Effective Field Theory for Few-Boson Systems

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EMMI Rapid Reaction Task Force:

The systematic treatment of the Coulomb interaction in few-body systems

May 31, 2016

GSI, Darmstadt

Motivation

- Pionless EFT is very effective for low-energy physics in the few-nucleon case.
- A 3-body interaction is needed at LO, but no 4-body interaction is needed.
- This holds also for the few-boson system.
- Are higher-body forces needed at LO to describe systems with more bodies?
- What is the regime of validity of the EFT as the number of particles increase?

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Universality

- Consider particles interacting through 2-body potential with range R.
- Classically, the particles 'feel' each other only within the potential range.
- But, in the case of resonant interaction, the wave function can have much larger extent.
- At low energies, the 2-body physics is completely govern by the scattering length, a_2.

$$\lim_{k\to 0} k \cot \delta(k) = -\frac{1}{a_2} + \frac{1}{2} r_2 k^2$$

From Sakurai's book

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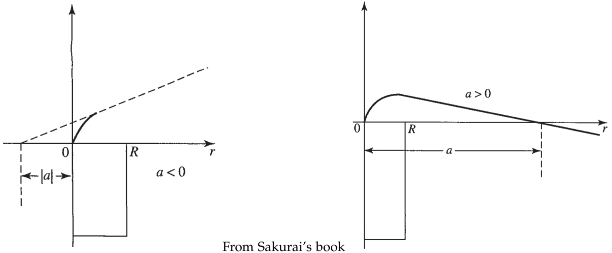

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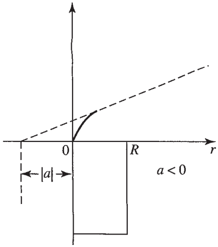

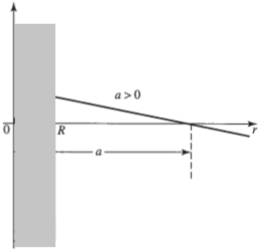

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Universality

- Generally, $a_2 \approx r_2 \approx R$.
  Universal systems are fine-tuned to get $a_2 \gg r_2, R$.
- Corrections to universal theory are of order of r_2/a_2 and R/a_2.
- For $a_2 > 0$, we have universal dimer with energy $E = -\hbar^2/m{a_2}^2$.
- ^{4}He Atoms: $a_2 \approx 170.9 a_0$, (a_0 = the Bohr radius), is much larger than its van der Waals radius, $r_{vdW} \approx 9.5 a_0$.
- Nucleus: $a_s \approx -23.4$ fm, $a_t \approx 5.42$ fm, $R = \hbar/m_\pi c \approx 1.4$ fm.
  Deuteron binding energy, 2.22 MeV, is close to $\hbar/m{a_t}^2 \approx 1.4$ MeV.
- Ultracold atoms near a Feshbach resonance,

$$a_2(B) = a_{bg}\left(1 + \frac{\Delta}{B - B_0}\right)$$

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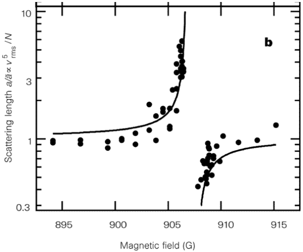
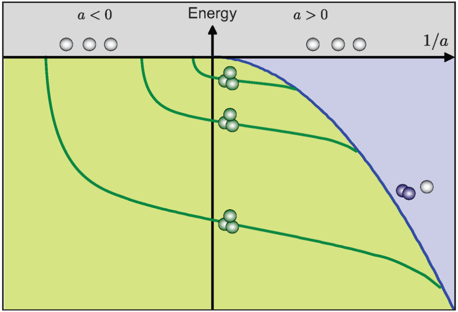


S. Inouye *et al.*, Nature 392, 151 (1998)

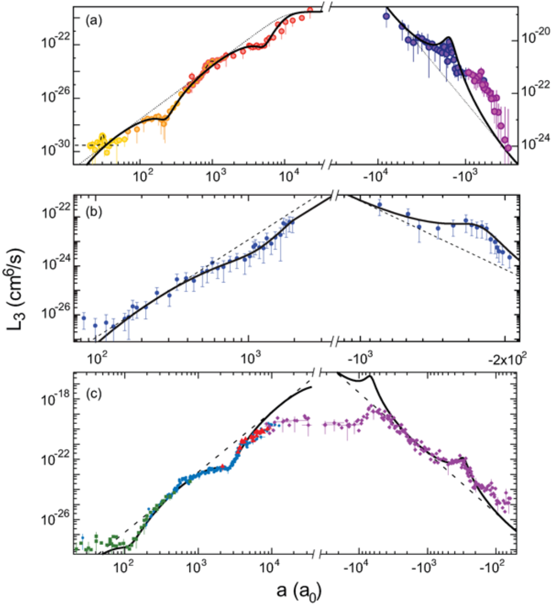
Efimov Physics

- *The unitary limit*: $E_2 = 0$, $a_2 \longrightarrow \infty$.
- In 1970 V. Efimov found out that if $E_2 = 0$ the 3-body system will have an **infinite** number of bound states.
- The 3-body spectrum is $E_n = E_0 e^{-2\pi n/s_0}$ with $s_0 = 1.00624$.



F. Ferlaino and R. Grimm, Physics **3**, 9 (2010)

Efimov Physics in Ultracold Atoms



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Effective Field Theory (EFT)

- Typically in physics we have an "underlying" theory, valid at a mass scale M_{hi}, but we want to study processes at momenta $Q \approx M_{lo} \ll M_{hi}$.
- For example, nuclear structure involves energies that are much smaller than the typical QCD mass scale, $M_{QCD} \approx 1$ GeV.
- Effective Field Theory (EFT) is a framework to construct the interactions systematically. The high-energy degrees of freedom are integrated out, while the effective Lagrangian have the same symmetries as the underlying theory.
- The details of the underlying dynamics are contained in the interaction strengths.

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Pionless EFT

- The degrees of freedom in pionless EFT are the nucleons.
- We have to include all terms conserving our theory symmetries, order by order.
- For nucleons, the Leading Order (LO) is,

$$V_{LO} = a_1 + a_2 \sigma_i \cdot \sigma_j + a_3 \tau_i \cdot \tau_j + a_4 (\sigma_i \cdot \sigma_j)(\tau_i \cdot \tau_j)$$

  where due to symmetry, only 2 are independent, corresponding to the two scattering lengths.
- The Next to Leading Order (NLO) is,

$$\begin{aligned} V_{NLO} &= b_1(k^2+q^2) + b_2(k^2+q^2)\sigma_i \cdot \sigma_j + b_3(k^2+q^2)\tau_i \cdot \tau_j \\ &+ b_4(k^2+q^2)(\sigma_i \cdot \sigma_j)(\tau_i \cdot \tau_j) \end{aligned}$$

$$\boldsymbol{q} = \boldsymbol{p}' - \boldsymbol{p}, \quad \boldsymbol{k} = \boldsymbol{p} + \boldsymbol{p}'$$

  here also only 2 parameters are independent, corresponding to the two effective ranges.
- p-wave enters at N^3LO!

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Short-Range EFT for Bosonic system

- For spinless bosons, most of the terms are dropped, and we have at LO,

$$V_{LO} = a_1.$$

- At NLO,

$$V_{NLO} = b_1(k^2 + q^2).$$

- Terms proportional to $\boldsymbol{k} \cdot \boldsymbol{q}$ are omitted due to time reversal symmetry.
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Stochastic Variational Method I

- To solve the N-body Schrodinger equation, we use correlated Gaussian basis,

$$\psi(\eta_1, \eta_2 ... \eta_{N-1}) = \sum_i c_i \mathcal{A} \exp(-\eta^T A_i \eta)$$

$\eta =$ Jacobi coordinates, $A_i =$ matrix of $(N-1) \times (N-1)$ numbers.

- Since $a \gg \Lambda^{-1}$, we need large spread of basis functions.
- Symmetrization gives factor of $N!$ which limits the number of particles.
- Works for bosons and fermions. Arbitrary angular momentum and parity, as well as spin and isospin, can be introduced.

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$$\langle A|A'\rangle = \left(\frac{(2\pi)^{N-1}}{\det B}\right)^{3/2} ; \quad B = A + A'$$

$$\langle A|T_{int}|A'\rangle = 3\langle A|A'\rangle \mathrm{Tr}[AB^{-1}A'\Pi]; \quad \Pi_{ij} = (2\mu_i)^{-1}\delta_{ij}$$

$$\langle A|V^{2b}|A'\rangle = \frac{\Lambda^2 C^{(0)}}{2m\sqrt{\pi}}\langle A|A'\rangle \sum_{i<j}\left(1 + f_{ij}\Lambda^2/2\right)^{-3/2} ; \; f_{ij} = \omega_{ij}^T B^{-1}\omega_{ij}, \; \mathbf{r}_{ij} = \omega_{ij}^T \boldsymbol{\eta}$$

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- To find the best A_i, we use the Stochastic Variational Method (SVM):
- We add basis function one by one, or try to replace an exist basis function by a new one.
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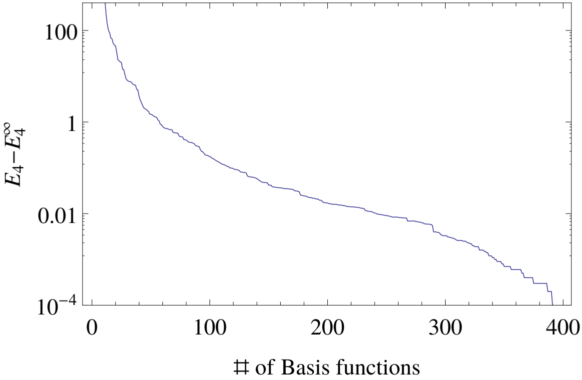

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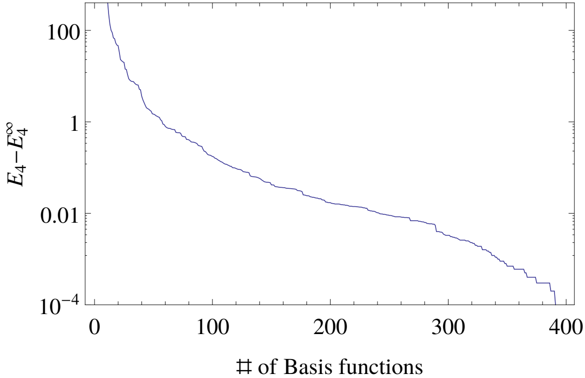

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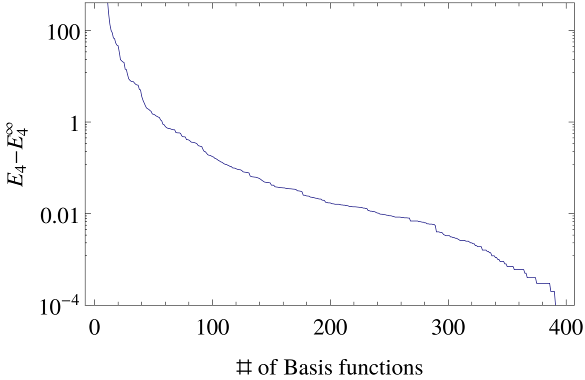

Regularization I: non local potential

- At LO, we have only contact interaction,

$$V(r_{ij}) = \tilde{C}^{(0)}\delta(r_{ij}).$$

- This interaction needs regularization and renormalization.
- The bound state of two identical bosons ($\hbar = c = 1$),

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and in momentum space,

$$\frac{p^2}{m}\phi(p) + \tilde{C}^{(0)}\int\frac{d^3p'}{(2\pi)^3}\phi(p') = -B_2\phi(p)$$

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R.F. Mohr *et al.*, Ann. Phys. **321**, 225 (2006).

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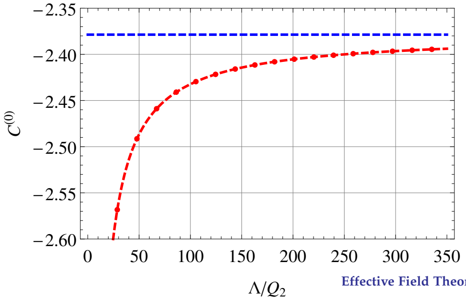

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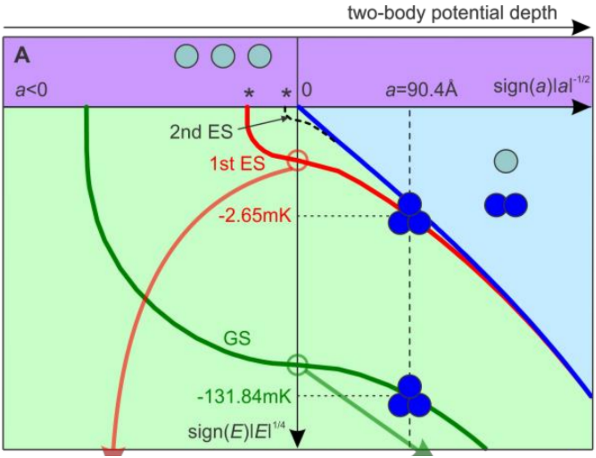
- To be definite, we focus on ^{4}He atomic systems.



from M. Kunitski *et al.*, Science **348** 551 (2015).

^{4}He Atoms

Length scales (in Å) for the ^{4}He atoms:

	LM2M2	TTY	PCKLJS
a_2	100.23	100.01	90.42(92)
r_2	7.326	7.329	7.27
r_{vdW}	5.378	5.378	5.378

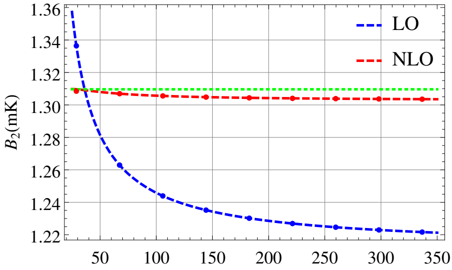
Binding energies (in mK) of ^{4}He clusters:

	LM2M2	TTY	PCKLJS	experiment
B_2	1.3094	1.3096	1.6154	$1.3^{+0.25}_{-0.19}$; 1.76(15)
B_3^*	2.2779	2.2761	2.6502	
$B_3^* - B_2$	0.9685	0.9665	1.0348	0.98(2)
B_3	126.50	126.16	131.84	
B_4^*	127.42		132.70	
B_4	559.22		573.90	

LM2M2: Aziz & Slaman, J. Chem. Phys. **94**, 8047 (1991). TTY: Tang, Toennies & Yiu, PRL **74**, 1546 (1995). PCKLJS: Przybytek, Cencek, Komasa, Lach, Jeziorski & Szalewicz, PRL **104**, 183003 (2010).

Two-boson system

- Breakdown scale $\sim r_{\text{vdw}} \sim r_2/2 \sim 7a_0$.



$$B_2 = B_{LO}\left[1 + \mathcal{O}\left(\frac{Q_2}{M}, \frac{Q_2}{\Lambda}\right)\right] ; \; B_2(\Lambda) \overset{\Lambda/Q_2}{=} B_{2\infty}\left[1 + \alpha\frac{Q_2}{\Lambda} + \beta\left(\frac{Q_2}{\Lambda}\right)^2 + \gamma\left(\frac{Q_2}{\Lambda}\right)^3\right]$$

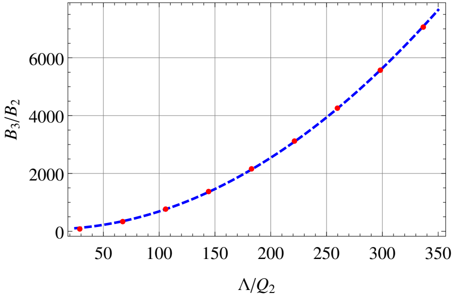
$B_2\infty(mK)$	α	β	γ
1.21	2.86	—	—
1.21	2.76	5.84	—
1.21	2.76	5.52	12.80
1.31			

$$\alpha Q_2 r_2/2 \approx 10\%$$

Three-boson system

Trying to calculate the trimer binding energy we get the Thomas collapse:

$$B_3 \propto \frac{\hbar\Lambda^2}{m}$$



- To stabilize the system, a 3-body counter term must be introduced at LO,

$$V_{LO}^{3N} = \frac{(4\pi)^2}{m\Lambda^4} D^{(0)} \sum_{i<j<k} \sum_{cyc} \delta_\Lambda(\mathbf{r}_{ij})\delta_\Lambda(\mathbf{r}_{jk}),$$

  Λ_* is a new momentum scale, $D^{(0)} = f(a\Lambda, \Lambda/\Lambda_*)$

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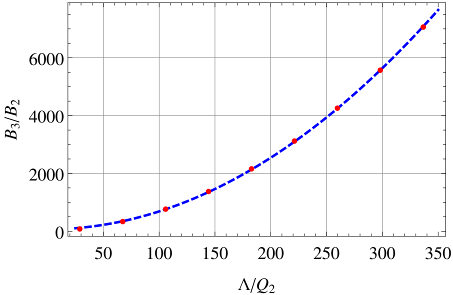

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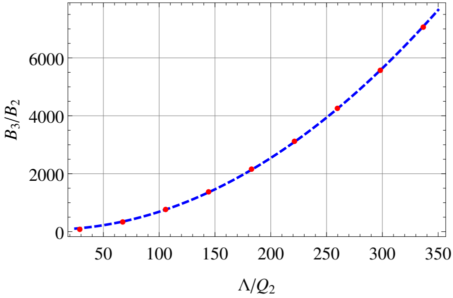

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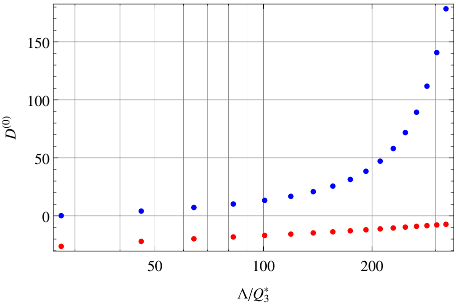
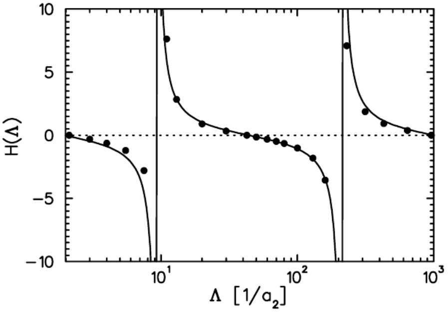


$\Lambda_3 = \Lambda_2$, local, smooth cutoff



$\Lambda_3 \ll \Lambda_2$, non-local, sharp cutoff

P. F. Bedaque, H.W. Hammer, and U. van Kolck

Phys. Rev. Lett. **82** 463 (1999).

$\Lambda_3 = \Lambda_2$, non-local, smooth cutoff

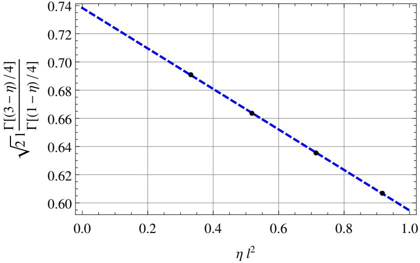
R.F. Mohr *et al.*, Ann. Phys. **321**, 225 (2006).

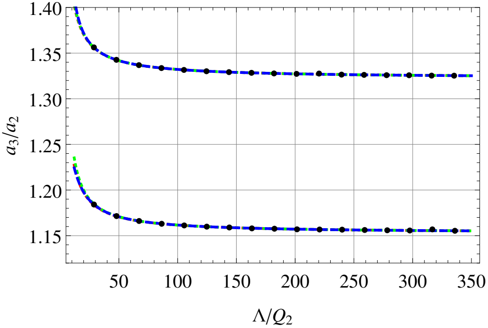
3-body LEC is fitted to B_3^*.
a_3, the atom-dimer scattering length is calculated in a trap, using

$$\sqrt{2}l\frac{\Gamma[(3-\eta)/4]}{\Gamma[(1-\eta)/4]} \simeq \frac{a_2}{a_3}\left(1-\frac{a_3 r_3}{4a_2^2}\eta l^2\right),$$

$$\eta = 2(E_3 - E_2)/\omega,\, l = a_2/a_{ho},\, a_{ho} = 1/\sqrt{2\mu\omega},\, \mu \simeq 2m/3$$

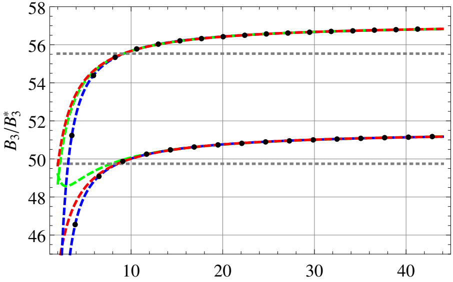
Busch *et al.*, Found. Phys. **28** 549 (1998); Stetcu *et al.*, Ann. Phys. **325**, 1644 (2010).





$$a_2 = 1.151(2)\ \text{(LM2M2)}$$

Trimer ground state



$$B_3(\Lambda) = B_{3\infty}\left[1 + \alpha\frac{Q_3}{\Lambda} + \overset{\Lambda/Q_3}{\beta}\left(\frac{Q_3}{\Lambda}\right)^2 + \gamma\left(\frac{Q_3}{\Lambda}\right)^3\right]$$

LM2M2

$B_3(\infty)/B_3^*$	α	β	γ
57.22	-0.26	—	—
57.23	-0.25	0.09	—
57.21	-0.26	0.20	-2.04
55.53			

PCKLJS

$B_3(\infty)/B_3^*$	α	β	γ
51.51	-0.29	—	—
51.56	-0.34	0.49	—
51.52	-0.31	0.47	-2.80
49.75			

$$\alpha Q_3 r_2/2 \approx 10\%$$

Four-boson system

Are more terms needed to stabilize heavier systems?

LM2M2

$B_4/B_3(\infty)$	α	β	γ
4.14	−1.38	—	—
4.21	−1.90	3.86	—
4.24	−2.02	4.07	4.31
4.42			

PCKLJS

$B_4/B_3(\infty)$	α	β	γ
4.09	−1.36	—	—
4.17	−1.90	4.02	—
4.16	−1.80	2.32	8.00
4.35			

$Q_4 r_2/2 \approx 55\%$

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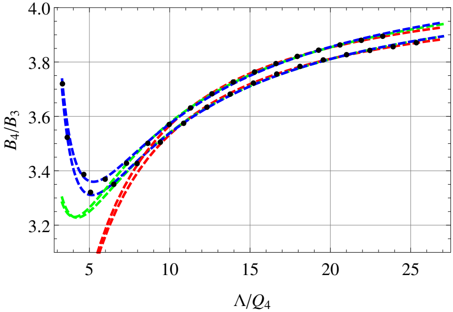

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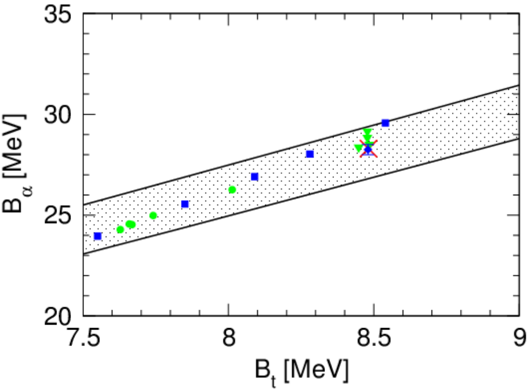
Tjon line

Another evidence is the Tjon line, the correlation between the binding energies of the triton and the α-particle.

J.A. Tjon, Phys. Lett. B **56**, 217 (1975).

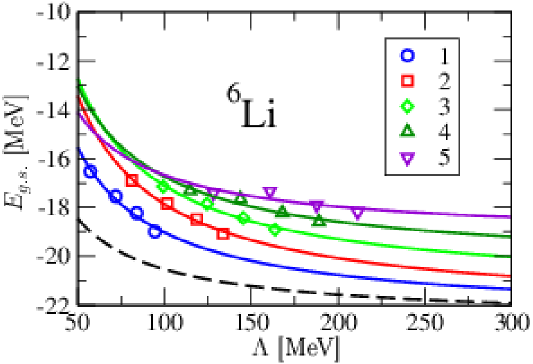


L. Platter, H.-W. Hammer, U.-G. Meissner, Phys. Lett. B **607**, 254 (2005).

Heavier system

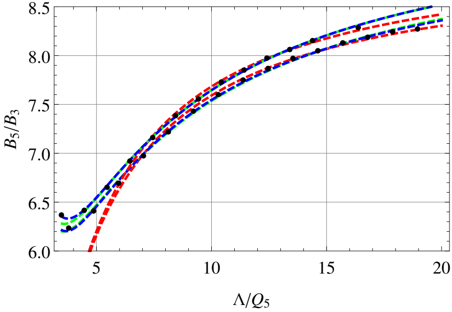
Are more terms needed to stabilize heavier systems?

For nucleons,



I. Stetcu, B.R. Barrett, and U. van Kolck, Phys. Lett. B **653** , 358 (2007).

Five-boson system



LM2M2

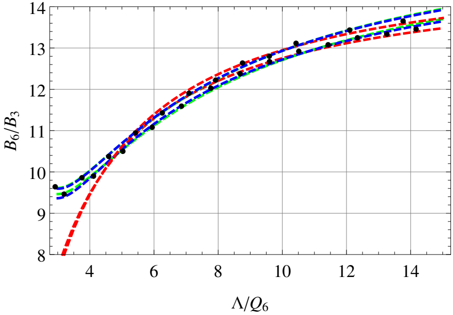
$B_5/B_3(\infty)$	α	β	γ
9.16	−1.61	—	—
9.61	−2.49	4.48	—
9.60	−2.46	4.06	1.38
10.33			

PCKLJS

$B_5/B_3(\infty)$	α	β	γ
9.02	−1.58	—	—
9.41	−2.43	4.31	—
9.32	−2.20	2.63	3.47

$$Q_5 r_2/2 \approx 75\%$$

Six-boson system



LM2M2

$B_6/B_3(\infty)$	α	β	γ
15.3	−1.53	—	—
16.4	−2.48	3.73	—
16.3	−2.35	2.99	1.15
18.41			

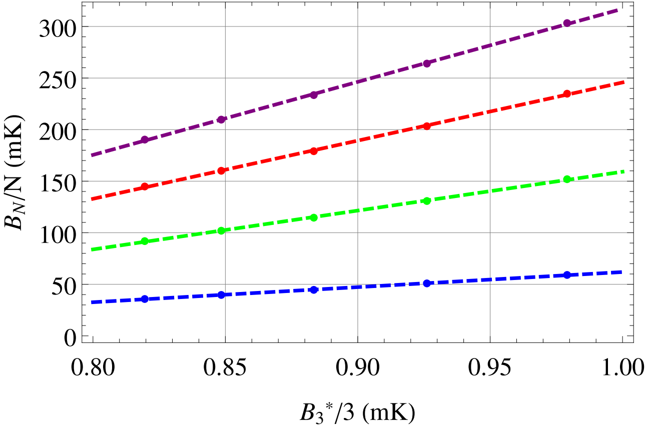
PCKLJS

$B_6/B_3(\infty)$	α	β	γ
14.9	−1.46	—	—
16.1	−2.47	3.69	—
15.7	−2.14	1.92	2.52

$$Q_6 r_2/2 \approx 85\%$$

Generalized Tjon-lines

Correlation between B_3^* to B_3, B_4, B_5, and B_6:



Conclusion

- A pionless EFT was constructed for few-body systems.
- The ^{4}He atomic system was studied, and our EFT fits nicely the known results.
- The convergence of pionless EFT for $A = 4, 5$ and 6 was studied.
- Generalized Tjon-lines were introduced, showing that at LO no 4,5 or 6-body term is needed.

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