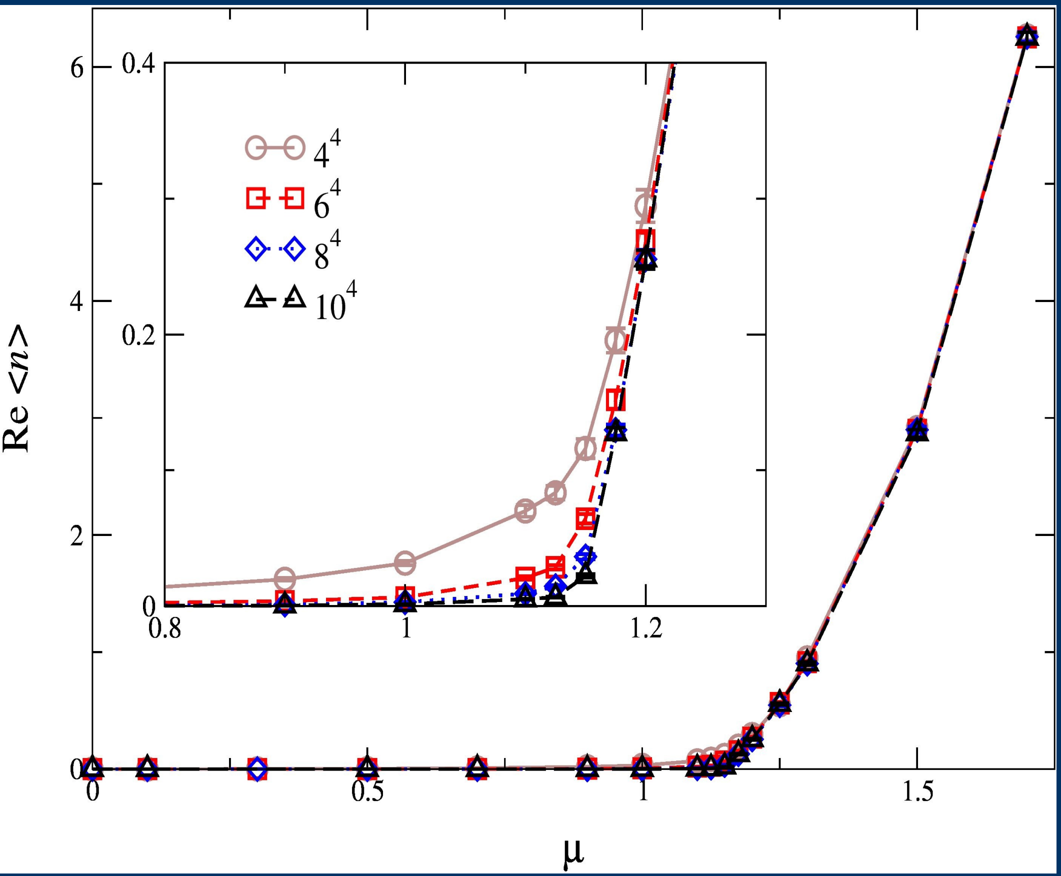


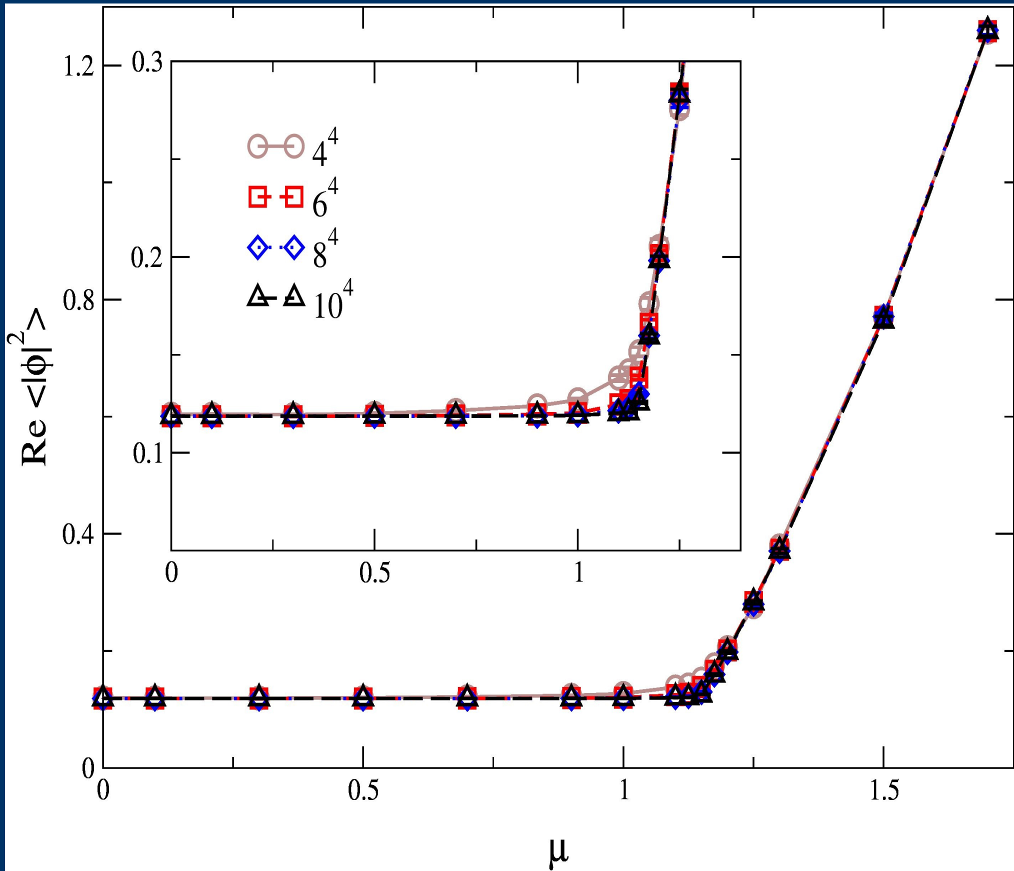
Silver Blaze:

Cancellations should destroy the apparent chemical potential dependence

$$\langle n \rangle = \frac{1}{\Omega} \frac{\partial \ln Z}{\partial \mu}$$

$$|\phi|^2 = \sum_{a=1,2} \left((\phi_a^R)^2 - (\phi_a^I)^2 + 2i\,\phi_a^R \phi_a^I \right)$$





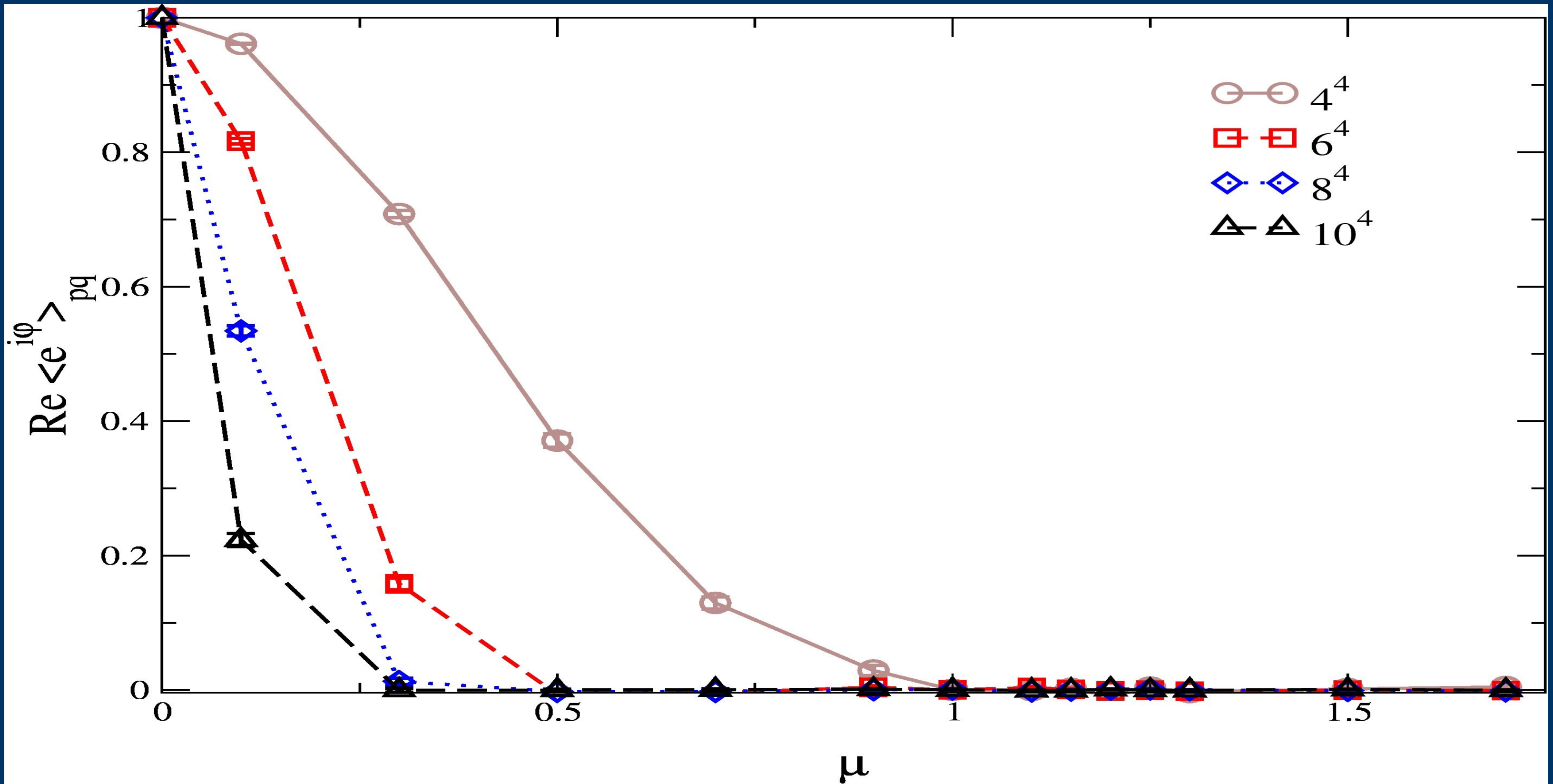
Sign problem

Complex action: $Z = \int D\phi\, e^{-S} = \int D\phi\, |e^{-S}|\, e^{i\varphi}$

Phase quenched theory $Z = \int D\phi\, |e^{-S}|$

Average phase factor $\langle e^{i\varphi} \rangle_{pq} = \frac{Z_{full}}{Z_{pq}} = e^{-\Omega \Delta f} \to 0$

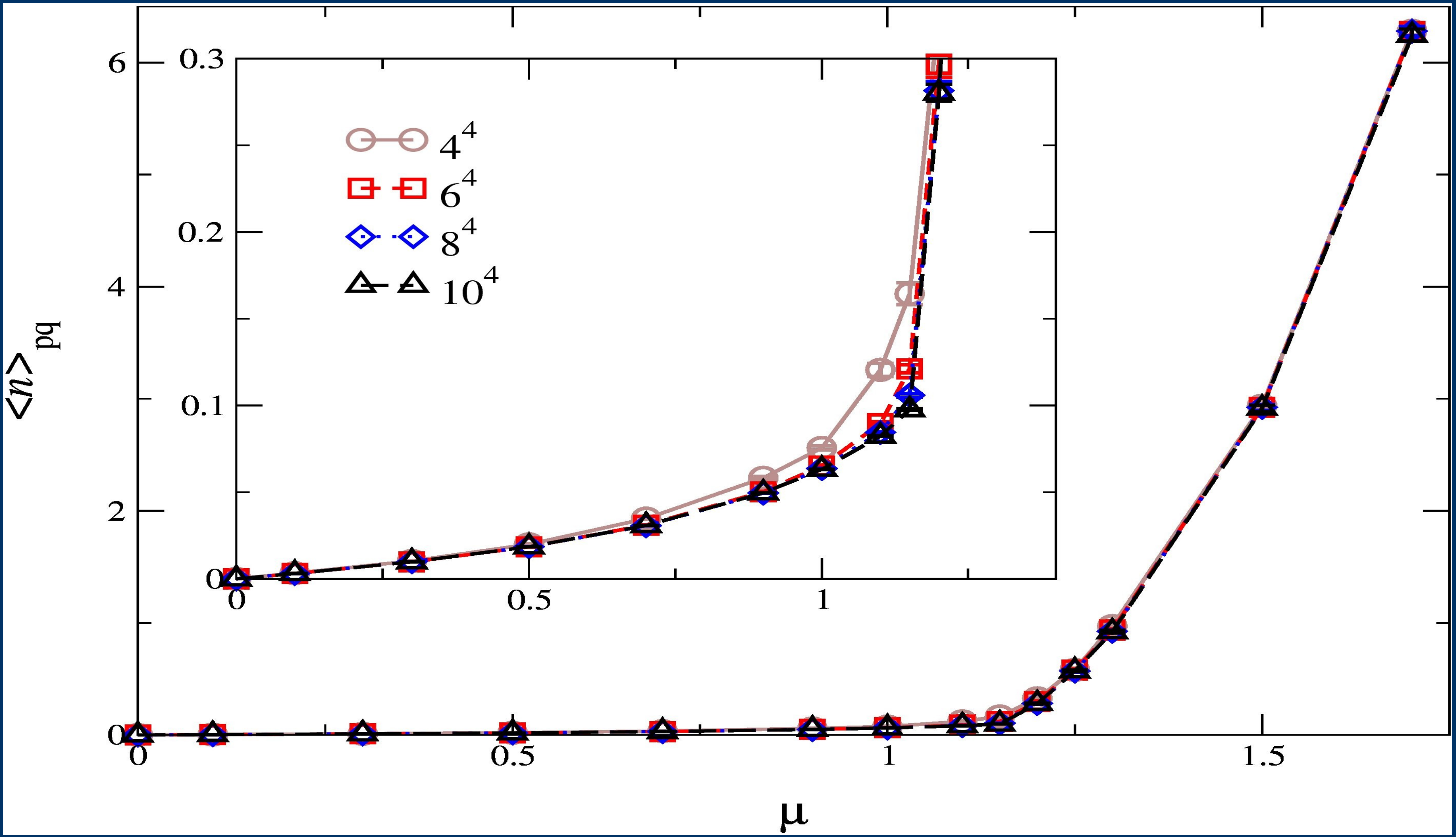
Sign problem is exponentially bad in thermodinamic limit



Phase quenched action $V(|\phi|)=\frac{1}{2}(m^2-\mu^2)|\phi|^2+\lambda|\phi|^4$

Also shows dependence on chemical potential

Phase factor is crucial for cancellation



Conclusions

Real-time lattice results

Without optimization: short real time simulation of scalar oscillator in equilibrium and non-eq. gives correct results (Schrodinger)

Langevin method: Schwinger Dyson equation solver

Optimization methods to reduce fluctuations:
- (reweighting)
- gaugefixing
- using small lattice-coupling

Method gives physical solution for SU(2) lattice gauge theory

Non-zero chemical potential

Encouraging results for
- U(1), SU(3) one plaquette models
- Bose Gas: sign problem of MC approach evaded