Renormalization-group flow of Φ^4 field theories and critical phenomena

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RG flow of LGW Φ^4 field theories and critical phenomena.

ABSTRACT: In the framework of the **renormalization-group (RG) theory**, several **critical phenomena** can be investigated by studying the **RG flow of an effective Landau-Ginzburg-Wilson (LGW) Φ^4 theory**, having an N-component order parameter as fundamental field, and containing up to 4th-order polynomials of the field. I discuss the **general properties of the RG flow of Φ^4 theories**, and present an **overview of RG field-theory results for physically interesting LGW Φ^4 theories**, whose results apply to liquids, magnets, disordered and/or frustrated systems, to the finite-T transition in hadronic matter, competition of different orderings, etc...

Critical phenomena are observed in many physical systems

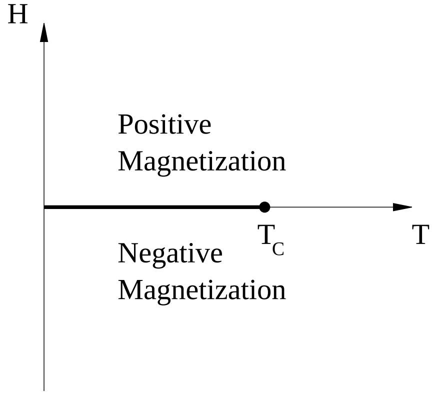
There are two broad classes of phase transitions:

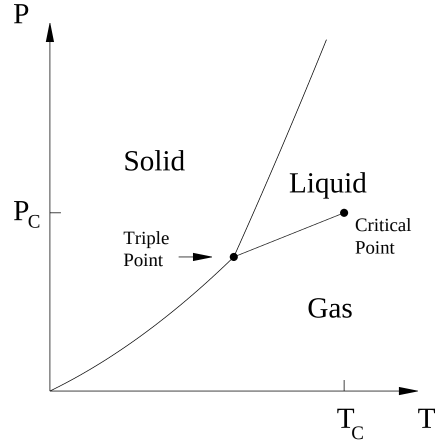
first order → discontinuity in thermodynamic quantities

continuous → nonanalytic behavior due to a diverging length

Examples of **continuous transitions**:

- magnetic transitions
- liquid-vapor in fluids





• first general framework was proposed by Landau (1937), based on an expansion of the free energy in powers of the order parameter, corresponding to mean-field approximation

• Renormalization-group (RG) theory by Wilson (1971)

In the framework of the RG theory, several critical phenomena can be investigated by studying the RG flow of Φ^4 theories with an N-component fundamental field Φ, and containing up to 4th-order polynomials of the field.

O(N)-symmetric models $\rightarrow \mathcal{L} = (\partial_\mu \vec{\Phi})^2 + r\vec{\Phi}^{\,2} + u\,(\vec{\Phi}^{\,2})^2$,

but also more complicated **multi-parameter Φ^4 theories** with several quadratic and quartic parameters, depending on the nature of the order parameter and the symmetry-breaking pattern

$$\mathcal{L} = \sum_i [(\partial_\mu \Phi_i)^2 + r_i \Phi_i^2] + \sum_{ijkl} u_{ijkl}\, \Phi_i \Phi_j \Phi_k \Phi_l$$

Results for their RG flow apply to several physical systems, such as liquids, magnets, superfluid transitions in ^{4}He and ^{3}He, disordered and/or frustrated systems, the finite-T transition in hadronic matter, quantum transitions in high-T superconductors, etc...

Plan of the talk

- RG theory of critical phenomena and universality
- Field-theory approach based on LGW Φ^4, in particular PFT
- RG flow of multiparameter Φ^4 theories
- Overview of results of physically interesting Φ^4 theories:

O(N) models and more complicated multiparameter Φ^4 theories, describing critical behaviors in

- liquids, superfluid transition in ^{4}He, magnets
- disordered systems, dilute antiferromagnets
- frustrated systems with noncollinear order
- in hadron matter
- the presence of competition of different orderings

Continuous transitions are characterized by power-law behaviors

- Disordered (symmetric) phase ($t \equiv T/T_c - 1 > 0,\ h = 0$):

$$\xi \sim t^{-\nu}, \quad C_H \sim t^{-\alpha}, \quad \chi \sim t^{-\gamma}, \quad \chi \sim \xi^{2-\eta}$$

- Ordered (broken) phase ($t < 0,\ h = 0^+$): $C_H \sim |t|^{-\alpha},\ M \sim |t|^{\beta}$
- Critical isotherm ($t = 0,\ h > 0$): $\chi \sim |h|^{-\gamma/\beta\delta},\ \widetilde{G}(q) \sim q^{-2+\eta}$
- Scaling equation of state: $h = t^{\beta\delta} F(z),\ z = M t^{-\beta}$
- Finite-size scaling, ex. $\chi \sim L^{2-\eta}$ at $t = 0$
- There are also critical behaviors characterized by exponential approaches: **LATTICE QCD** where $\xi \sim \exp(c\beta)$, and also 2D σ models, 2D KT transition

Main ideas to describe the critical behavior at a continuous transition

- **Order parameter** which effectively describes the critical modes
- **Scaling hypothesis**: singularities arise from the long-range correlations of the order parameter, diverging length scale
- **Universality**: the critical behavior is essentially determined by a few global properties: the space dimensionality, the nature and the symmetry of the order parameter, the symmetry breaking

RENORMALIZATION-GROUP THEORY

- RG flow in a Hamiltonian space
- the critical behavior is associated with a fixed point of the RG flow
- only a few perturbations are relevant, the corresponding positive eigenvalues are related to the critical exponents ν, η, etc...

The Gibbs free energy obeys a scaling law

$$\mathcal{F}_{\text{sing}}(u_1, u_2, \ldots, u_k, \ldots) = b^{-d}\mathcal{F}_{\text{sing}}(b^{y_1}u_1, b^{y_2}u_2, \ldots, b^{y_k}u_k, \ldots)$$

u_k are nonlinear scaling fields (analytic functions of the model parameters)

In a standard continuous transition: **two relevant scaling fields** $u_t \sim t = T/T_c - 1$ (with $y_t = 1/\nu$) and $u_h \sim h$ (external field, with $y_h = (\beta + \gamma)/\nu$), and irrelevant u_i ($i \geq 3$) with $y_i < 0$.

When $u_t, t \to 0$ and $u_h, h \to 0$

$$\mathcal{F}_{\text{sing}} \approx \xi^{-d}\left[f(h\xi^{y_h}) + \xi^{-\omega} f_\omega(h\xi^{y_h}) + ...\right], \qquad \xi \sim t^{-\nu}$$

$O(\xi^{-\omega})$ arises from the leading irrelevant u_3, and $\omega = -y_3$.

The presence of other relevant perturbations beside t and h gives rise to **multicritical behaviors**. In the case of one more relevant field g and for $h = 0$: $\mathcal{F}_{\text{sing}} \approx t^{d\nu} f(g t^{-\phi})$, where $\phi > 0$ is the crossover exponent.

The RG theory provides the basis for the field-theory approaches.

Many critical phenomena can be described by LGW Φ^4 theories

$$\mathcal{L} = \sum_i [(\partial_\mu \Phi_i)^2 + r_i \Phi_i^2] + \sum_{ijkl} u_{ijkl}\, \Phi_i \Phi_j \Phi_k \Phi_l$$

where Φ is a N-component field. They are constructed by requiring a few global properties of the system, keeping terms up to 4th order.

UNIVERSALITY CLASSES within which the critical behavior is universal: • spatial dimension • nature of the critical modes and order parameter • symmetry and symmetry-breaking pattern

Ex: SUPERFLUID transition in ^{4}He along the λ-line: D=3, quantum amplitude of helium atoms as order parameter, U(1) symmetry

3-D XY UNIVERSALITY CLASS: $\mathcal{L} = |\partial_\mu \varphi|^2 + r\,|\varphi|^2 + u\,|\varphi|^4$ with a complex field φ, characterized by the critical exponents:
$\nu = 0.6717(1)$, $\alpha = -0.0151(3)$, $\eta = 0.0381(2)$ (Campostrini, etal, 2006)

Perturbative schemes in field-theory approach

We are interested in the critical behavior of the "bare" correlation functions $\Gamma_n(p; r, u, \Lambda)$ of the ϕ^4 theory $\mathcal{L} = (\partial_\mu \vec{\varphi})^2 + r\vec{\varphi}^{\,2} + u\,(\vec{\varphi}^{\,2})^2$

- Massive zero-momentum scheme defined in the disordered phase

$$\Gamma_2(p) = Z_\varphi^{-1}[m^2 + p^2 + O(p^4)], \quad \Gamma_4(0) = Z_\varphi^{-2} m^{4-d} g, \quad \Gamma_{2,1}(0) = Z_t^{-1}$$

which relate the renormalized quantities m, g to the bare ones r, u.

- The critical limit $m \to 0$ (corresponding to $\xi \to \infty$) can be studied by Callan-Symanzik RG equations for $\Gamma_n^{(r)}(p; m, g)$

$$\left[m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} - \frac{1}{2}n\eta_\varphi(g)\right]\Gamma_n^{(r)}(p) = [2 - \eta_\varphi(g)]m^2\Gamma_{n,1}^{(r)}(p; 0)$$

- The RG functions $\beta(g) = m\partial g/\partial m$ and $\eta_{\varphi,t}(g) = \partial \ln Z_{\varphi,t}/\partial \ln m$ can be computed as power series of g (computed up to six, seven loops by Nickel etal for O(N) models)

- when $m \to 0$ the coupling g is driven toward an infrared-stable fixed point, i.e. a zero g^* of the β-function $\beta(g) \approx -\omega(g^* - g)$

- Using the RG equations, $\eta = \eta_\varphi(g^*), \quad 1/\nu = 2 - \eta_\varphi(g^*) + \eta_t(g^*)$

- The perturbative FT expansions are asymptotic: $S(g) = \sum_n s_n g^n$, $s_n \sim n^b(-a)^n n!$, $a > 0$. They must be resummed before evaluating at g^*, exploiting Borel summability and knowledge of the large-order behavior by computing instanton semiclassical solutions, which provide important nonperturbative information

- Alternative $\overline{\text{MS}}$ renormalization scheme defined at $T = T_c$, $\epsilon \equiv 4 - d$ expansion, but also exp in the coupling setting $\epsilon = 1$

Many results for the **3D Ising universality class**
(liquid-vapor systems, fluid mixtures, uniaxial magnets)
corresponding to $\mathcal{L} = (\partial_\mu \varphi)^2 + r\varphi^2 + u\varphi^4$ with $\varphi \in \Re$

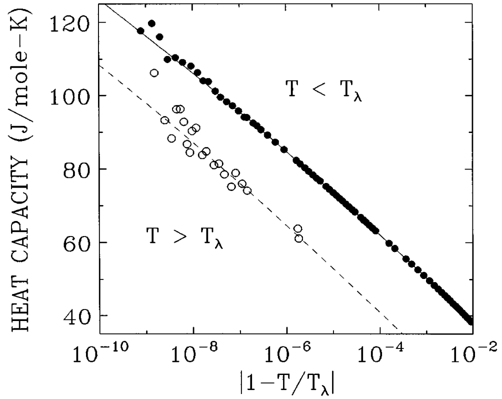
		ν	η	β
EXPT	liq-vap	0.6297(4)*	0.042(6)	0.324(2)
	fluid mix	0.6297(7)*	0.038(3)	0.327(3)
	magnets	0.6300(17)*		0.325(2)
Lattice	HT exp[1]	0.63012(16)	0.0364(2)	0.3265(1)
	MC[2]	0.63020(12)	0.0368(2)	0.3267(1)
PFT	6,7-l MZM[3]	0.6304(13)	0.034(3)	0.326(1)
	$O(\epsilon^5)$ exp[3]	0.6290(25)	0.036(5)	0.326(3)

* By using the hyperscaling relation $\alpha = 2 - 3\nu$. [1] M. Campostrini, A. Pelissetto, P. Rossi, EV (2002). [2] Y. Deng, HWJ Blöte, (2003) [3] R. Guida, J. Zinn-Justin, (1998)

3D XY universality class

$\mathcal{L} = |\partial_\mu \varphi|^2 + r\,|\varphi|^2 + u\,|\varphi|^4$ (complex φ)

The superfluid transition in ^{4}He is an exceptional experimental opportunity, exploiting also a microgravity environment using the Space Shuttle (data up to a few nK from T_c)



		α	ν	η
EXPT	^{4}He [1]	$-0.0127(3)$	$0.6709(1)^*$	
Lattice	MC+HT [3]	$-0.0151(3)^*$	$0.6717(1)$	$0.0381(2)$
	MC [4]	$-0.0151(9)^*$	$0.6717(3)$	
PFT	6,7-l MZM[2]	$-0.011(4)$	$0.6703(15)$	$0.035(3)$
	$O(\epsilon^5)$ exp[2]	$-0.004(11)$	$0.6680(35)$	$0.038(5)$

* By $\alpha = 2 - 3\nu$. [1] J.A. Lipa, etal, PRB 68 (2003) 174518; PRL 76 (1996) 944. [2] R. Guida, J. Zinn-Justin, (1998). [3] M. Campostrini, M. Hasenbusch, A. Pelissetto, EV (2006). [4] E. Burovski, etal, (2006)

$\longrightarrow$ Significant discrepancy between **EXPT** and **Lattice** results

There are also several critical phenomena which are described by more general multi-parameter Φ^4 theories:

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{N}(\partial_\mu\varphi_i)^2 + r_i\varphi_i^2] + \frac{1}{4!}\sum_{ijkl=1}^{N} u_{ijkl}\,\varphi_i\varphi_j\varphi_k\varphi_l$$

• The parameter r_i and u_{ijkl} depend on the symmetry.

• If criticality is driven by one T-like parameter, and all φ_i become critical, $\sum_i \varphi_i^2$ must be the only invariant quadratic term. Thus $r_i = r$, $\sum_i u_{iikl} \propto \delta_{kl}$, etc...

• In the absence of a large symmetry like $O(N)$, several quartic couplings must be considered.

• all Φ^4 theories are expected to be trivial for $D = 4$ like $\mathrm{O}(N)$ models

Examples of physically interesting LGW Φ^4 theories

- MN model with a real $M \times N$ matrix field ϕ_{ai}

$$\mathcal{L} = \sum_{i,a} \left[(\partial_\mu \phi_{ai})^2 + r\phi_{ai}^2 \right] + \sum_{ij,ab} \left(u_0 + v_0 \delta_{ij} \right) \phi_{ai}^2 \phi_{bj}^2$$

For $N \to 0$, disordered spin systems at magnetic transitions.

For $M = 1$, $N = 2, 3$, magnets with cubic anisotropy.

- O(M)⊗O(N) model, fields ϕ_a are M sets of N-comp vectors

$$\mathcal{L} = \sum_a [(\partial_\mu \phi_a)^2 + r\phi_a^2] + u_0 (\sum_a \phi_a^2)^2 + v_0 \sum_{a,b} (\phi_a \cdot \phi_b)^2$$

For $M = 2, N = 3$, $v_0 < 0$, $\mathrm{U}(2) \to \mathrm{O}(2)$, superfluid transitions in ^{3}He.

For $M = 2$, $v_0 > 0$, $O(2) \otimes O(N) \to O(2) \otimes O(N-2)$, noncollinear frustrated magnets (stacked triangular antiferromagnets).

- Spin-density wave model (Φ_a are complex N-comp vectors)

$$|\partial_\mu \Phi_1|^2 + |\partial_\mu \Phi_2|^2 + r(|\Phi_1|^2 + |\Phi_2|^2) + u_{1,0}(|\Phi_1|^4 + |\Phi_2|^4)$$
$$+u_{2,0}(|\Phi_1^2|^2 + |\Phi_2^2|^2) + w_{1,0}|\Phi_1|^2|\Phi_2|^2 + w_{2,0}|\Phi_1 \cdot \Phi_2|^2 + w_{3,0}|\Phi_1^* \cdot \Phi_2|^2$$

Critical behavior in spin-density wave systems.

Quantum transitions in high-T_c superconductors (cuprates).

- U(N)⊗U(N) models (Φ is a complex N×N matrix)

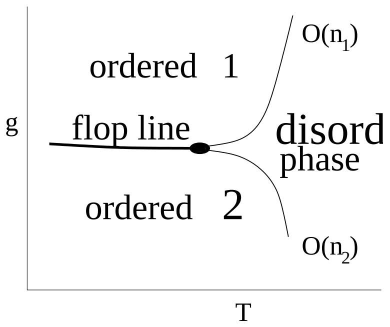
$$\mathcal{L}_U = \mathrm{Tr}\partial_\mu \Phi^\dagger \partial_\mu \Phi + r\mathrm{Tr}\Phi^\dagger\Phi + u_0 \left(\mathrm{Tr}\Phi^\dagger\Phi\right)^2 + v_0 \mathrm{Tr}\left(\Phi^\dagger\Phi\right)^2$$

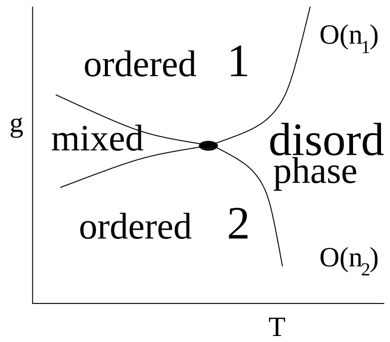
Finite-T transition in QCD with N quarks, neglecting U(1)$_A$ anomaly

- SU(N)⊗SU(N) models: $\mathcal{L}_{SU} = \mathcal{L}_U + w_0 \left(\det\Phi^\dagger + \det\Phi\right)$

Finite-T transition in QCD taking into account the U(1)$_A$ anomaly effects

Multicritical behaviors arising from the **competition of different orderings**, ex. with symmetries O(n_1) and O(n_2), at the point where the corresponding transition lines meet (by tuning two relevant parameters T and g)





In high-T_c superconductors, anisotropic antiferromagnets, etc...

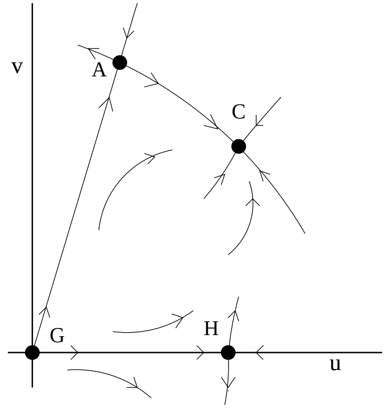
- O(n_1)⊕O(n_2) theory with two O(n_1) and O(n_2) vector fields

$$\mathcal{L} = (\partial_\mu \vec{\phi}_1)^2 + (\partial_\mu \vec{\phi}_2)^2 + r_1 \vec{\phi}_1^{\,2} + r_2 \vec{\phi}_2^{\,2} + u_1 (\vec{\phi}_1^{\,2})^2 + u_2 (\vec{\phi}_2^{\,2})^2 + w \vec{\phi}_1^{\,2} \vec{\phi}_2^{\,2}$$

- Coupled N-comp fields with $(Z_2)_{\rm par} \otimes O(N)$ symmetry

$$\partial_\mu \phi \cdot \partial_\mu \phi + \partial_\mu \psi \cdot \partial_\mu \psi + r_1 \phi^2 + r_2 \psi^2 + u_0 \phi^4 + v_0 \psi^4 + w_0 \phi^2 \psi^2 + z_0 (\phi \cdot \psi)^2$$

• The RG flow is determined by its FPs, common zeroes g^*_{ijkl} of $\beta_{ijkl}(g_{abcd}) \equiv \mu \partial g_{ijkl}/\partial\mu$. A FP is stable if all eigenvalues of $S_{ij} = \partial\beta_i/\partial g_j|_{g=g^*}$ have positive real part



• **The existence of a stable FP** implies that systems with the given global properties can undergo a continuous transition, whose asymptotic behavior is controlled by the stable FP.

• **The absence of a stable FP** predicts 1st-order transitions

• Even in the presence of a stable FP, systems that are outside its attraction domain undergo 1st-order transitions, which means that the nature of the transition is not a universal feature

• η conjecture (EV, Zinn-Justin, 2006): *In Φ^4 theories the stable FP is the one corresponding to the fastest decay of correlations, i.e. maximum η*

RG flow, critical exponents, etc..., by **FT perturbative methods**

$$\mathcal{L} = \sum_i [(\partial_\mu \varphi_i)^2 + r_i \varphi_i^2] + \sum_{ijkl} u_{ijkl}\, \varphi_i \varphi_j \varphi_k \varphi_l$$

- **Massive (disordered-phase) MZM scheme**: expansion in powers of the MZM quartic couplings g_{ijkl}

$$\Gamma^{(2)}_{ij}(p) = \delta_{ij} Z_\varphi^{-1} \left[m^2 + p^2 + O(p^4) \right], \qquad \Gamma^{(4)}_{ijkl}(0) = m\, Z_\varphi^{-2}\, g_{ijkl}$$

- **Massless (critical) $\overline{\rm MS}$ scheme:** Minimal subtraction within the dimensional regularization, ϵ expansion, $d = 3$ $\overline{\rm MS}$ exp

- High-order computations for several LGW Φ^4 theories, to six loops, requiring the calculation $\gtrsim 1000$ diagrams (Pelissetto, EV)

- Resummation exploiting Borel summability and calculation of the large-order behavior, by instanton semiclassical calculation

- The comparison of MZM and $\overline{\rm MS}$ expansions checks the results

Magnetic transitions in disordered systems

Spin models with impurities: mixing of antiferromagnetic materials with non magnetic ones, $Fe_u Zn_{1-u} F_2$, $Mn_u Zn_{1-u} F_2$ (uniaxial), $Fe_x Er_z$, $Fe_x Mn_y Zr_z$ (isotropic), ^4He in porous materials.

Modeled by $\mathcal{H} = -J \sum_{\langle ij \rangle} \rho_i \, \rho_j \; \vec{s}_i \cdot \vec{s}_j$, where $\rho_i = 1, 0$ with probability p and $1 - p$ respectively

Quenched disorder: the relaxation of impurities is very slow, thus the free energy $F(\rho) \propto \ln Z(\rho)$ must be averaged over the disorder, thus thermal and then disorder averages

$$\langle \mathcal{O} \rangle (\beta, \{\rho\}) = \frac{\sum_{\{s\}} \mathcal{O} e^{-\beta \mathcal{H}(s;\rho)}}{\sum_{\{s\}} e^{-\beta \mathcal{H}(s;\rho)}}, \qquad \overline{\langle \mathcal{O} \rangle} = \int [d\rho] P(\rho) \langle \mathcal{O} \rangle (\beta, \{\rho\})$$

Φ^4 **theory** with quenched disorder coupled to the energy density

$$\mathcal{H}_\psi = \partial_\mu \vec{\varphi}(x)^{\,2} + (r + \psi(x)) \vec{\varphi}(x)^{\,2} + g_0 (\vec{\varphi}(x)^{\,2})^2$$

$\psi(x)$ is a spatially uncorrelated random field, $P(\psi) \sim \exp(-\psi^2/4w)$

The replica trick, $\ln Z = \lim_{n\to 0}(Z^n - 1)/n$, allows us to integrate out disorder, obtaining a translation invariant Hamiltonian

$$\mathcal{H}_{MN} = \sum_{i,a} \left[(\partial_\mu \phi_{a,i})^2 + r\phi_{a,i}^2\right] + \sum_{ij,ab} (u_0 + v_0\delta_{ij})\, \phi_{a,i}^2 \phi_{b,j}^2$$

$a, b = 1, ...M$, $i, j = 1, ...N$, $u_0 < 0$. The original system is recovered **in the limit** $N \to 0$.

The critical behavior is determined by the RG flow of the MN model in the limit $N \to 0$, and in particular by analyzing the high-order MZM and $\overline{\mathrm{MS}}$ series for $N = 0$.

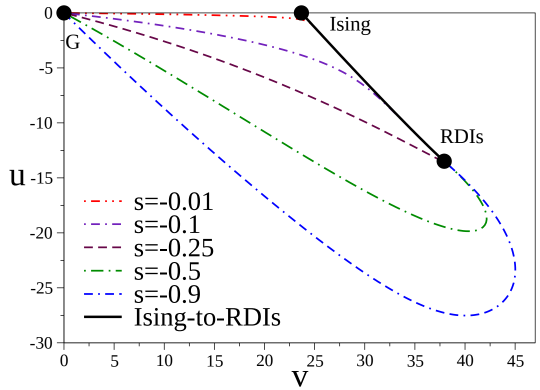
More generally: Universality classes for magnetic transitions where disorder does not break $O(N)$ symmetry, even in the presence of **frustration**, e.g. Edwards-Anderson models $\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij}\vec{s}_i \cdot \vec{s}_j$ where $J_{ij} = \pm 1$ with $P(J_{ij}) = p\delta(J_{ij} - J) + (1-p)\delta(J_{ij} + J)$ and small p

Question: Is the critical behavior affected by disorder?

The critical behavior of the pure system is stable if $\alpha_{\rm pure} < 0$ (Harris, 1974), as in multicomponent systems.

In Ising-like systems the pure Ising FP is unstable since $\alpha_{\rm Is} = 0.1096(5)$. Another stable FP exists, implying the existence of a new 3D RDI universality class.

Experiments confirm it.



^{4}He in porous materials and isotropic magnets show the same critical behavior as pure systems. Ising-like systems behave differently, showing $\nu > \nu_{\rm Ising} \simeq 0.630$

RDI exp	ν	β
experiments [1]	0.69(1)	0.359(9)
PFT[2]	0.678(10)	0.349(5)
MC RSIM [3]	0.683(2)	0.354(1)
MC $\pm J$ **EA** [3]	0.682(3)	0.353(2)

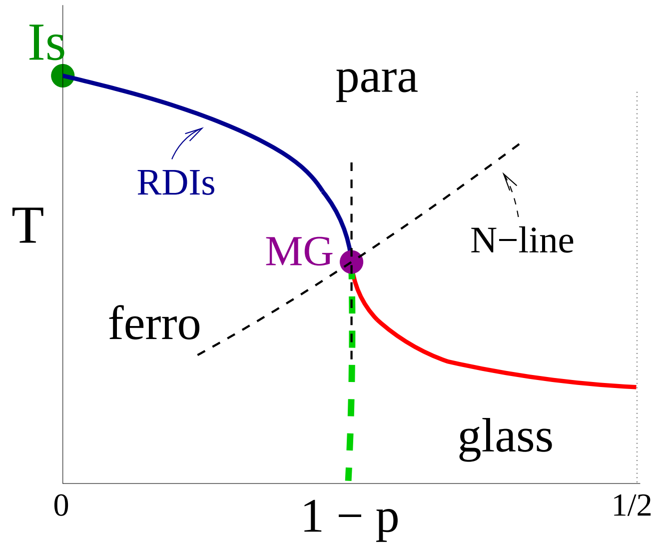
[1] Uniaxial antiferromagnets $Fe_xZn_{1-x}F_2$ (Slanič etal, 1999) [2] To six loops (Pelissetto, EV, 2000) [3] Finite-size scaling analysis of MC data (Hasenbusch, Parisen Toldin, Pelissetto, EV 2007)

The para-ferromagn. transition in **the 3D $\pm J$ Edwards-Anderson Ising model** belong to the RDI universality class.

$H = -\sum_{\langle xy \rangle} J_{xy}\sigma_x\sigma_y$, on a simple cubic lattice, where $\sigma_x = \pm 1$, and $J_{xy} = \pm 1$ are uncorrelated quenched random variables with probability distribution $P(J_{xy}) = p\delta(J_{xy} - 1) + (1-p)\delta(J_{xy} + 1)$.

Simplified model for disordered uniaxial materials which show glassy behavior in their phase diagram, such as $Fe_xMn_{1-x}TiO_3$.

The high-T phase is **paramagnetic**. The low-T phase depends on p: it is **ferromagnetic** for small values of $1-p$, while it is **glassy with vanishing magnetization** for larger values. **High-T** and **low-T** phase are separated by **para-ferro** and **para-glassy** transition lines, which meet at a **magnetic-glassy MCP** (Hasenbusch, etal, 2008)



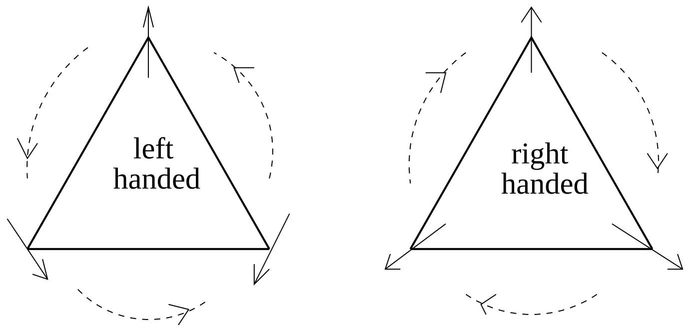
O(M)⊗O(N) theory (ϕ_a are M sets of N-component vectors)

$$\mathcal{L} = \sum_a [(\partial_\mu \phi_a)^2 + r\phi_a^2] + u_0 (\sum_a \phi_a^2)^2 + v_0 \sum_{a,b} [(\phi_a \cdot \phi_b)^2 - \phi_a^2 \phi_b^2]$$

• For $M = 2$, $v_0 > 0$, $O(N) \to O(N-2)$

→ transitions in frustrated systems with noncollinear order, stacked triangular antiferromagnets (STAs), such as $CsMnBr_3$, $CsVBr_3$, modeled by $H = J_\parallel \sum_{\langle vw \rangle_{xy}} \vec{s}(v) \cdot \vec{s}(w) - J_\perp \sum_{\langle vw \rangle_z} \vec{s}(v) \cdot \vec{s}(w) + D \sum_v s_z(v)^2$

Frustration arises from the special geometry → ordered ground state with a chiral 120^o structure

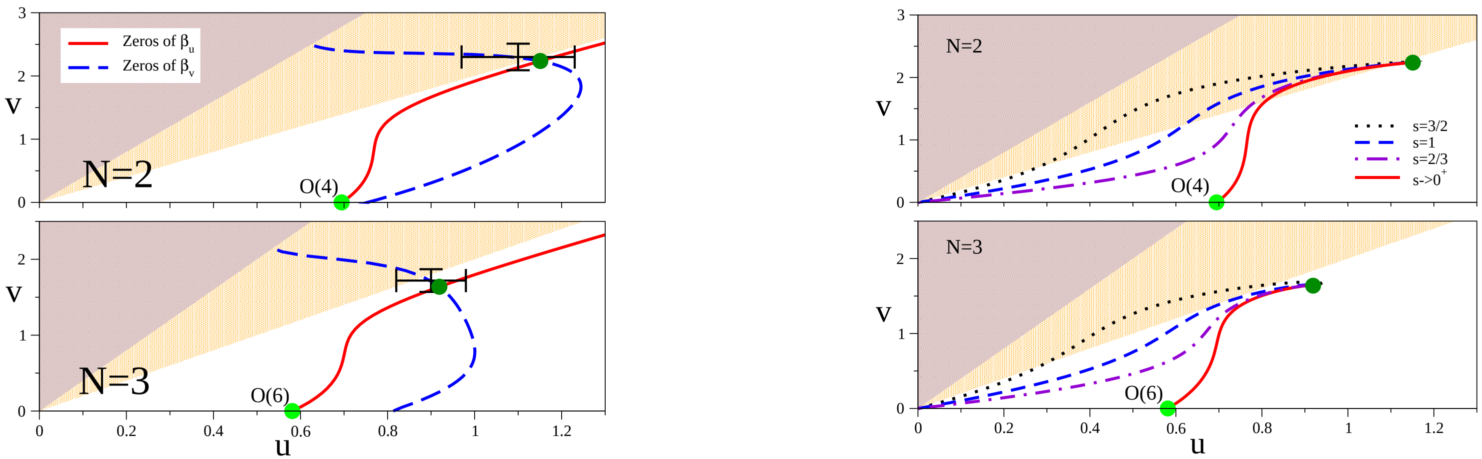


Experiments on STAs show continuous transitions, and evidences of the existence of new chiral universality classes.

• For $M = 2, N = 3$, $v_0 < 0$, $\mathrm{U}(2) \to \mathrm{O}(2)$, transition in ^{3}He

- No stable FP found for $d \lesssim 4$, within $\epsilon = 4 - d$ exp, predicting first-order transitions for all systems, thus leading to an apparent contradiction. **But** the extension to $d = 3$ is not guaranteed: new FPs may appear going from $d \lesssim 4$ to $d = 3$ (this also occurs in superconductors)

- High-order computations within 3D FT schemes show the existence of a stable FP (Pelissetto, Rossi, EV, 2001; Calabrese etal 2004), in agreement with experiments.



Zeroes of the $\overline{\rm MS}$ β functions (left) and RG trajectories in the u, v plane from the Gaussian to the stable chiral FP (right), with $\nu = 0.57(3),\ 0.55(3)$ for $N = 2,\ 3$.

Finite-T transition of QCD with N_f light quarks

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{f=1}^{N_f} \bar\psi_f \left(i\gamma_\mu D_\mu - m_f \right) \psi_f$$

$$Z = {\rm Tr}\, e^{-\beta H} = \int DAD\bar\psi D\psi \, \exp(-S/g^2), \qquad S = \int_0^\beta dt \int d^3x \, \mathcal{L}_{\rm QCD}$$

Chiral symmetry for $m_f = 0$: $\psi_{L,R} \to U(N_f)_{L,R} \psi_{L,R}$

$U(N_f)_L \otimes U(N_f)_R \;\; \simeq U(1)_V \otimes U(1)_A \otimes SU(N_f)_L \otimes SU(N_f)_R$

$U(1)_V$ quark-number conservation, $U(1)_A$ broken by the anomaly,
$SU(N_f)_L \otimes SU(N_f)_R$ broken to $SU(N_f)_V$ due to a nonzero $\langle \bar\psi\psi \rangle$

Phase transition at $T_c \simeq 200$ Mev restoring chiral symmetry

- Order parameter $\to$ $\Phi_{ij} = \bar\psi_{L,i} \psi_{R,j}$, a $N_f \times N_f$ complex matrix
- SB due to $\langle \bar\psi\psi \rangle$: $SU(N_f)_L \otimes SU(N_f)_R \to SU(N_f)_V$

The nature of the transition depends on N_f

The nature of the finite-T transition in QCD can be investigated using RG methods (originally applied by Pisarski, Wilczek, 1984)

Let us assume that the transition is continuous ...

When $\xi \gg 1/T_c$ the system is effectively 3D, then its critical behavior belongs to a 3D universality class characterized by a complex $N_f \times N_f$ matrix order parameter Φ_{ij} and symmetry breaking $\mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R \to \mathrm{SU}(N_f)_V$, or $\mathrm{U}(N_f)_L \otimes \mathrm{U}(N_f)_R \to \mathrm{U}(N_f)_V$ if the $\mathrm{U}(1)_A$ is effectively restored at T_c.

The most general 3D LGW Φ^4 theory compatible with the above properties provides an effective theory of the critical modes at T_c.

No anomaly: $\mathrm{Tr}\partial_\mu \Phi^\dagger \partial_\mu \Phi + r\mathrm{Tr}\Phi^\dagger\Phi + u_0(\mathrm{Tr}\Phi^\dagger\Phi)^2 + v_0\mathrm{Tr}(\Phi^\dagger\Phi)^2$ where $\mathrm{U}(N)_L \otimes \mathrm{U}(N)_R \rightarrow \mathrm{U}(N)_V$ is realized for $v_0 > 0$.

Due to anomaly: $\mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R \rightarrow \mathrm{SU}(N_f)_V$, achieved by adding determinant terms, such as $\det\Phi$.

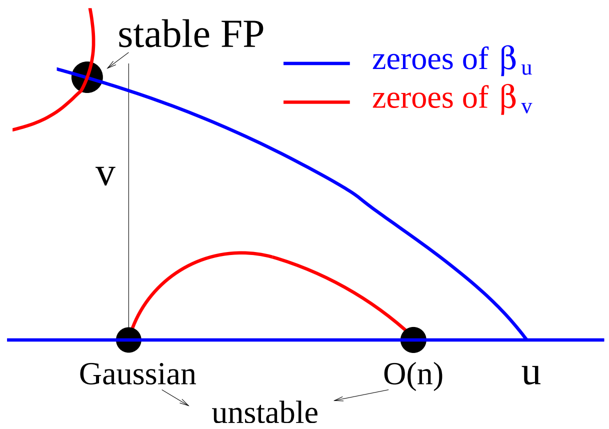
Nonvanishing quark masses correspond to an external field H, i.e. by adding $\mathrm{Tr}H\Phi$

Consistency with the hypothesis of continuous transition requires a stable FP: (i) if no stable FPs exist, the transition of QCD is predicted to be first order; (ii) if a stable 3D FP exists, it can be continuous and its universal critical behavior is determined by the FP. But, it may still be first order if the system is outside the attraction domain of the stable FP. In this case the nature of the transition is not a universal feature.

• **QCD$_{N_f=2}$ neglecting anomaly:** The corresponding universality class exists if there is a stable FP in the 3D U(2)⊗U(2) theory with a complex 2×2 matrix field Φ

$$\mathcal{L}_{\rm U(2)} = {\rm Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r{\rm Tr}\Phi^\dagger\Phi + u_0 \left({\rm Tr}\Phi^\dagger\Phi\right)^2 + v_0 {\rm Tr}\left(\Phi^\dagger\Phi\right)^2$$

In both MZM and 3D $\overline{\rm MS}$ schemes, the analysis of high-order series show the presence of a stable FP (Basile, Pelissetto, EV 2005) → 3D U(2)⊗U(2)/U(2) universality class with $\nu \approx 0.7$, $\eta \approx 0.1$.



This implies that the transition **can** be continuous.

No stable FP is found close to 4D within the ϵ expansion (Pisarski, Wilczek, 1984), thus this FP is peculiar of 3D critical behaviors (as finite-T QCD)

• QCD$_{N_f=2}$ taking into account the U(1)$_A$ anomaly

Symmetry breaking → SU(2)⊗SU(2)/SU(2) ≃ O(4)/O(3), which corresponds to the O(4) universality class.

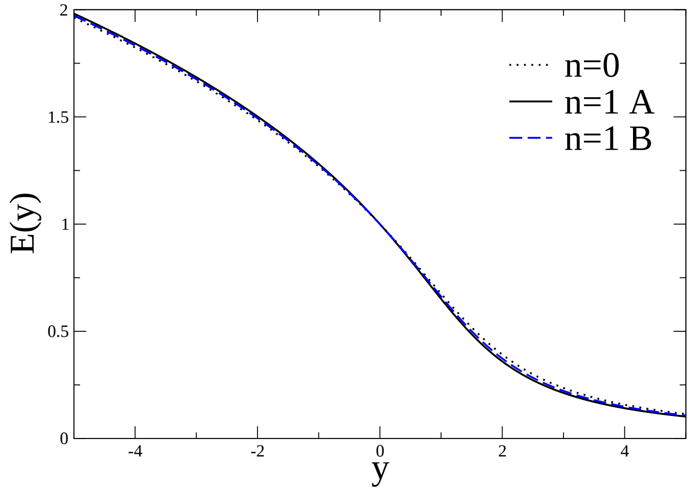
If the transition is continuous, it must show the O(4) scaling behavior

$$\vec{M} \propto \vec{H}|H|^{(1-\delta)/\delta}E(y)$$

$$y \propto t|H|^{-1/(\beta+\delta)}$$

$$\langle\bar{\psi}\psi\rangle \propto |M|, \quad m_f \propto |H|$$

$$\delta = 4.789(6), \quad \beta = 0.3882(10)$$



(O(4) results from Parisen Toldin etal, 2003, and Hasenbusch 2001)

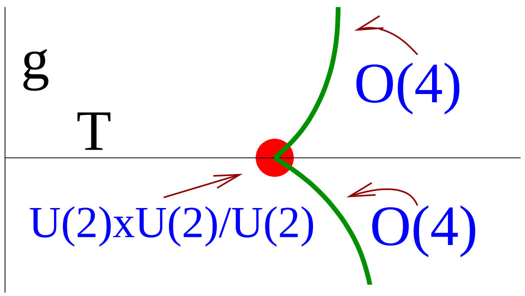
The effects of the anomaly can be also investigated by considering the most general Φ^4 theory with symmetry SU(2)$\otimes$SU(2):

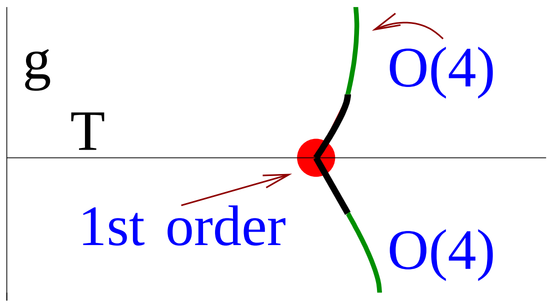
$$\mathcal{L}_{SU(2)} = \mathrm{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r \mathrm{Tr} \Phi^\dagger \Phi + u_0 \left(\mathrm{Tr}\Phi^\dagger \Phi\right)^2 + v_0 \mathrm{Tr}\left(\Phi^\dagger \Phi\right)^2 +$$
$$w_0 \left(\det\Phi^\dagger + \det\Phi\right) + x_0 \left(\mathrm{Tr}\Phi^\dagger\Phi\right)\left(\det\Phi^\dagger + \det\Phi\right) + y_0 \left[(\det\Phi^\dagger)^2 + (\det\Phi)^2\right]$$

where w_0,x_0,$y_0 \sim g$ $\longrightarrow$ effective breaking of $\mathrm{U}(1)_A$ (Pelissetto, EV, 2005)

There are 2 quadratic terms: transition lines in the T-g plane meeting at a MCP controlled by the $\mathrm{U}(2)_L \otimes \mathrm{U}(2)_R$ theory for $g = 0$

in the case of continuous (left) or first order (right) at $g = 0$





O(4) critical behavior if the transition is continuous and $g \neq 0$.

• QCD with $N_f \geq 3 \rightarrow \mathcal{L} = \mathrm{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r\mathrm{Tr}\Phi^\dagger\Phi +$
$u_0 \left(\mathrm{Tr}\Phi^\dagger\Phi\right)^2 + v_0 \mathrm{Tr}\left(\Phi^\dagger\Phi\right)^2 + w_0 \left(\det\Phi^\dagger + \det\Phi\right)$

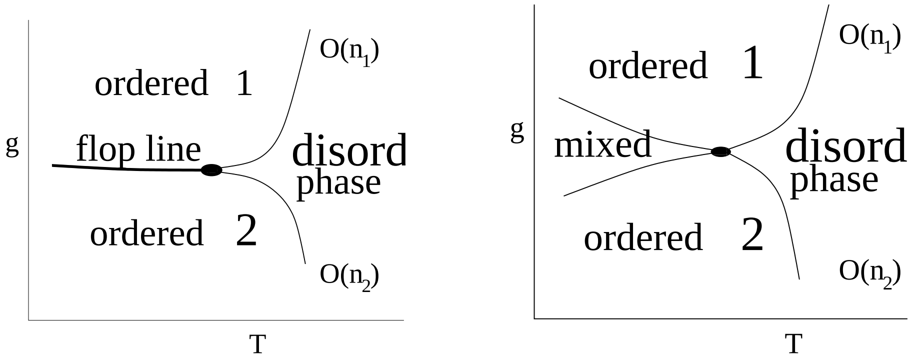
High-order FT analyses do not show any stable FP for $N \geq 3$, therefore the transition QCD is predicted to be first order

Summary of predictions for the finite-T transition of QCD

QCD		no anomaly, $N_c \rightarrow \infty$
	$\mathrm{SU}(N_f) \otimes \mathrm{SU}(N_f)$	$\mathrm{U}(N_f) \otimes \mathrm{U}(N_f)$
$N_f = 1$	crossover or first order	O(2) or first order
$N_f = 2$	O(4) or first order	$\mathrm{U}(2)_L \otimes \mathrm{U}(2)_R / \mathrm{U}(2)_V$ or first order
$N_f \geq 3$	first order	first order

Lattice MC results are substantially consistent with these scenarios

Multicritical behavior arising from the competition of different orderings with symmetries O(n_1) and O(n_2), at the point where the transition lines meet, tuning two relevant parameters T and g



In high-T_c superconductors, anisotropic antiferromagnets, etc...

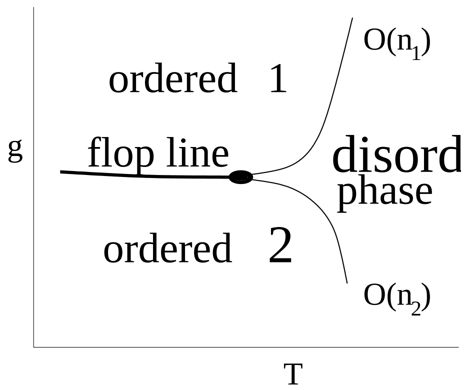
O(n_1)⊕O(n_2) theory with two O(n_1) and O(n_2) vector fields

$$\mathcal{L} = (\partial_\mu \vec{\phi}_1)^2 + (\partial_\mu \vec{\phi}_2)^2 + r_1 \vec{\phi}_1^2 + r_2 \vec{\phi}_2^2 + u_1 (\vec{\phi}_1^2)^2 + u_2 (\vec{\phi}_2^2)^2 + w \vec{\phi}_1^2 \vec{\phi}_2^2$$

Mean-field analysis: if $\delta = u_1 u_2 - 9w^2 < 0$ the MCP is bicritical, for $\delta > 0$ it is tetracritical. **Scaling at the multicritical point**: $\mathcal{F}_{\rm s} \approx t^{d\nu} f(g t^{-\phi})$ where ν and ϕ are universal critical exponents.

The multicritical behavior at the MCP is determined by the stable FP of the RG flow in the quartic-coupling space when $r_{1,2}$ are tuned to their critical values.

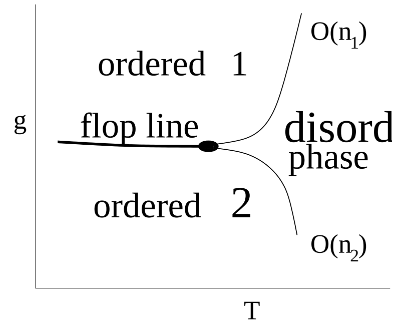
FPs from the zeroes of the $\overline{\rm MS}$ β functions $\beta_{u_1} = \mu\partial u_1/\partial\mu$, $\beta_{u_2} = \mu\partial u_2/\partial\mu$, $\beta_w = \mu\partial w/\partial\mu$. The critical exponents ν and ϕ by evaluating the RG dimensions of the operators ϕ_i and $\phi_i\phi_j$ at the stable FP. The PFT expansion has been computed up to five loops in $\overline{\rm MS}$ expansion (Calabrese etal, 2003)



An interesting possibility is that at the MCP an effective enlargement of the symmetry, $O(n_1) \oplus O(n_2) \longrightarrow O(n_1 + n_2)$, is realized. This requires the stability of the $O(n_1 + n_2)$-symmetric FP.

• In 3D systems, FT calculations show that the $O(n_1 + n_2)$ FP is stable for $n_1 = n_2 = 1$ and unstable when $n_1 + n_2 \geq 3$.

The enlargement of the symmetry can generally occur only when two Ising lines meet at a MCP. In the other cases it requires a tuning of an additional relevant parameter.



• For $n_1 = 1$ (Ising), $n_2 = 2$ (XY), $n_1 + n_2 = 3$, a "biconical" FP is stable (its critical exponents are however very close to the O(3) ones). This result applies to anisotropic antiferromagnets in a uniform magnetic field H along the anisotropy axis

• For $n_1 + n_2 \geq 4$ a decoupled FP is stable. High-T_c superconductors (cuprates, such as $La_{2-x}Sr_xCuO_4$) at low-T exhibit both superconductivity ($n_1 = 2$) and antiferromagnetism ($n_2 = 3$) depending on doping x. Their competition may give rise to a MCP in the T-x phase diagram.

some conclusions ...

• **RG flows of generalized Φ^4 theories describe many critical phenomena.**

• **Field-theory approaches are effective, even in complex cases with several quadratic and quartic parameters**

• **Accurate results are obtained by perturbative expansions and high-order computations, after resummation exploiting Borel summability and the knowledge of their large-order behavior. Satisfactory comparisons with experiments**

Results from PFT may improve by extending the series. This is a very hard numerical task, essentially limited by the computation of the multivariable integrals associated with the huge number of diagrams (they are already $\gtrsim 1000$ at six loops).