Large Nc confinement, universal shocks, and random matrices

EMMI - Workshop

Phase Transitions

in Particle, Nuclear and Condensed Matter Systems

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Outline

- Phase transition in large Nc Yang-Mills theories, in d=2 (Durhuus-Olesen), possible universality and simulations in d=3,4 (Narayanan-Neuberger)

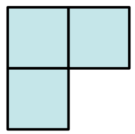
-(Fluid)-dynamics of eigenvalues of Wilson loops, (complex) Burgers equation - Dyson's fluid

- Further developments : finite Nc effects as viscosity corrections - relation with random matrix theory.

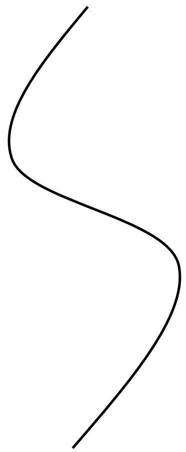
Work done in collaboration with Maciej Nowak
(arXiv: 0801.1859 - PRL 101:102001 (2008)
and arXiv: 0902.2223)

The Wilson loop in SU(N) gauge theories

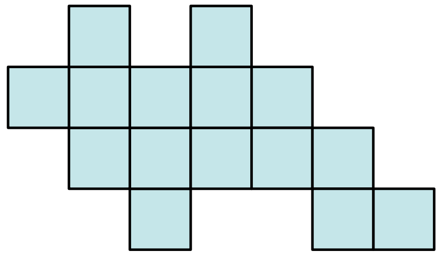
$$W(C) = P\exp\left(i\oint_C A_\mu \mathrm{d}x^\mu\right)$$



Small loops - Short distance physics - weak interactions



Cross-over



large loops

long distance physics

strong interactions

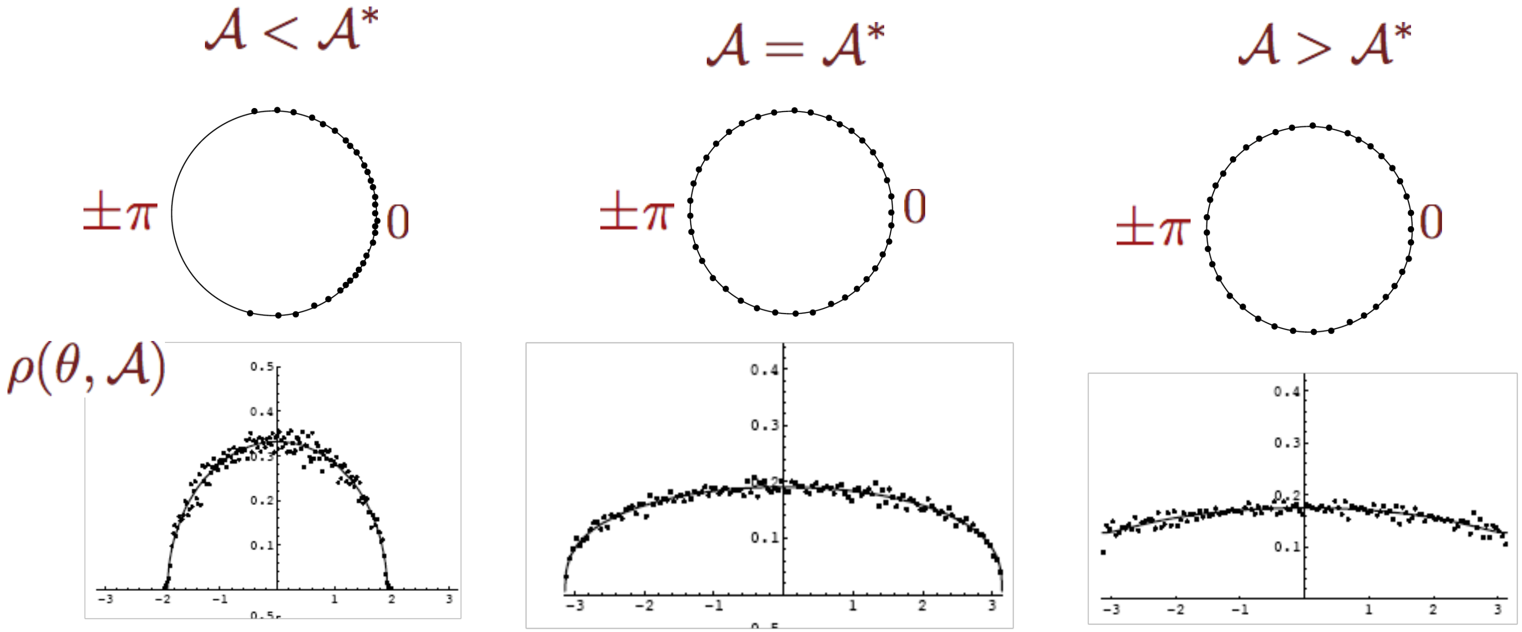
Becomes a sharp transition as $N_c \to \infty \quad (g^2 N_c \ \text{fixed})$

"Confinement-deconfinement" transition (?)

Universal properties in d=2,3,4 (?)

Spectral density as a function of the area A (d=2)

$$\rho(\lambda) \sim \langle \delta(\lambda - W(\mathcal{A})) \rangle_{YM} \qquad \lambda = e^{i\theta} \qquad -\pi \leq \theta \leq \pi$$



Durhuus-Olesen phase transition

(P. Durhuus and P. Olesen, NPB 1981)

The moments of the spectral density $\rho(\theta, \mathcal{A})$

The moments are related to Laguerre polynomials

$$w_n(\mathcal{A}) \equiv \langle \mathrm{Tr}[W(\mathcal{A})^n] \rangle = \int_{-\pi}^{+\pi} \mathrm{e}^{in\theta} \rho(\theta, \mathcal{A})\, d\theta$$
$$= \frac{1}{n} L^1_{n-1}(n\mathcal{A}) \mathrm{e}^{-n\mathcal{A}/2}$$

Remarkable properties at large n $(\mathcal{A}^* = 4)$

$$w_n(\mathcal{A}) \longrightarrow \begin{cases} \frac{1}{n^{3/2}} \times \text{ oscill} & \mathcal{A} < \mathcal{A}^* \\ \frac{1}{n^{4/3}} & \mathcal{A} = \mathcal{A}^* \\ \frac{1}{n^{3/2}} \mathrm{e}^{-nf(A)} & \mathcal{A} > \mathcal{A}^* \end{cases}$$

Much studied

Olesen, Durhuus, Rossi, Kazakov, Douglas, Gross, Gopakumar, Matytsin, Migdal, etc.

A random matrix perspective

(R. Janick and W. Wieczorek, math-ph/0312043)

A model for the Wilson loop

$$W = \lim_{M\to\infty} \lim_{N\to\infty} \prod_{i=1}^{M} U_i \qquad U_i = \mathrm{e}^{i\sqrt{(t/M)}H_i}$$

$$H_i: \quad P(H) \sim \mathrm{e}^{-N\mathrm{Tr}V(H)} \quad \langle \frac{1}{N}\mathrm{Tr}H \rangle = 0 \quad \langle \frac{1}{N}\mathrm{Tr}H^2 \rangle = m_2$$

Resolvent and spectral density

$$G(z) = \int_{-\pi}^{+\pi} d\theta \, \frac{\rho(\theta)}{z - \mathrm{e}^{i\theta}} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{w_n}{z^{n+1}}$$

$$G(z,t) = \frac{1 + f(z,t)}{z}$$

Evolution equation

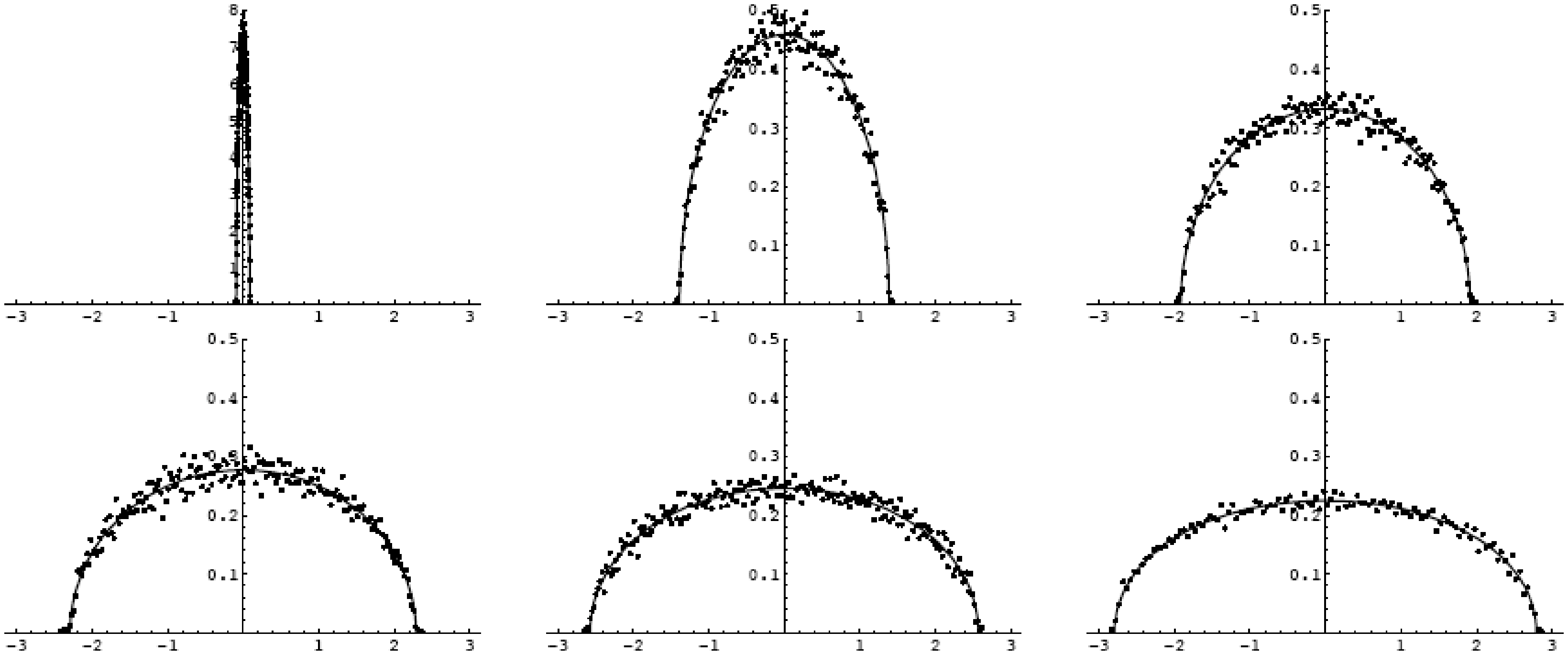
$$zf = (1+f)\mathrm{e}^{-t(f+\frac{1}{2})m_2}$$

$$t \leftrightarrow \mathcal{A} \qquad\qquad t^* = \frac{4}{m_2} \;\leftrightarrow \mathcal{A}^*$$

Evolution from small areas to large areas (1)

$$t = 0 \quad \rho(\theta) = \delta(\theta)$$

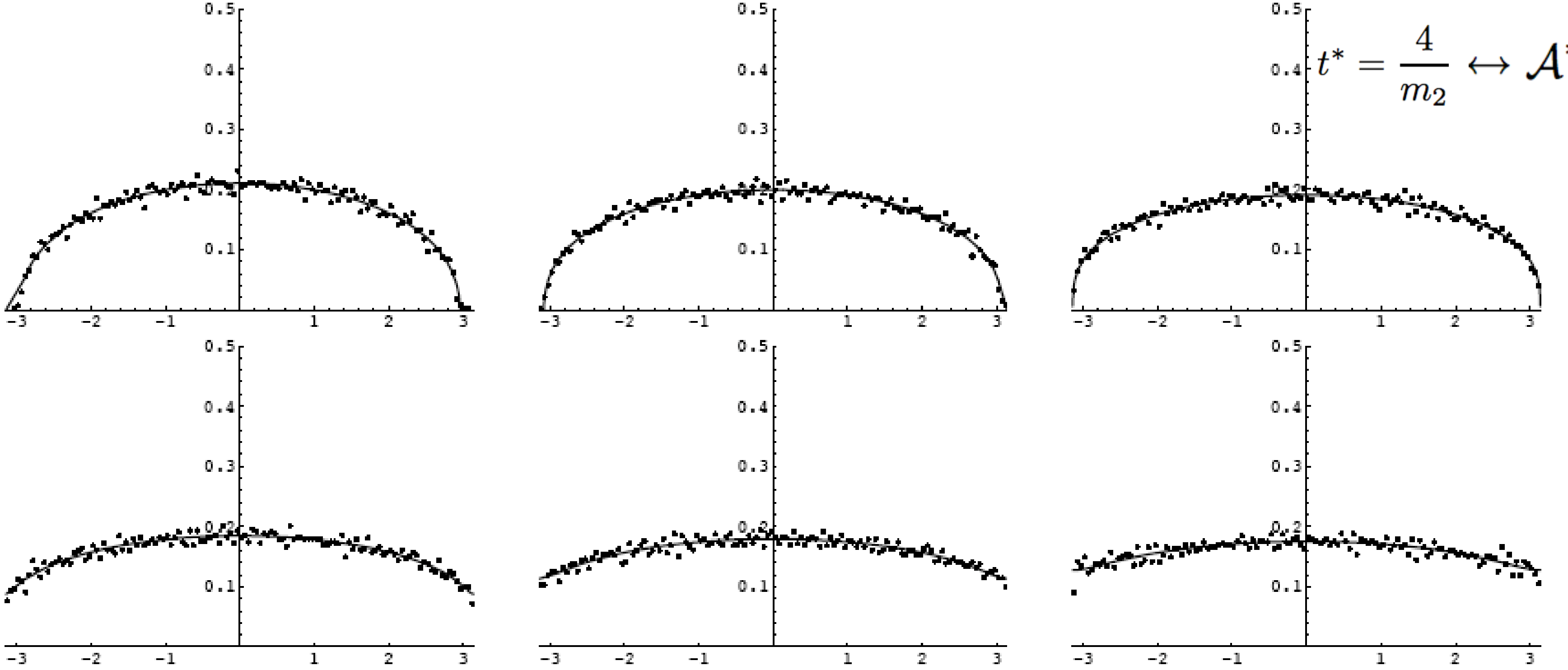
$$\left(f(z) = G(z) = \frac{1}{z-1} \right)$$



(from R. Janick and W. Wieczorek, math-ph/0312043)

Evolution from small areas to large areas (2)

$$t^* = \frac{4}{m_2} \leftrightarrow \mathcal{A}^*$$



(from R. Janick and W. Wieczorek, math-ph/0312043)

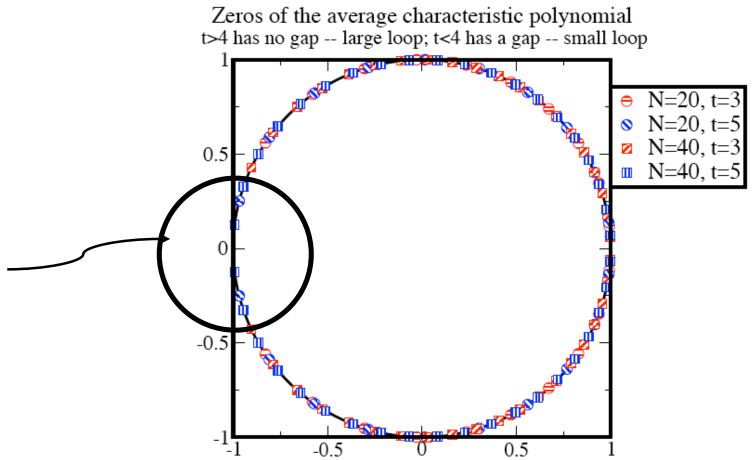
Numerical studies on the lattice

Average characteristic polynomial

$$Q_N(z,t) \equiv \langle \det(z - W(t)) \rangle$$

Closing of the gap is universal

$$d=2, d=3, d=4(?)$$



Double scaling limit

$$z = -\mathrm{e}^{-y}, \qquad y = \frac{2}{12^{1/4} N^{3/4}} \xi, \qquad \frac{1}{t} = \frac{1}{t^*} + \frac{\alpha}{4\sqrt{3N}},$$

$$Q_N(z,t) \to \lim_{N\to\infty} \left(\frac{4N}{3}\right)^{1/4} Z_N(\theta, \mathcal{A}) = \int_{-\infty}^{+\infty} du\, \mathrm{e}^{-u^4 - \alpha u^2 + \xi u}$$

(R.Narayanan and H.Neuberger, arXiv:0711.4551)

(Fluid) dynamics of eigenvalues and the Burgers equation

Define

$$F(\theta) \equiv i\left(zG(z) - \frac{1}{2}\right) = i\left(\frac{1}{2} + \sum_{n=1}^{+\infty} w_n \mathrm{e}^{-in\theta}\right), \qquad z = \mathrm{e}^{i\theta}$$

(Olesesn, Gross, Gopakumar, etc)

$$G(z) = \int_{-\pi}^{+\pi} d\theta \, \frac{\rho(\theta)}{z - \mathrm{e}^{i\theta}} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{w_n}{z^{n+1}}$$

$$F(\theta) = \int_{-\infty}^{+\infty} d\alpha \, \frac{\rho(\alpha)}{\theta - \alpha} = \frac{1}{2}\int_{-\pi}^{+\pi} d\alpha \, \rho(\alpha) \cot\left(\frac{\theta - \alpha}{2}\right)$$

$$H\rho(\theta) = \frac{1}{2\pi} P.V. \int_{-\pi}^{+\pi} d\alpha \, \rho(\alpha) \cot\left(\frac{\theta - \alpha}{2}\right)$$

(Hilbert transform)

$$\rho(\theta) = \frac{1}{\pi} \mathrm{Im} F(\theta - i0_+) = \frac{1}{2\pi}\left(1 + \sum_{n=1}^{+\infty} 2w_n \cos(n\theta)\right)$$

The function $\frac{1}{\pi}F(\theta) = H\rho(\theta) + i\rho(\theta)$ obeys the Burgers equation

$$\boxed{\partial_A F + F \partial_\theta F = 0}$$

Burgers equation
and equations of fluid dynamics

$$\partial_{\mathcal{A}} F + F \partial_\theta F = 0$$

$$(\mathcal{A} \sim \text{time}, \quad \theta \sim \text{position}, \quad F \sim \text{velocity field})$$

Navier-Stokes equation

$$\partial_t u + (u \cdot \nabla) u - \nu \nabla^2 u = -\frac{\nabla P}{\rho}$$

Euler equation

$$\partial_t u + (u \cdot \nabla) u = -\frac{\nabla P}{\rho}$$

Burger=s equation is often used as a model for turbulence (shocks)

Important: F is here complex $\frac{1}{\pi} F(\theta) = H\rho(\theta) + i\rho(\theta)$

(Hopf-Burgers eqn, free random variable calculus, Voiculescu ...)

Solution of the complex Burgers equation

$$\partial_{\mathcal{A}} F + F \partial_\theta F = 0$$

$(\mathcal{A} \sim \text{time}, \quad \theta \sim \text{position}, \quad F \sim \text{velocity field})$

Can be solved with Characteristics

$$F(\mathcal{A}, \theta) = F_0(\xi(\mathcal{A}, \theta)), \qquad F_0(\theta) = F(\mathcal{A} = 0, \theta)$$

$$\theta = \xi + \mathcal{A} F_0(\xi)$$

Characteristics (= Lines of constant 'velocity')

Singularity when

$$0 = \frac{d\theta}{d\xi} = 1 + A F_0'(\xi)$$

The qualitative behavior of the solution is determined by the location of the singularities in the complex plane

$$1 + \mathcal{A}F_0'(\xi_c) = 0$$

Expanding near a singularity $\theta = \xi + \mathcal{A}F_0(\xi)$

$$\theta = \theta_c + \frac{1}{2}(\xi - \xi_c)^2 \mathcal{A}F_0''(\xi_c) + \frac{1}{6}(\xi - \xi_c)^3 \mathcal{A}F_0'''(\xi_c)$$

One can invert the characteristic equation and get

$$\xi(\mathcal{A}, \theta)$$

Asymptotic behavior of moments

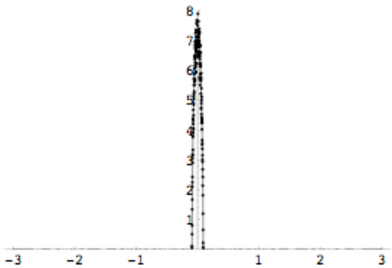
$$\theta_c = \theta^*(\mathcal{A}) + i\Delta(\mathcal{A}) \qquad F \sim (\theta - \theta_c)^\mu$$

$$w_n = |n|^{-(\mu+1)} \mathrm{e}^{-n\Delta(\mathcal{A})} \mathrm{Re}\, \mathrm{e}^{in\theta^*}$$

Gapped phase

$\mathcal{A} < \mathcal{A}^*$

$$\rho(0,\theta) = \delta(\theta)$$



Characterisics

$$\theta = \xi + \mathcal{A}F_0(\xi) \qquad F_0(\xi) = \frac{A}{2}\cot\frac{\xi}{2}$$

singularity

$$1 + \mathcal{A}F_0'(\xi) = 0 \qquad \xi_c = 2\arcsin\frac{\sqrt{\mathcal{A}}}{2}$$

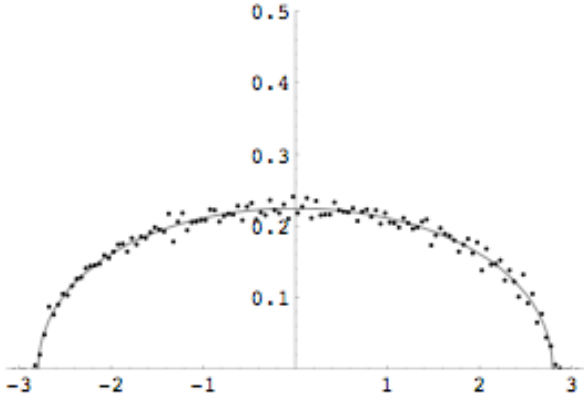
$$\theta_c = \xi_c + \sqrt{\mathcal{A}(1-\mathcal{A}/4)}$$

Vicinity of the singularity

$$\theta = \theta_c + (\xi - \xi_c)^2\sqrt{1/\mathcal{A} - 1/4}$$

$$\xi - \xi_c = \frac{1}{(\mathcal{A}(1-\mathcal{A}/4))^{1/4}}(\theta - \theta_c)^{1/2}$$

$$\longrightarrow \mu = \frac{1}{2}, \Delta(\mathcal{A}) = 0 \qquad w_n \sim \frac{1}{n^{3/2}}$$



$$\rho(\theta) = \frac{1}{\pi}\mathrm{Im}F(\theta - i0_+) = \frac{1}{\pi}\frac{1}{(\mathcal{A}(1-\mathcal{A}/4))^{1/4}}(\theta_c - \theta)^{1/2} \qquad (\theta < \theta_c)$$

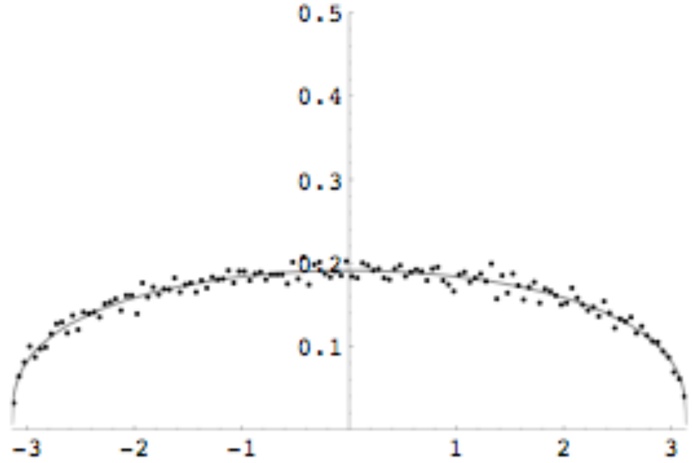
Closing the gap

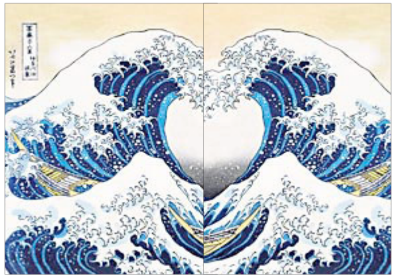
$\mathcal{A} = \mathcal{A}^*$

$$\theta = \pi - \frac{1}{3A^*}(\xi - \xi_c)^3$$

$$\rho(\theta) \sim (\pi - \theta)^{1/3}$$

$$\mu = \frac{1}{3} \longrightarrow w_n \sim n^{-3/4}$$





Gapless phase

$\mathcal{A} > \mathcal{A}^*$

Start with initial conditions

$$\rho(\theta, \mathcal{A}_0) = \frac{1}{2\pi}\left(1 + 2\epsilon\cos\theta\right), \qquad \mathcal{A}_0 \gg 1$$

$$F_0(\xi) = \frac{i}{2}\left(1 + 2\epsilon \mathrm{e}^{-i\xi}\right) \qquad \theta = \xi + (\mathcal{A} - \mathcal{A}_0)F_0(\xi)$$

singularity

$$\mathrm{e}^{i\xi_c} = -\epsilon(\mathcal{A} - \mathcal{A}_0)$$

In vicinity of the singularity $\qquad \theta = \theta_c + \frac{i}{2}(\xi - \xi_c)^2$

Two cases

$$\mathcal{A} > \mathcal{A}_0 \qquad \theta_c = \pi - i\left(1 - \frac{\mathcal{A} - \mathcal{A}_0}{2} + \ln\epsilon(\mathcal{A} - \mathcal{A}_0)\right)$$

$$\mathcal{A} < \mathcal{A}_0 \qquad \theta_c = -i\left(1 - \frac{\mathcal{A} - \mathcal{A}_0}{2} + \ln\epsilon(\mathcal{A}_0 - \mathcal{A})\right)$$

In the second case, the singularity hits the real axis in a finite time A1

$$0 = 1 - \frac{\mathcal{A}_1 - \mathcal{A}_0}{2} + \ln\epsilon(\mathcal{A}_0 - \mathcal{A}_1)$$

Approaching $N_c \to \infty$ (1)

(arXiv 0902.2223 and work in progress)

Dyson's Brownian motion (hermitian matrices)

$$\langle \delta x_i \rangle = E(x_i)\,\Delta t \qquad \langle (\delta x_i)^2 \rangle = \Delta t$$

$$E(x_j) = \sum_{i \neq j} \left(\frac{1}{x_j - x_i} \right)$$

Fokker-Planck equation for the joint probability $P(x_1, \cdots, x_N, t)$

$$\frac{\partial P}{\partial t} = \frac{1}{2} \sum_i \frac{\partial^2 P}{\partial x_i^2} - \sum_i \frac{\partial}{\partial x_i} \left(E(x_i) P \right)$$

Whose solution reads

$$P(x_1, \cdots, x_N, t) = C \prod_{i<j} (x_i - x_j)^2\, \mathrm{e}^{-\sum_i \frac{x_i^2}{2t}}$$

Approaching $N_c \to \infty$ (2)

Average density of eigenvalues ("one-particle density")

$$\tilde{\rho}(x,t) = \int \prod_{k=1}^{N} dx_k \, P(x_1, \cdots, x_N, t) \sum_{l=1}^{N} \delta(x - x_l)$$

Infinite hierarchy of equations

$$\frac{\partial \tilde{\rho}(x,t)}{\partial t} = \frac{1}{2} \frac{\partial^2 \tilde{\rho}(x,t)}{\partial \lambda^2} - \frac{\partial}{\partial \lambda} \mathrm{PV} \int dy \, \frac{\tilde{\rho}(x,y,t)}{x-y}$$

$$\tilde{\rho}(x,y) = \tilde{\rho}(x)\tilde{\rho}(y) + \tilde{\rho}_{con}(x,y)$$

To study the large N limit, rescale

$$\tilde{\rho}(x) = N\rho(x) \qquad \tau = Nt$$

and get

$$\frac{\partial \rho(x)}{\partial \tau} + \frac{\partial}{\partial x} \rho(x) \mathrm{PV} \int dy \, \frac{\rho(y)}{x-y} = \frac{1}{2N} \frac{\partial^2 \rho(x)}{\partial x^2} + \mathrm{PV} \int dy \, \frac{\rho_{con}(x,y)}{x-y}$$

Approaching $N_c \to \infty$ (3)

Resolvent

$$G(z,\tau) = \left\langle \frac{1}{N} \mathrm{Tr} \frac{1}{z - H(\tau)} \right\rangle = \int dy \, \frac{\rho(y,\tau)}{z-y}$$

Average of the characteristic polynomial $\quad \langle \det(z - H(\tau)) \rangle = \prod_{i=1}^{N} (z - \bar{x}_i)$

Equation for $\rho(x,\tau)$ reduces to (inviscid) Burgers eqn. for G (in large N limit)

$$\partial_\tau G(z,\tau) + G(z,\tau)\,\partial_z G(z,\tau) = 0$$

Note that

$$G(z,\tau) = \left\langle \frac{1}{N} \mathrm{Tr} \frac{1}{z - H(\tau)} \right\rangle = \frac{\partial}{\partial z} \left\langle \frac{1}{N} \mathrm{Tr} \ln(z - H(\tau)) \right\rangle = \frac{\partial}{\partial z} \left\langle \frac{1}{N} \ln\det(z - H(\tau)) \right\rangle$$

$$F(z,\tau) = \frac{\partial}{\partial z} \frac{1}{N} \ln \langle \det(z - H(\tau)) \rangle \qquad F(z,\tau) \approx G(z,\tau) \text{ as } N \to \infty$$

F fulfills the viscid Burgers equation (EXACTLY !)

$$\partial_\tau F(z,\tau) + F(z,\tau)\,\partial_z F(z,\tau) = -\frac{1}{2N} \partial_z^2 F(z,\tau)$$

(see also Neuberger, arXiv 0806.0149 , 0809.1238)

Conclusions

- Many features of the large Nc transition are coded in the solution of a Simple Burgers equation (universal shocks, etc).

- Provides a simple understanding for the remarkable universality that is emerging from lattice calculations

- Finite Nc corrections appears as « viscous » effects in the fluid of eigenvalues

- A general picture emerges in the framework of random matrix theory