Universal Range Effects to Efimov Features

Chen Ji

ECT* / INFN-TIFPA

EMMI RRTF, 02.06.2016

In Collaboration with: Bijaya Acharya, Eric Braaten,
Lucas Platter, Daniel Phillips

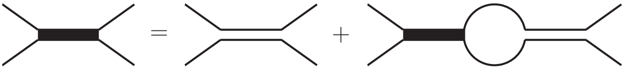
EFT for 3 identical bosons

- **LO $(r/a)^0$ EFT Lagrangian for 3 identical bosons**

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m} \right) \psi - d^\dagger \left(i\partial_0 + \frac{\nabla^2}{4m} - \Delta \right) d - \frac{g}{\sqrt{2}} \left(d^\dagger \psi\psi + \mathrm{h.c} \right) + h d^\dagger d \psi^\dagger \psi + \cdots$$

  - terms with more derivatives are at higher orders $(r/a)^n$

- **Non-perturbative features at LO**
  - atom-atom (dimer) scattering (tune g)



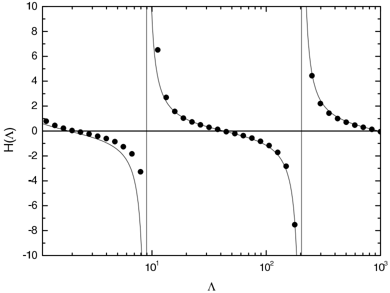
$\propto \frac{1}{1/a+ik}$

  - atom-dimer scattering (tune h)

Bedaque, Hammer, van Kolck '99

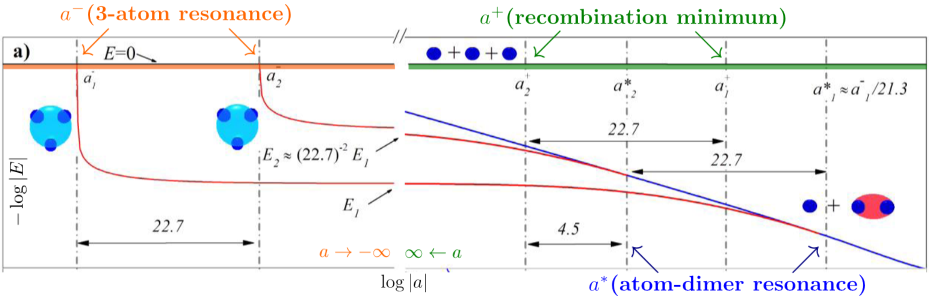
LO renormalization

- **LO 3BF h:**
  - tune $H(\Lambda) = \Lambda^2 h/2mg^2$:
    fix one 3-body observable
  - limit cycle:
    $H(\Lambda)$ periodic for $\Lambda \to \Lambda(22.7)^n$
    Bedaque *et al.* '00
  - scaling invariance $\to$ Efimov physics
    Efimov '71



Universal physics at LO

- **Universal features in three-body systems (Efimov effects)**
  - 3-body spectrum: a function of scattering length a
- **geometric spectrum**
  - $E_n = (22.7)^{-2} E_{n-1}$ in the limit $a \to \infty$
- **universal relation of recombination features**
  - $a^* = a^+/4.5 = -a^-/21.3$
  - *i.e.* $a^-_{(n)} = 22.7 a^-_{(n-1)}$



Zaccanti *et al.* '09

Range effects in EFT

- range effects on universal physics
  - 2-body observable: $k \cot \delta_0 = -\frac{1}{a} + \frac{r}{2} k^2 + \cdots$
  - 3-body observables: $\rightarrow$ in r/a expansion

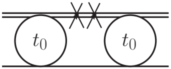
- r/a corrections (fixed a):
  - Hammer, Mehen '01 (NLO)
  - Bedaque, Rupak, Griesshammer, Hammer '03 (N^2LO, partial resummation)
  - Platter, Phillips '06 (N^2LO, partial resummation)
  - CJ, Phillips '13 (N^2LO, full perturbation)

- r/a corrections (variable a):
  - Platter, CJ, Phillips '09 (NLO, partial resummation)
  - CJ, Platter, Phillips '10; '12 (NLO, full perturbation)

N^{2}LO $(r/a)^2$ range effects (fixed a)

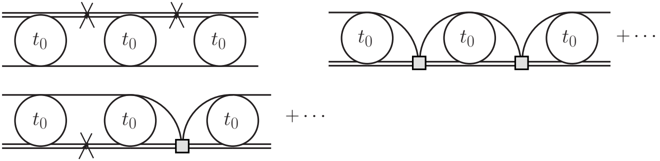
- N^{2}LO corrections to atom-dimer scattering amplitude:
  - in 2$^{\text{nd}}$ order perturbation theory ($\sim r^2/a^2$):

N^{2}LO dimer:



N^{2}LO 3BF:

two NLO terms:



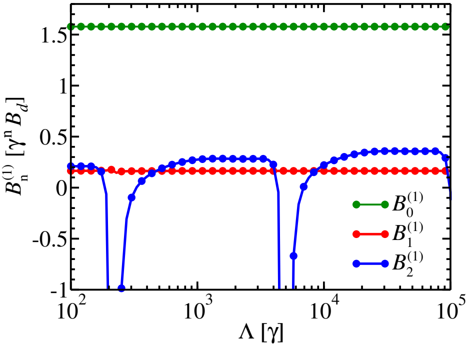
- N^{2}LO 3-body force:
  $H_2(E,\Lambda) = r^2\Lambda^2\ h_{20}(\Lambda) + r^2 m E_3\ h_{22}(\Lambda)$
  $\rightarrow$ one additional 3-body input is needed

CJ, Phillips FBS '13

c.f. Bedaque *et al.* '03 & Platter, Phillips '06

N^{2}LO Renormalization in Helium Trimers

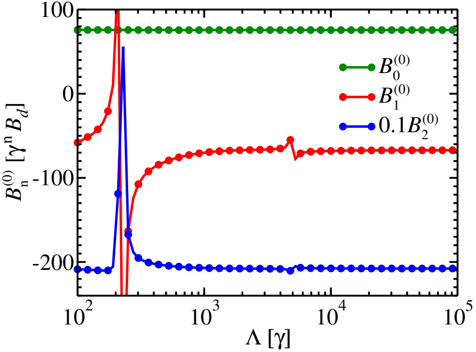
- $H_2 = r^2\Lambda^2 \; h_{20}$



$a_{ad} \rightarrow$ LO/NLO/N^{2}LO

$B_t^{(1)} = B_0^{(1)} + rB_1^{(1)} + r^2 B_2^{(1)}$

- $H_2 = r^2\Lambda^2 \; h_{20} + r^2 m E_t \, h_{22}$



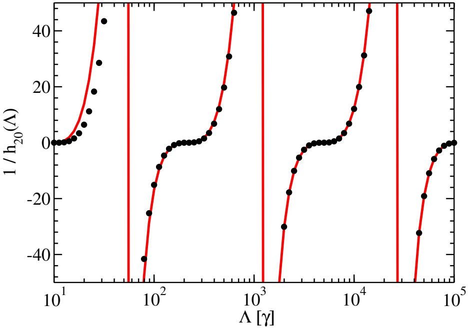
$a_{ad} \rightarrow$ LO/NLO/N^{2}LO

$B_t^{(1)} \rightarrow$ N^{2}LO

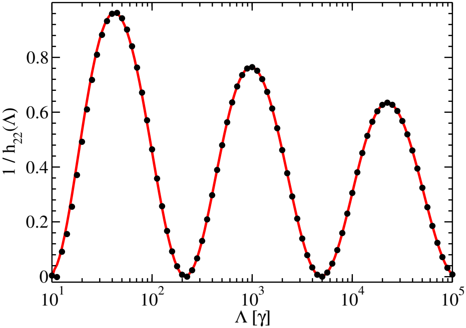
$B_t^{(0)} = B_0^{(0)} + rB_1^{(0)} + r^2 B_2^{(0)}$

N^{2}LO three-body 3BF

- $H_2(\Lambda) = r^2\Lambda^2\ h_{20}(\Lambda) + r^2 m E_t\ h_{22}(\Lambda)$



$1/h_{20}(\Lambda)$



$1/h_{22}(\Lambda)$ [partial version]

^{4}He trimer

Input	$B_t^{(1)}$ [B_d]	$B_t^{(0)}$ [B_d]	a_{ad} [γ^{-1}]	r_{ad} [γ^{-1}]
TTY potential	1.738	96.33	1.205	

CJ, Phillips FBS '13

^{4}He trimer

Input		$B_t^{(1)}$ $[B_d]$	$B_t^{(0)}$ $[B_d]$	a_{ad} $[\gamma^{-1}]$	r_{ad} $[\gamma^{-1}]$
TTY potential		1.738	96.33	1.205	
a_{ad}	LO	1.723	97.12	1.205	0.8352
a_{ad}	NLO	1.736	89.72	1.205	0.9049
a_{ad}, $B_t^{(1)}$	N^{2}LO	1.738	116.9	1.205	0.9132

CJ, Phillips FBS '13

^{4}He trimer

Input		$B_t^{(1)}$ $[B_d]$	$B_t^{(0)}$ $[B_d]$	a_{ad} $[\gamma^{-1}]$	r_{ad} $[\gamma^{-1}]$
TTY potential		1.738	96.33	1.205	
a_{ad}	LO	1.723	97.12	1.205	0.8352
a_{ad}	NLO	1.736	89.72	1.205	0.9049
a_{ad}, $B_t^{(1)}$	N^2LO	1.738	116.9	1.205	0.9132
$B_t^{(1)}$	LO	1.738	99.37	1.178	0.8752
$B_t^{(1)}$	NLO	1.738	89.77	1.201	0.9130
$B_t^{(1)}$, a_{ad}	N^2LO	1.738	115.9	1.205	0.9135

CJ, Phillips FBS '13

^{4}He trimer

Input		$B_t^{(1)}$ $[B_d]$	$B_t^{(0)}$ $[B_d]$	a_{ad} $[\gamma^{-1}]$	r_{ad} $[\gamma^{-1}]$
TTY potential		1.738	96.33	1.205	
a_{ad}	LO	1.723	97.12	1.205	0.8352
a_{ad}	NLO	1.736	89.72	1.205	0.9049
a_{ad}, $B_t^{(1)}$	N^{2}LO	1.738	116.9	1.205	0.9132
$B_t^{(1)}$	LO	1.738	99.37	1.178	0.8752
$B_t^{(1)}$	NLO	1.738	89.77	1.201	0.9130
$B_t^{(1)}$, a_{ad}	N^{2}LO	1.738	115.9	1.205	0.9135

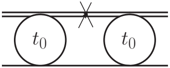
- **Difference in 2 renormalization schemes (LO→NLO→N^{2}LO):**
  - atom-dimer effective range r_{ad}: 5% → 0.9% → 0.02%
  - ground-state trimer $B_t^{(0)}$: 2% → 0.07% → 0.9%

CJ, Phillips FBS '13

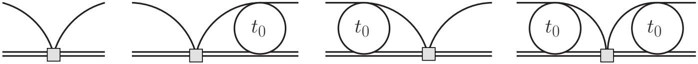
NLO (r/a) range effects

- **Calculate r/a correction to atom-dimer amplitude**

NLO dimer:



NLO 3BF:



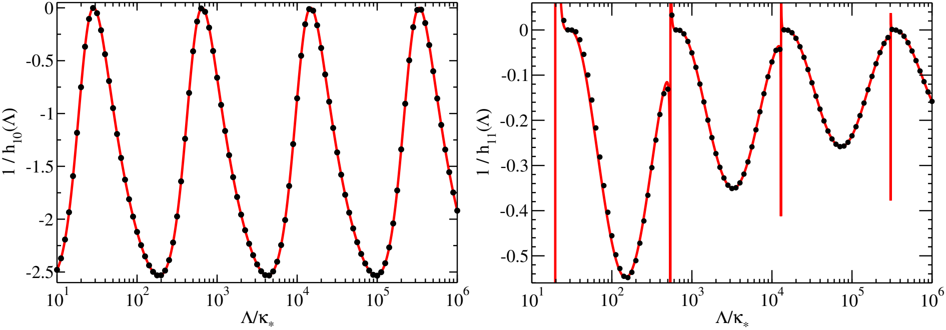
- **NLO 3-body force:**

$$H_1(\Lambda) = r\Lambda\ h_{10}(\Lambda) + r/a\ h_{11}(\Lambda)$$

  - a **fixed:** h_{11} is absorbed (no additional 3-body input is needed)
  - a **varies:** one additional 3-body input is needed

NLO three-body 3BF

- $H_1(\Lambda) = r\Lambda\ h_{10}(\Lambda) + r/a\ h_{11}(\Lambda)$



$1/h_{10}(\Lambda)$ $1/h_{11}(\Lambda)$

Recombination of ^{7}Li Atoms

	Experiment[†‡]	LO	LO	NLO[⋆]
3A res $a_{-,0}$ $[a_B]$	-264[†]	-264	-244	-264
rec min $a_{+,1}$ $[a_B]$	1160[†]	1254	1160	1160
Ad res $a_{*,1}$ $[a_B]$	180[‡]	281	259	210(44)

† Gross *et al.* '09 ‡ Machtey *et al.* '12 ⋆ Ji, Phillips, Platter '10

Universal Relations at NLO

- recombination features are correlated by

$$a_{i,n} = \lambda^n \theta_i \kappa_*^{-1} + (J_i + n\,\sigma) r; \qquad i = *, +, -; \quad \lambda = 22.694$$

- J_i is non-universal but $J_i - J_j$ is a universal number
  - *e.g.* $J_+ - J_- = \sigma/2$, $\quad \sigma = 1.095$
- $\kappa_* r$ and J_i can be fixed by the ratio of two Efimov features
  - *e.g.* fix $(a_{-,1}/a_{-,0})/\lambda$, $(a_{-,2}/a_{-,1})/\lambda$
    deviate from 1 due to range effects
- predict ratios of any other features $(a_{i,n+1}/a_{i,n})/\lambda$

CJ, Braaten, Phillips, Platter, PRA 2015

Three-Body Force (3BF) at NLO

- LO 3BF: RG limit cycle → discrete scaling symmetry λ^n
- NLO 3BF: RG range-modification → discrete scaling breaking $n\sigma$

- 3BF up to NLO can be divided into 4 different contributions

$$H(\Lambda) = H_0(\Lambda/\Lambda_*) + h_{10}(\Lambda/\Lambda_*)\Lambda r + \left[\eta\, H_0'(\Lambda/\Lambda_*)\ln(\Lambda/\mu) + \tilde{h}_{11}(\Lambda/\Lambda_*)\right]\frac{r}{a}$$

with $H_0' = (\Lambda d/d\Lambda)H_0$

Three-Body Force (3BF) at NLO

- LO 3BF: RG limit cycle $\rightarrow$ discrete scaling symmetry λ^n
- NLO 3BF: RG range-modification $\rightarrow$ discrete scaling breaking $n\sigma$

- 3BF up to NLO can be divided into 4 different contributions

$$H(\Lambda) = \underbrace{H_0(\Lambda/\Lambda_*)}_{\text{log periodic}} + \underbrace{h_{10}(\Lambda/\Lambda_*)\Lambda r}_{\text{linear divergence}} + \left[\underbrace{\eta\, H_0'(\Lambda/\Lambda_*)\ln(\Lambda/\mu)}_{\text{log divergence}} + \underbrace{\tilde{h}_{11}(\Lambda/\Lambda_*)}_{\text{log periodic}}\right]\frac{r}{a}$$

with $H_0' = (\Lambda d/d\Lambda)H_0$

Three-Body Force (3BF) at NLO

- LO 3BF: RG limit cycle $\rightarrow$ discrete scaling symmetry λ^n
- NLO 3BF: RG range-modification $\rightarrow$ discrete scaling breaking $n\sigma$

- 3BF up to NLO can be divided into 4 different contributions

$$H(\Lambda) = \underbrace{H_0(\Lambda/\Lambda_*)}_{\text{log periodic}} + \underbrace{h_{10}(\Lambda/\Lambda_*)\Lambda r}_{\text{linear divergence}} + \left[\underbrace{\eta\, H_0'(\Lambda/\Lambda_*)\ln(\Lambda/\mu)}_{\text{log divergence}} + \underbrace{\tilde{h}_{11}(\Lambda/\Lambda_*)}_{\text{log periodic}}\right]\frac{r}{a}$$

with $H_0' = (\Lambda d/d\Lambda)H_0$

- Rewrite 3BF

$$\begin{aligned} H(\Lambda) &= H_0\left[\ln(\Lambda/\Lambda_*)\right] + \frac{r}{a}\eta\, H_0'\left[\ln(\Lambda/\Lambda_*)\right]\,\ln(\Lambda/\Lambda_*) \\ &= H_0\left[(1+\eta r/a)\ln(\Lambda/\Lambda_*)\right] \\ &= H_0\left[\ln(\Lambda/\Lambda_*)^{1+\eta r/a}\right] \end{aligned}$$

Three-Body Force (3BF) at NLO

- LO 3BF: RG limit cycle → discrete scaling symmetry λ^n
- NLO 3BF: RG range-modification → discrete scaling breaking $n\sigma$

- 3BF up to NLO can be divided into 4 different contributions

$$H(\Lambda) = \underbrace{H_0(\Lambda/\Lambda_*)}_{\text{log periodic}} + \underbrace{h_{10}(\Lambda/\Lambda_*)\Lambda r}_{\text{linear divergence}} + \left[\underbrace{\eta\, H_0'(\Lambda/\Lambda_*)\ln(\Lambda/\mu)}_{\text{log divergence}} + \underbrace{\tilde{h}_{11}(\Lambda/\Lambda_*)}_{\text{log periodic}}\right]\frac{r}{a}$$

with $H_0' = (\Lambda d/d\Lambda)H_0$

- Rewrite 3BF

$$\begin{aligned} H(\Lambda) &= H_0\left[\ln(\Lambda/\Lambda_*)\right] + \frac{r}{a}\eta\, H_0'\left[\ln(\Lambda/\Lambda_*)\right]\ln(\Lambda/\Lambda_*) \\ &= H_0\left[(1+\eta r/a)\ln(\Lambda/\Lambda_*)\right] \\ &= H_0\left[\ln(\Lambda/\Lambda_*)^{1+\eta r/a}\right] \end{aligned}$$

- Running 3B parameter at NLO: $\kappa_*(Q,a) = (Q/\kappa_*)^{-\eta r/a}\kappa_*$

RG Improvement

- insert running parameter into LO universal relation
- leads to Renormalization-Group Improved universal relations

$$a_{i,n} = \lambda^n \theta_i \left(\lambda^n |\theta_i|\right)^{\eta r \kappa_* / (\lambda^n \theta_i)} \kappa_*^{-1} + \tilde{J}_i r$$

RG Improvement

- insert running parameter into LO universal relation
- leads to Renormalization-Group Improved universal relations

$$a_{i,n} = \lambda^n \theta_i \left(\lambda^n |\theta_i|\right)^{\eta r \kappa_* / (\lambda^n \theta_i)} \kappa_*^{-1} + \tilde{J}_i r$$

- expand $a_{i,n}$ up to linear-in-r correction

$$\text{c.f.} \quad a_{i,n} = \lambda^n \theta_i \kappa_*^{-1} + (J_i + n\,\sigma) r$$

$$\text{requires} \quad \sigma = \eta\pi / s_0$$

RG Improvement

- insert running parameter into LO universal relation
- leads to Renormalization-Group Improved universal relations

$$a_{i,n} = \lambda^n \theta_i \left(\lambda^n |\theta_i|\right)^{\eta r \kappa_* / (\lambda^n \theta_i)} \kappa_*^{-1} + \tilde{J}_i r$$

- expand $a_{i,n}$ up to linear-in-r correction

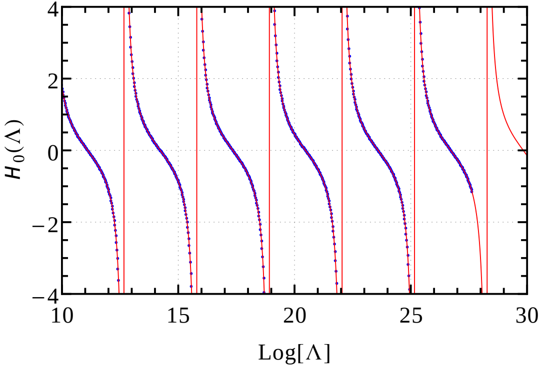
$$\text{c.f.} \quad a_{i,n} = \lambda^n \theta_i \kappa_*^{-1} + (J_i + n\,\sigma) r$$

$$\text{requires} \quad \sigma = \eta\pi/s_0$$

- verify η by calculating it from three-body force

LO 3BF

$$H_0(\Lambda) = c_K \frac{\sin[s_0 \log(\Lambda/\Lambda_*) + \tan^{-1} s_0]}{\sin[s_0 \log(\Lambda/\Lambda_*) - \tan^{-1} s_0]}$$

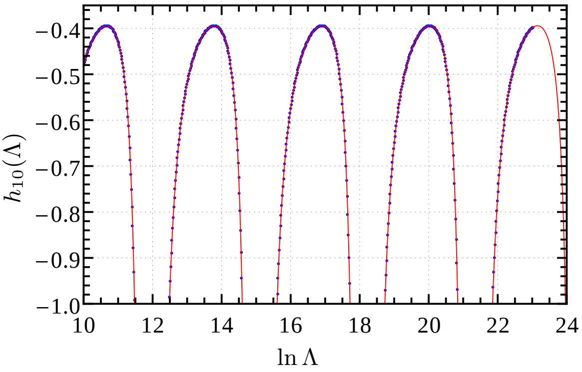


$c_K = 0.879$ due to corrections from regulator effects

NLO 3BF ($\Lambda_{NLO} \ll \Lambda_{LO}$)

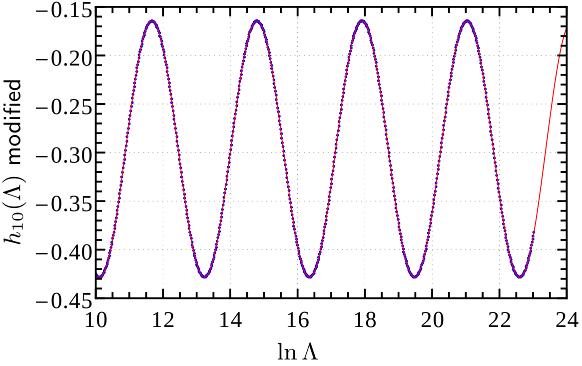
$\Lambda_{NLO} \ll \Lambda_{LO}$ to get rid of remaining regulator effects

$$h_{10}(\Lambda) = -\frac{3\pi(1+s_0^2)}{64\sqrt{1+4s_0^2}} \frac{\sqrt{1+4s_0^2} - \cos\left(2s_0 \ln\frac{\Lambda}{\Lambda_*} - \tan^{-1} 2s_0\right)}{\sin^2\left(s_0 \ln\frac{\Lambda}{\Lambda_*} - \tan^{-1} s_0\right)}$$

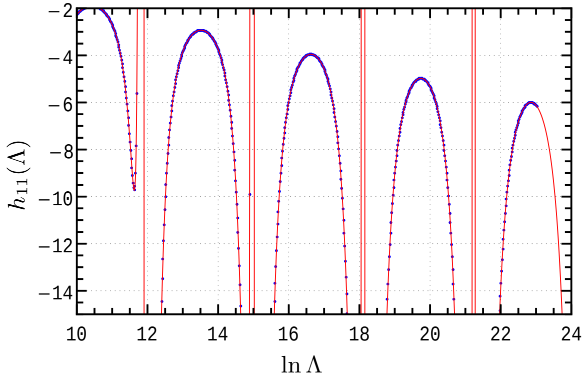


NLO 3BF ($\Lambda_{NLO} \ll \Lambda_{LO}$)

$$h_{10} \times \sin^2\left(s_0 \ln(\Lambda/\Lambda_*) - \tan^{-1} s_0\right)$$
$$= -\frac{3\pi(1+s_0^2)}{64\sqrt{1+4s_0^2}}\left[\sqrt{1+4s_0^2} - \cos\left(2s_0 \ln\frac{\Lambda}{\Lambda_*} - \tan^{-1} 2s_0\right)\right]$$



$$
\begin{aligned}
h_{11}(\Lambda) \quad = \quad & -\frac{\sqrt{3}\pi(1+s_0^2)}{16}\,\frac{(1+\mathrm{Re}\,C_1)}{\sin^2\left(s_0\ln(\Lambda/\Lambda_*)-\tan^{-1}s_0\right)}\ln\frac{\Lambda}{\mu} \\
& +\frac{\sqrt{3}\pi(1+s_0^2)}{32 s_0}\,\frac{\sin\left(2s_0\ln(\Lambda/\Lambda_*)\right)+|C_1|\sin\left(2s_0\ln(\Lambda/\Lambda_*)+\arg C_1\right)}{\sin^2\left(s_0\ln(\Lambda/\Lambda_*)-\tan^{-1}s_0\right)} \\
& -\frac{\sqrt{3}\pi(1+s_0^2)^{3/2}}{16 s_0}\,\frac{\cos\left(s_0\ln(\Lambda/\Lambda_*)\right)+|C_1|\cos\left(s_0\ln(\Lambda/\Lambda_*)+\arg C_1\right)}{\sin^3\left(s_0\ln(\Lambda/\Lambda_*)-\tan^{-1}s_0\right)} \\
& \times\left[1-\cos\left(2s_0\ln(\Lambda/\Lambda_*)-\tan^{-1}(2s_0)\right)/\sqrt{1+4s_0^2}\right]
\end{aligned}
$$



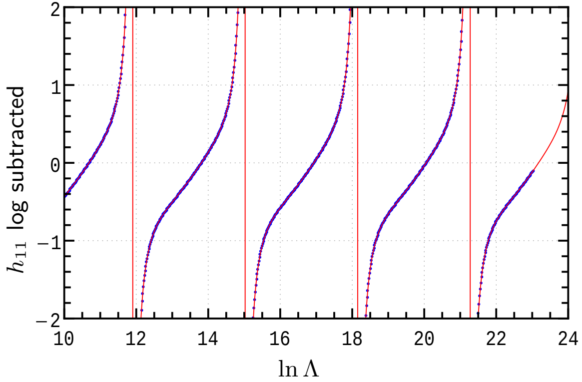
$\Lambda_{NLO} \ll \Lambda_{LO}$

$$[h_{11} - h_{11}(\log)] \times \sin^2\left(s_0 \ln(\Lambda/\Lambda_*) - \tan^{-1} s_0\right)$$

$$= \frac{\sqrt{3}\pi(1+s_0^2)}{32 s_0}\left[\sin\left(2s_0\ln(\Lambda/\Lambda_*)\right) + |C_1|\sin\left(2s_0\ln(\Lambda/\Lambda_*) + \arg C_1\right)\right]$$

$$- \frac{\sqrt{3}\pi(1+s_0^2)^{3/2}}{16 s_0}\ \frac{\cos\left(s_0\ln(\Lambda/\Lambda_*)\right) + |C_1|\cos\left(s_0\ln(\Lambda/\Lambda_*) + \arg C_1\right)}{\sin\left(s_0\ln(\Lambda/\Lambda_*) - \tan^{-1} s_0\right)}$$

$$\times \left[1 - \cos\left(2s_0\ln(\Lambda/\Lambda_*) - \tan^{-1}(2s_0)\right)/\sqrt{1+4s_0^2}\right]$$

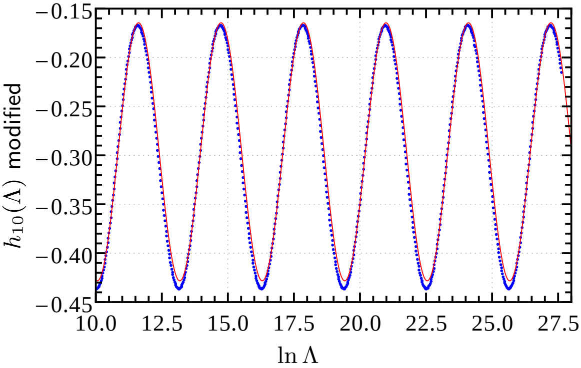


$\Lambda_{NLO} \ll \Lambda_{LO}$

NLO 3BF ($\Lambda_{NLO} = \Lambda_{LO}$)

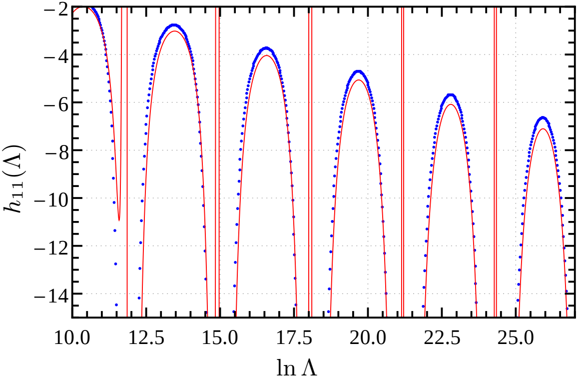
- $\Lambda_{NLO} = \Lambda_{LO}$ is required for analyzing running LO/NLO 3BF at the same time

$$h_{10} \times \sin^2\left(s_0 \ln(\Lambda/\Lambda_*) - \tan^{-1} s_0\right)$$



NLO 3BF ($\Lambda_{NLO} = \Lambda_{LO}$)

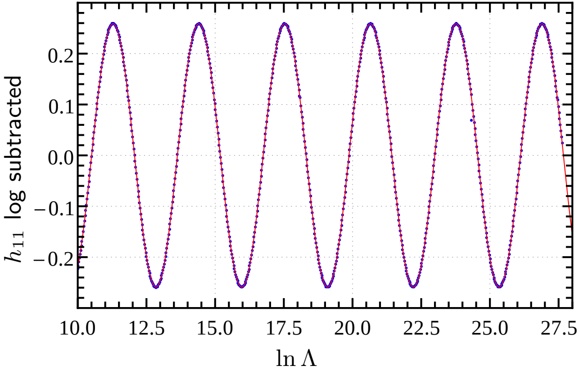
$h_{11}(\Lambda)$



NLO 3BF ($\Lambda_{NLO} = \Lambda_{LO}$)

$$\begin{aligned} & [h_{11} - d_K h_{11}(\log)] \times \sin^2\left(s_0 \ln(\Lambda/\Lambda_*) - \tan^{-1} s_0\right) \\ = \quad & -0.2588 \sin\left(2s_0 \ln(\Lambda/\Lambda_*) + \arg C_1 + \tan^{-1}(s_0/3)\right) \end{aligned}$$

$d_K = 0.949$ is the NLO regulator compensation factor



LO/NLO Three-Body Force

- LO 3BF

$$H_0(\Lambda) = c_K \frac{\sin[s_0 \log(\Lambda/\Lambda_*) + \tan^{-1} s_0]}{\sin[s_0 \log(\Lambda/\Lambda_*) - \tan^{-1} s_0]}$$

- NLO 3BF

$$\begin{aligned} h_{11} \log \text{ term} &= -d_K \frac{\sqrt{3}\pi(1+s_0^2)}{16} \frac{(1+\mathrm{Re}\, C_1)}{\sin^2[s_0 \ln(\Lambda/\Lambda_*) - \tan^{-1} s_0]} \ln \frac{\Lambda}{\mu} \\ &= \eta\, H_0'(\Lambda) \ln(\Lambda/\mu) \end{aligned}$$

$$\text{with } \eta = \frac{d_K}{c_K} \frac{\sqrt{3}\pi}{32} \left(\frac{s_0^2+1}{s_0}\right)^2 (1+\mathrm{Re}\, C_1) = 0.351$$

LO/NLO Three-Body Force

- LO 3BF

$$H_0(\Lambda) = c_K \frac{\sin[s_0 \log(\Lambda/\Lambda_*) + \tan^{-1} s_0]}{\sin[s_0 \log(\Lambda/\Lambda_*) - \tan^{-1} s_0]}$$

- NLO 3BF

$$\begin{aligned} h_{11} \log \text{ term} &= -d_K \frac{\sqrt{3}\pi(1+s_0^2)}{16} \frac{(1+\mathrm{Re}\, C_1)}{\sin^2[s_0 \ln(\Lambda/\Lambda_*) - \tan^{-1} s_0]} \ln \frac{\Lambda}{\mu} \\ &= \eta\, H_0'(\Lambda) \ln(\Lambda/\mu) \end{aligned}$$

$$\text{with } \eta = \frac{d_K}{c_K} \frac{\sqrt{3}\pi}{32} \left(\frac{s_0^2+1}{s_0}\right)^2 (1+\mathrm{Re}\, C_1) = 0.351$$

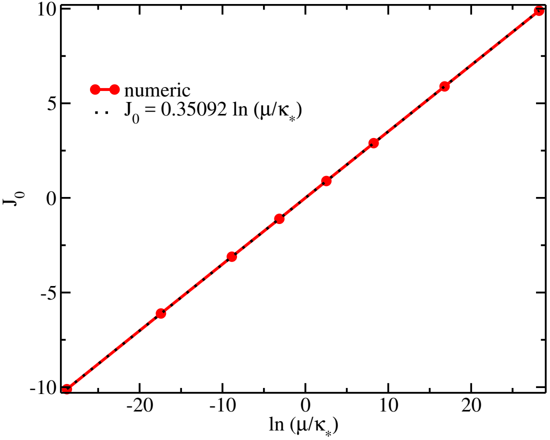
- therefore $\sigma = \eta\pi/s_0 = 1.095$
- running 3BF matches observation in NLO predictions

J_0 vs μ

- NLO universal relations: $a_{i,n} = \lambda^n \theta_i \kappa_*^{-1} + (J_i + n\sigma)r$
- NLO three-body parameter: $H_1 = \frac{r}{a} 0.351 H_0'(\Lambda) \ln(\Lambda/\mu) + \cdots$
- correlations exist between J_i and μ

J_0 vs μ

- NLO universal relations: $a_{i,n} = \lambda^n \theta_i \kappa_*^{-1} + (J_i + n\sigma)r$
- NLO three-body parameter: $H_1 = \frac{r}{a} 0.351 H_0'(\Lambda) \ln(\Lambda/\mu) + \cdots$
- correlations exist between J_i and μ



Benchmarking the Relations

- Compare the universal relations to calculations that employ finite range interaction
- **Deltuva**: Momentum space calculations with short-range separable interaction
- **Schmidt, Rath & Zwerger**: Two-channel model with 2 parameters and a form factor

Deltuva PRA 2012

$$(a_{-,n+1}/a_{-,n})/\lambda$$

n	0	1	2	3	4
Deltuva 2012	0.7822	0.9665	0.9976	0.9999	1.0000
NLO	0.7822	0.9665	0.9975	0.9998	1.0000
RG-NLO	0.7822	0.9665	0.9975	0.9998	1.0000

Schmidt, Rath & Zwerger 2012

$s_{\rm res} = 100$

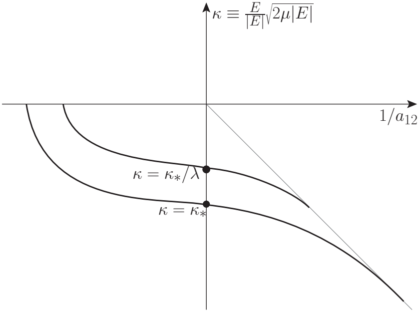
$(a_{-,n+1}/a_{-,n})/\lambda$				$(a_{*,n+1}/a_{*,n})/\lambda$		
n	***0***	***1***	***2***	***0***	***1***	***2***
Schmidt et al.	***0.753***	***0.962***	***0.998***	***0.175***	***1.764***	***1.029***
NLO	***0.753***	***0.962***	***0.997***	***-8.1***	***1.150***	***1.032***
RG-NLO	***0.753***	***0.962***	***0.997***	***0.0002***	***1.206***	***1.034***

$s_{\rm res} = 1$

$(a_{-,n+1}/a_{-,n})/\lambda$				$(a_{*,n+1}/a_{*,n})/\lambda$		
n	***0***	***1***	***2***	***0***	***1***	***2***
Schmidt et al.	***1.008***	***0.998***	***0.9998***	***0.757***	***0.983***	***1.001***
NLO	***1.008***	***0.998***	***0.9998***	***-0.431***	***0.986***	***1.002***
RG-NLO	***1.008***	***0.998***	***0.9998***	***0.240***	***0.986***	***1.002***

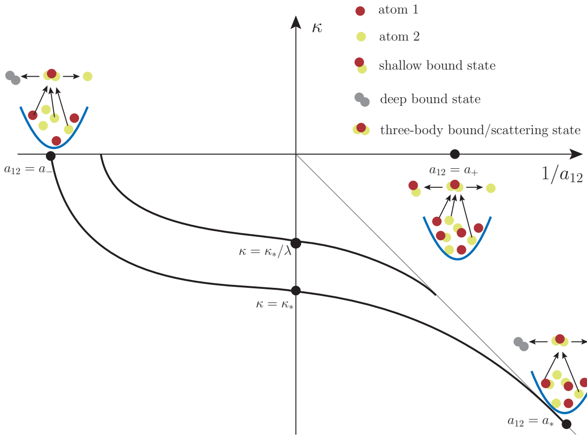
Efimov Effect in Heteronuclear Mixtures

- two heavy atoms (2) and one light atom (1)
- large a_{12} near Feshbach resonance
- small a_{22}



- λ varies with m_1/m_2.

Observable Features of The Efimov Spectrum



Universal Relations at Leading Order

At $r_0 \to 0$, $a_{22} \to 0$,

$$a_{i,n} = \lambda^n \theta_i \kappa_*^{-1}; \qquad i = *, +, -$$

System	m_1/m_2	λ	θ_+	θ_*	θ_-
^{6}Li-Cs-Cs	4.511×10^{-2}	4.865	0.6114	3.388×10^{-2}	-1.349
^{7}Li-Cs-Cs	5.263×10^{-2}	5.465	0.5887	3.392×10^{-2}	-1.376
^{6}Li-Rb-Rb	6.897×10^{-2}	6.835	0.5492	3.367×10^{-2}	-1.436
^{7}Li-Rb-Rb	8.046×10^{-2}	7.864	0.5266	3.328×10^{-2}	-1.477
^{40}K-Rb-Rb	0.4598	122.7	0.2194	1.014×10^{-2}	-2.430
^{41}K-Rb-Rb	0.4713	131.0	0.2142	9.705×10^{-3}	-2.451

Universal relations at Next-to-Leading Order

corrections from

- r_{12}/a_{12}
- a_{22}/a_{12}

Up to linear terms,

$$a_{i,n} = \lambda^n \theta_i \kappa_*^{-1} + (J_i + n\sigma) r_{12} + (Y_i + n\bar{\sigma}) a_{22}$$

- σ and $\bar{\sigma}$ are universal numbers for given mass ratio.
- difference btw J_i and Y_i (e.g. $J_* - J_-$, $Y_* - Y_+$) are universal
- we find $J_+ - J_- = \sigma/2$ and $Y_+ - Y_- = \bar{\sigma}/2$ for all mass ratios.

Universal relations at Next-to-Leading Order

System	$\sigma = 2(J_0 - J_-)$	$J_* - J_0$	$\bar{\sigma} = 2(Y_0 - Y_-)$	$Y_* - Y_0$
^{6}Li-Cs-Cs	0.693	0.840	0.141	0.680
^{7}Li-Cs-Cs	0.743	0.828	0.204	0.821
^{6}Li-Rb-Rb	0.840	0.820	0.367	1.11
^{7}Li-Rb-Rb	0.904	0.823	0.502	1.30
^{40}K-Rb-Rb	2.74	1.52	12.1	8.74
^{41}K-Rb-Rb	2.80	1.54	12.7	9.07

Acharya, CJ, Platter, *in preparation*

Conclusion

- Range corrections to Efimov physics in perturbation:
  - NLO for varying a:

$$H_1(\Lambda) = r\Lambda \; h_{10}(\Lambda/\Lambda_*) + r/a \; h_{11}(\Lambda/\mu)$$

  - N^2LO for fixed a:

$$H_2(\Lambda) = r^2\Lambda^2 \; h_{20}(\Lambda/\Lambda_*) + r^2 m E_3 \; h_{22}(\Lambda/\mu)$$

- Universal relations in range effects are connected with running three-body parameters

$$a_{i,n} = \lambda^n \theta_i \kappa_*^{-1} + (J_i + n\sigma) r$$

$$\kappa_*(Q, a) = (Q/\kappa_*)^{-\eta r/a} \kappa_*$$

- Universal range effects and short-a effects in heteronuclear mixtures