Effects of the U(1) Anomaly at the QCD Chiral Phase Transition

Shailesh Chandrasekharan
Duke University

(in collaboration with Abhijit Mehta)
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Phys. Rev. Lett. 99 142004 (2007)

Phase Transitions in Particle, Nuclear and Condensed Matter Systems
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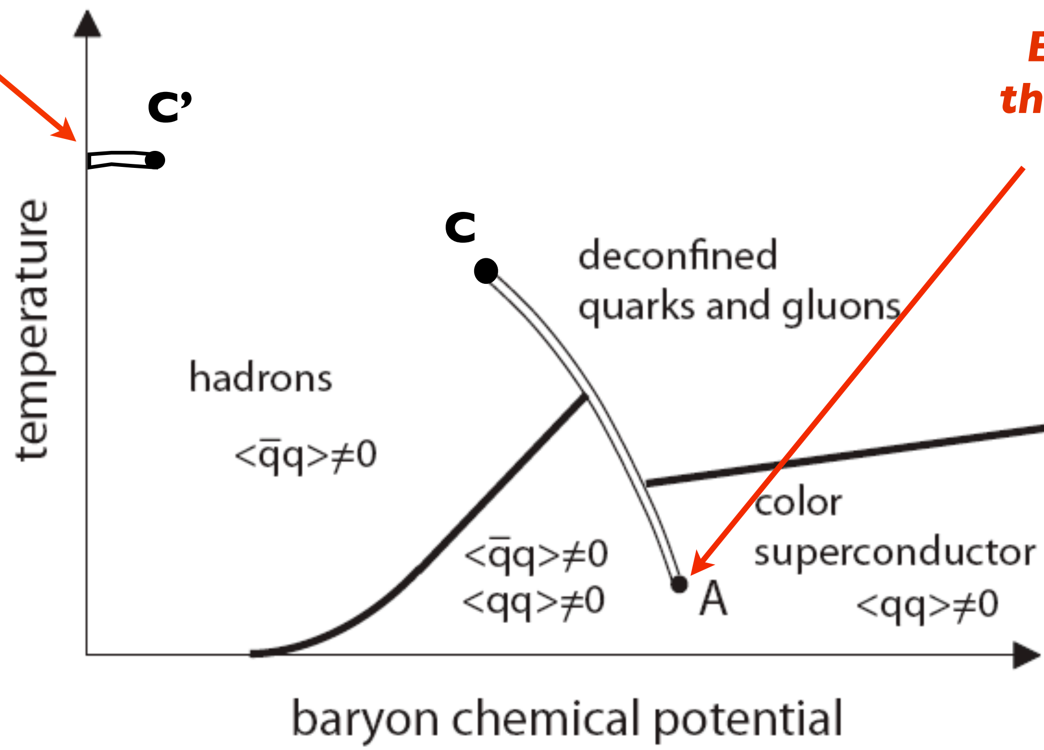
Phase Diagram of QCD

Baym, Hatsuda, Tachibana and Yamamoto, (2008)

Effects of the anomaly

Effects of the anomaly



Anomaly in QCD

- Anomaly is the name given to a classical U(1) axial symmetry of the action that is broken by quantum effects. (A cutoff breaks this symmetry!).
- The "Lattice Action" breaks the symmetry explicitly!
- Need the continuum limit to get the right anomaly.
- Yesterday we learned that the anomaly can change the nature of the chiral phase transition.
- So unless we get the anomaly right we may not have a complete understanding of the chiral transition.
- Yet another reason to worry about continuum extrapolations.

Predictions from RG

Basile, Pelisetto and Vicari, PoS(LAT2005) 199.

Ettore Vicari (this workshop)

Order parameter $\left\langle \psi_i(x)\overline{\psi}_j(x) \right\rangle = \Sigma_{ij}(x)$ ➡ N_f x N_f complex matrix

chiral symmetry transformation: $\Sigma \implies L\ \Sigma\ R^\dagger e^{i\theta}$

The Landau-Ginzburg Lagrangian

$$\mathcal{L} = \mathrm{Tr}\left(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger\right) + t\,\mathrm{Tr}\left(\Sigma\Sigma^\dagger\right) + g_1\left(\mathrm{Tr}\left[\Sigma\Sigma^\dagger\right]\right)^2 + g_2\mathrm{Tr}\left(\Sigma\Sigma^\dagger\Sigma\Sigma^\dagger\right) + \underset{\substack{\uparrow \\ \text{Anomaly Strength}}}{c}\left(\mathrm{Det}(\Sigma) + \mathrm{Det}(\Sigma^\dagger)\right)$$

c = 0 : Invariance under $SU(N_f)$ x $SU(N_f)$ x $U(1)$

c ≠ 0 : Invariance under $SU(N_f)$ x $SU(N_f)$ x Z_{N_f}

Predictions continued....

- $N_f \geq 3$: First order
- $N_f = 2$ + large anomaly : Second order O(4) universality
- $N_f = 2$ + small anomaly : (O(4) or first order)?

More relevant to QCD

If first order how large should c be?
(non-universal question!)

Relevant to large N_c QCD

- $N_f = 2$ + no anomaly (New FP or first order)?

If a new FP, can we find it?

Leneghan, PRD 63, 037901 (2001)
Mean Field Theory

Status of Lattice QCD Calculations

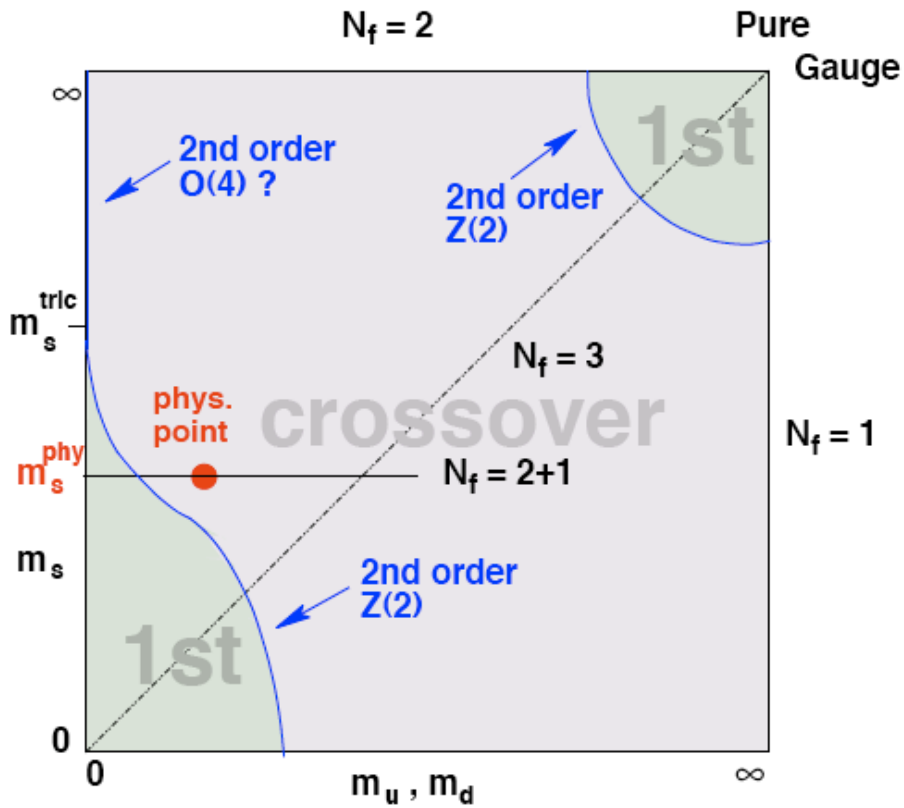
Currently lattice calculations suggest that the physical world is in a crossover region of the phase diagram.

But many uncertainties remain!

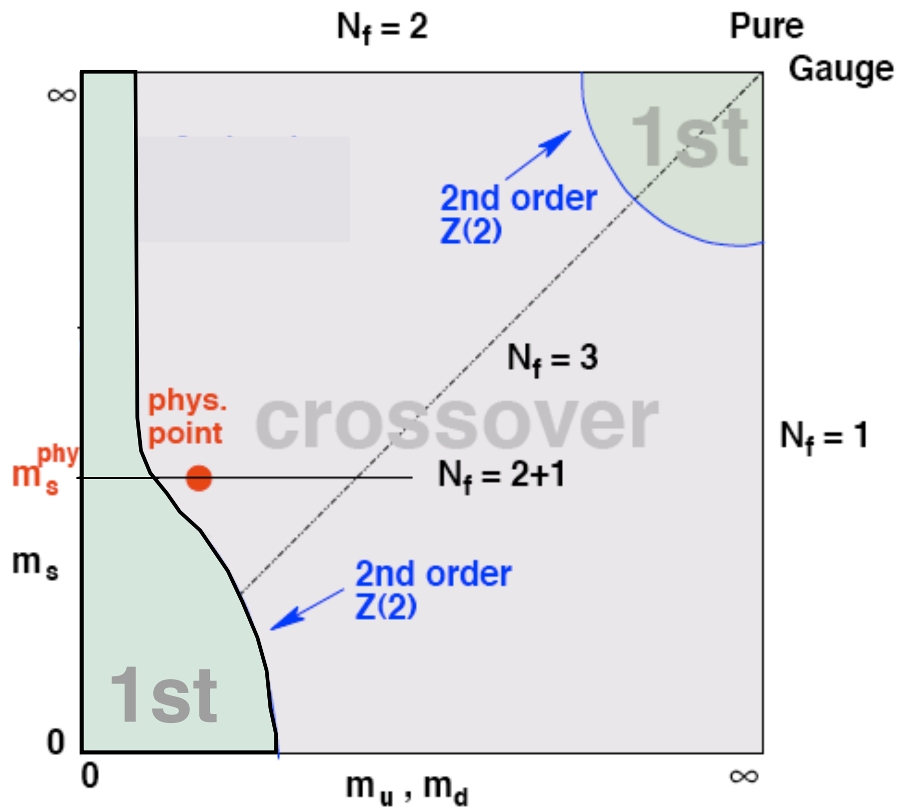
1. Continuum extrapolations are difficult
2. Chiral symmetries do not emerge without continuum limit
3. Difficult to make pions very light
4. Chiral extrapolations are difficult
5. ***Anomaly strength uncontrolled: depends on lattice artifacts***

Summary of results : Columbia Plot

U. Heller, PoS (LATTICE 2006).



Columbia Plot again but with a weak anomaly and a first order transition !



How weak should the anomaly be for this change to occur?

Problems with Monte Carlo!

- Spin models are usually well studied with Monte Carlo methods.
- But comparitively very little work on the phase structure in chiral models (like the one relevant for QCD with an anomaly).
- The new FP's which have been proposed in many cases remain unexplored
- Lack of efficient algorithms
- In particular cluster algorithms do not work.
- A new approach is necessary

An algorithmic motivation for our work!

Our model

- Model is constructed with fermionic degrees of freedom.
- Pions arise from confinement and chiral symmetry breaking just like in QCD.
- A well defined lattice field theory with all the chiral symmetries of QCD intact.
- Has a parameter to tune the strength of the anomaly.
- Can be studied with efficient "cluster" algorithms in the chiral limit and close to it.

Our Model: Strongly coupled QED

d=3

Action

$$S = -\sum_x \sum_{\mu=1}^{d+1} \eta_{\mu,x} \left[\mathrm{e}^{i\phi_{\mu,x}} \overline{\psi}_x \psi_{x+\hat{\mu}} - \mathrm{e}^{-i\phi_{\mu,x}} \overline{\psi}_{x+\hat{\mu}} \psi_x \right] - \sum_x \left[m \overline{\psi}_x \psi_x + \frac{\tilde{c}}{2} \left(\overline{\psi}_x \psi_x \right)^2 \right]$$

U(1) gauge theory

no gauge action ⟶ strongly coupled!

mass

Anomaly

$$\psi_x = \begin{pmatrix} u_x \\ d_x \end{pmatrix}$$ ⟵ Two flavors ⟶ $$\overline{\psi}_x = \begin{pmatrix} u_x & d_x \end{pmatrix}$$

$$\left(\eta_{\mu,x} \right)^2 = 1, \mu = 1, 2, 3$$

$$\left(\eta_{4,x} \right)^2 = T$$

Lattice size L x L x L x L_t

$L_t = 4$

Temperature parameter

Model has symmetries of N_f=2 QCD

Symmetries

Our model contains all the chiral symmetries

$$S = -\sum_x \sum_{\mu=1}^{d+1} \eta_{\mu,x} \left[\mathrm{e}^{i\phi_{\mu,x}} \overline{\psi}_x \psi_{x+\hat{\mu}} - \mathrm{e}^{-i\phi_{\mu,x}} \overline{\psi}_{x+\hat{\mu}} \psi_x \right] - \sum_x \left[m \overline{\psi}_x \psi_x + \frac{\tilde{c}}{2} \left(\overline{\psi}_x \psi_x \right)^2 \right]$$

When $m = 0$ and $\tilde{c} = 0$

the action is invariant under $SU(2) \times SU(2) \times U(1)$.

If x is even

$$\psi_x \longrightarrow \mathrm{e}^{i\theta} \ L \ \psi_x,$$

$$\overline{\psi}_x \longrightarrow \overline{\psi}_x \ R^\dagger \ \mathrm{e}^{i\theta},$$

$L \ , R \in SU(2)$

If x is odd

$$\psi_x \longrightarrow \mathrm{e}^{-i\theta} \ R \ \psi_x,$$

$$\overline{\psi}_x \longrightarrow \overline{\psi}_x \ L^\dagger \ \mathrm{e}^{-i\theta},$$

When $\tilde{c} \neq 0$ then the $U(1)$ symmetry is broken to Z_2.

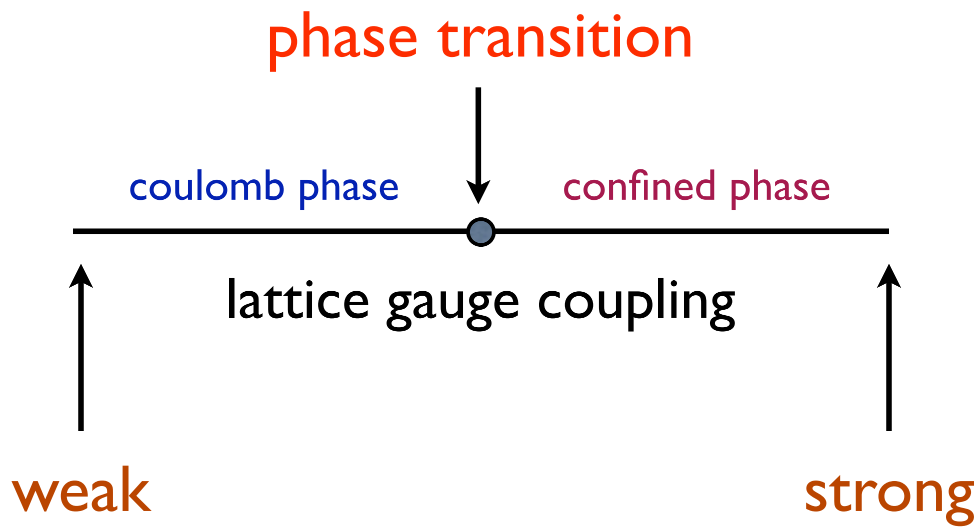
$$\frac{1}{2}(\overline{\psi}_x \psi_x)^2 \equiv \overline{u}_x u_x \overline{d}_x d_x$$

$$\overline{u}_x u_x \overline{d}_x d_x \rightarrow \mathrm{Det}(R^\dagger L \mathrm{e}^{i2\theta}) \overline{u}_x u_x \overline{d}_x d_x$$

$$\overline{u}_x u_x \overline{d}_x d_x \rightarrow \mathrm{Det}(L^\dagger R \mathrm{e}^{i2\theta}) \overline{u}_x u_x \overline{d}_x d_x$$

How can QED show confinement and chiral symmetry breaking?

On the lattice there is a confined phase at strong couplings!



The world-line approach

The partition function of our model is given by

$$Z \;=\; \int [d\overline{\psi}\; d\psi] \int [d\phi]\; \mathrm{e}^{-S(\overline{\psi},\psi,\phi)}$$

It is usually believed that

"... there is no way to represent Grassmann variables on a computer so we need to integrate them away! ... "

So in the conventional approach

$$Z = \int [d\phi] \sum_{[\theta]} \;\mathrm{Det}(M([\theta,\phi]))$$

auxiliary field

For Monte Carlo purposes this is a very inefficient approach in many cases

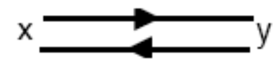
But, recently a new and a more natural approach has emerged!

Grassmann variables can be used to generate fermion world line configurations

For example consider a single bond term with one flavor

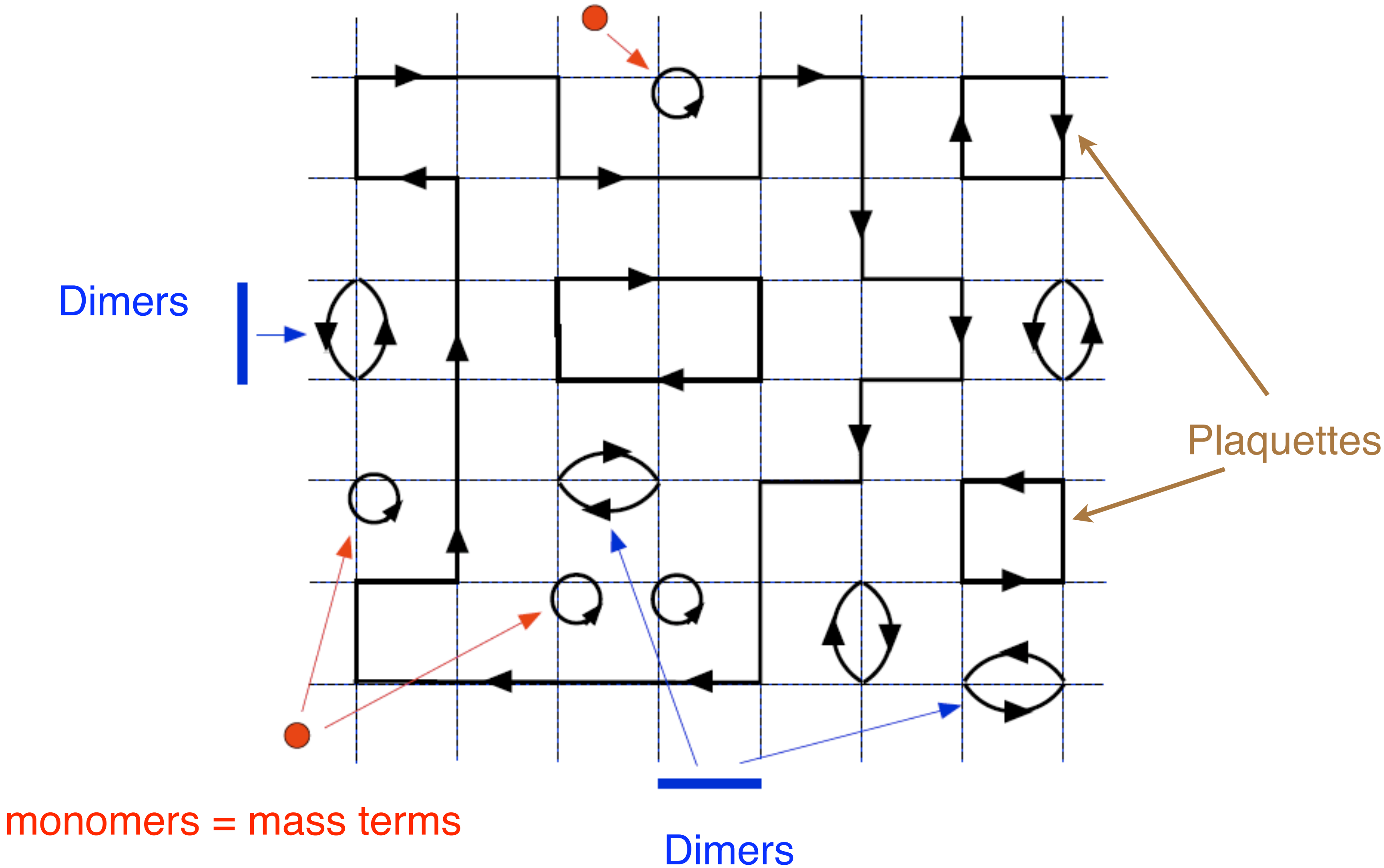
$$\mathrm{e}^{\eta_{xy}[\overline{\psi}_x\psi_y\mathrm{e}^{i\phi_{xy}}-\overline{\psi}_y\psi_x\mathrm{e}^{-i\phi_{xy}}]} = 1+\eta_{xy}\overline{\psi}_x\psi_y\mathrm{e}^{i\phi_{xy}} - \eta_{xy}\overline{\psi}_y\psi_x\mathrm{e}^{-i\phi_{xy}} + \overline{\psi}_x\psi_x\overline{\psi}_y\psi_y$$

Pictorially the terms with Grassmann variables can be represented as

		
$\eta_{xy}\overline{\psi}_x\psi_y\mathrm{e}^{i\phi_{xy}}$	$-\eta_{yx}\overline{\psi}_y\psi_x\mathrm{e}^{-i\phi_{xy}}$	$-\overline{\psi}_x\psi_y\overline{\psi}_y\psi_x$ $= \psi_x\overline{\psi}_x\psi_y\overline{\psi}_y$

This representation can be generalized to all models!

Fermion World-line configurations



Thus, the fermionic partition function can be written as

$$Z = \sum_{C \in \text{fermion loops}} \text{Sign}([C])\, W([C])$$

The sign function depends on the topology of the loop:

1. There is a negative sign for every backward bond.
2. There is a sign factor that comes from local phases.
3. Every fermion loop is given a negative sign.

This representation is useful in Monte Carlo only if sign problems can be solved!

Research over the past decade shows that in many models the sign problem can be solved in novel ways!

In our model "all" signs cancel and the problem is equivalent to the statistical mechanics of a class monomer-dimer configurations which we call MDPI configurations.

MDPI configurations

Rossi and Wolff (1984), Karsch and Mutter (1991)

MDPI: Monomer Dimer Pion-loop Instanton

Site variables:

Instanton: $I(x) = 0, 1$

u-monomers: $n_u(x) = 0, 1$

d-monomers: $n_d(x) = 0, 1$

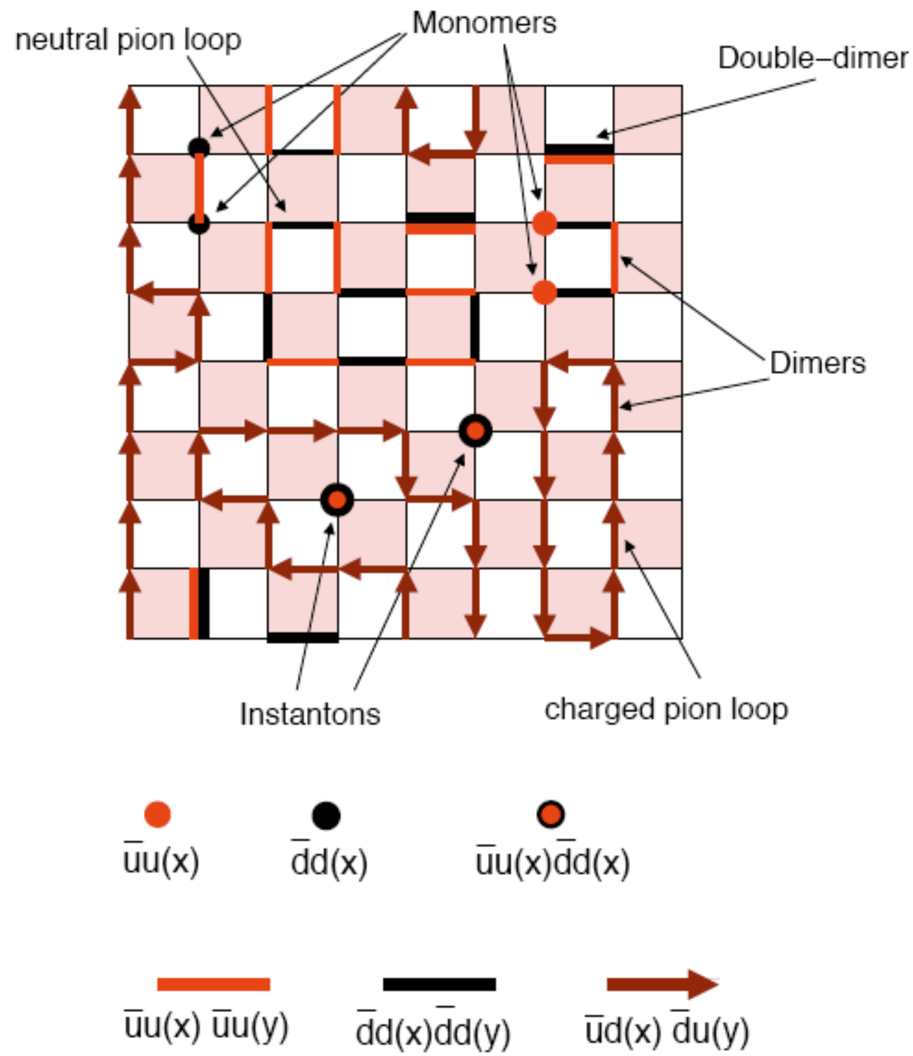
link variables:

uu-dimers $\pi^u_\mu(x) = 0, 1$

dd-dimers $\pi^d_\mu(x) = 0, 1$

ud-dimers $\pi^1_\mu(x) = -1, 0, 1$

There are constraints!



Constraints of MDPI configurations

$$\sum_\mu \pi^1_\mu(x) \;=\; 0$$

$$2I(x) + \sum_\mu \left[\pi^u_\mu(x) + \pi^d_\mu(x) + n^u(x) + n^d(x)\right] + \sum_\mu \left|\pi^1_\mu(x)\right| \;=\; 2$$

$$n_u(x) + \sum_\mu \left[\pi^u_\mu(x) - \pi^d_\mu(x)\right] - n_d(x) \;=\; 0$$

MDPI representation of the model

$$Z = \sum_{\mathcal{K}} \prod_x m^{n_u(x)+n_d(x)} c^{I(x)} \; T^{|\pi^u_5(x)+\pi^d_5(x)+\pi^1_5(x)|}$$

$$c \equiv \tilde{c} + m^2.$$

Sum is over allowed MDPI configurations

Update Algorithms

Prof'ev & Svustinov (2000), Sandvik (2001), Adams & S.C (2002)

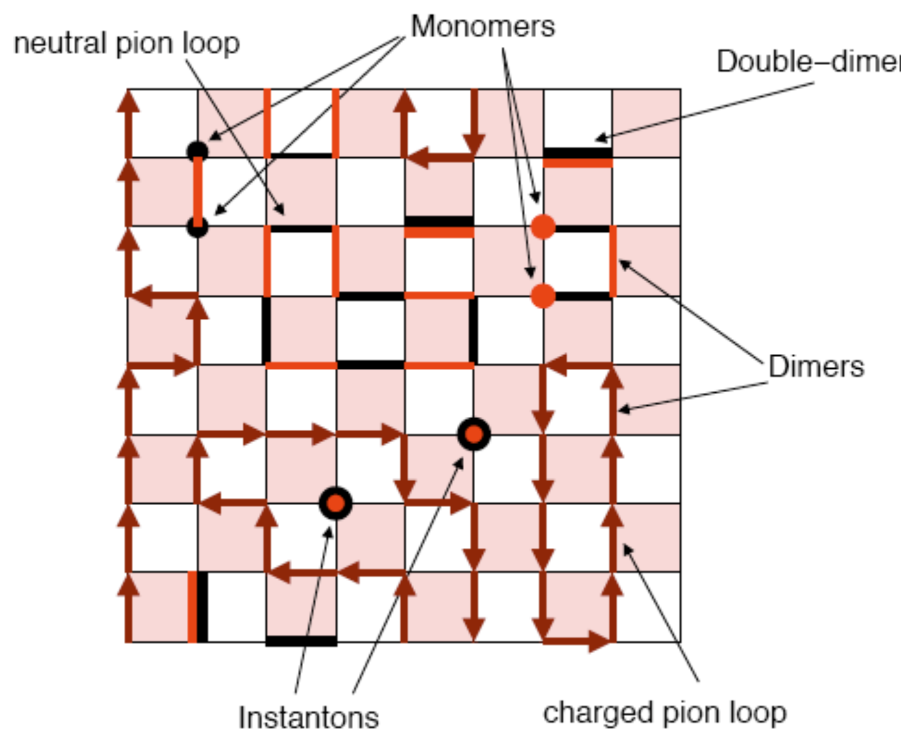
There are four updates in the algorithm

A. Loop Flip update

B. Loop Swap update

C. Directed Loop update
of charged pions

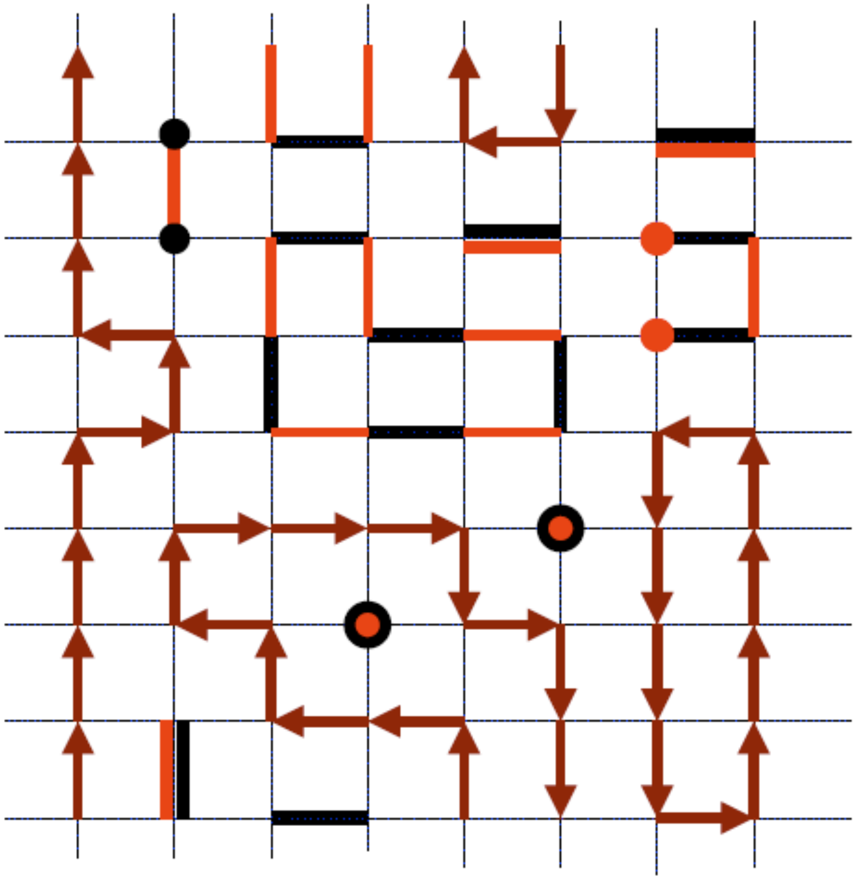
D. Directed Loop update
of neutral pions

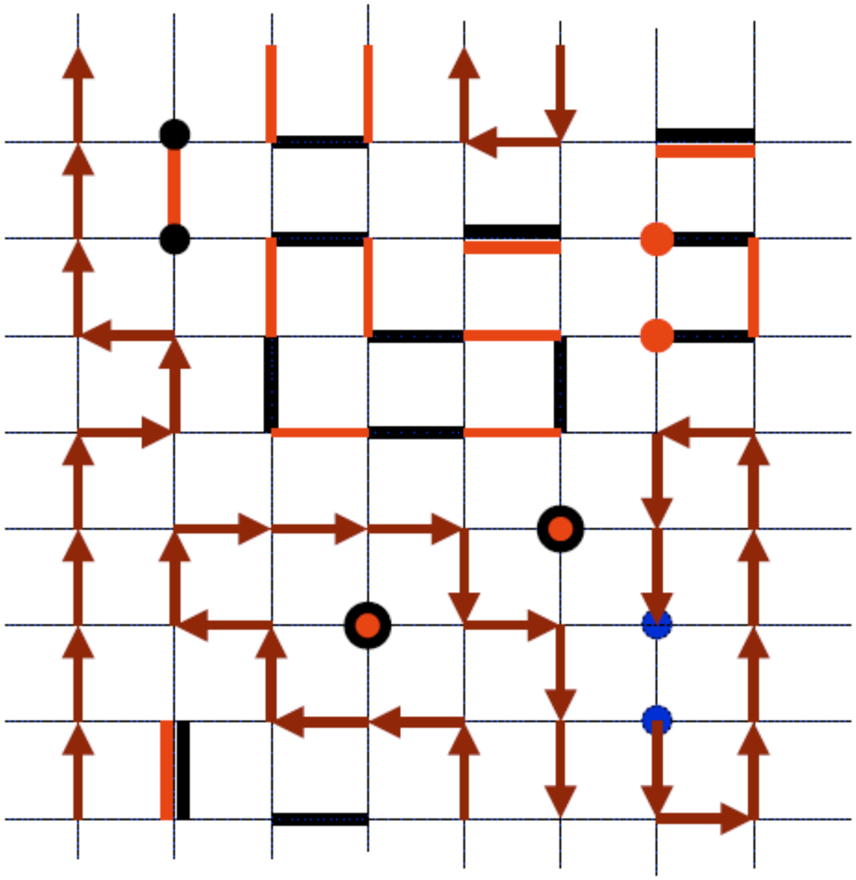


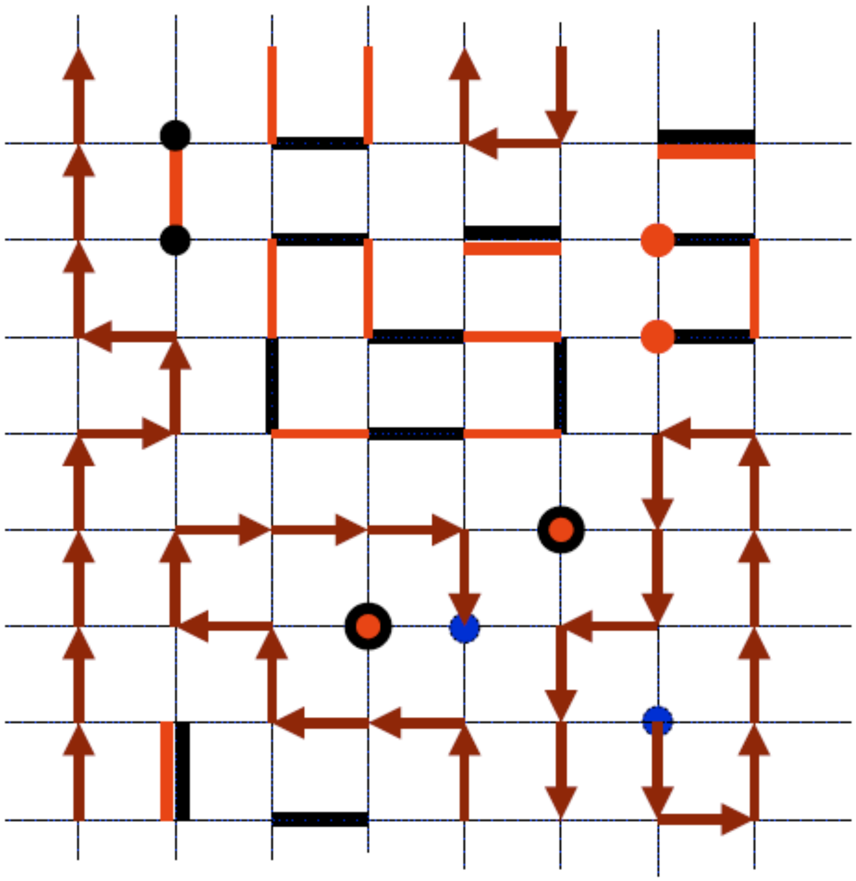
chiral limit poses no difficulties!

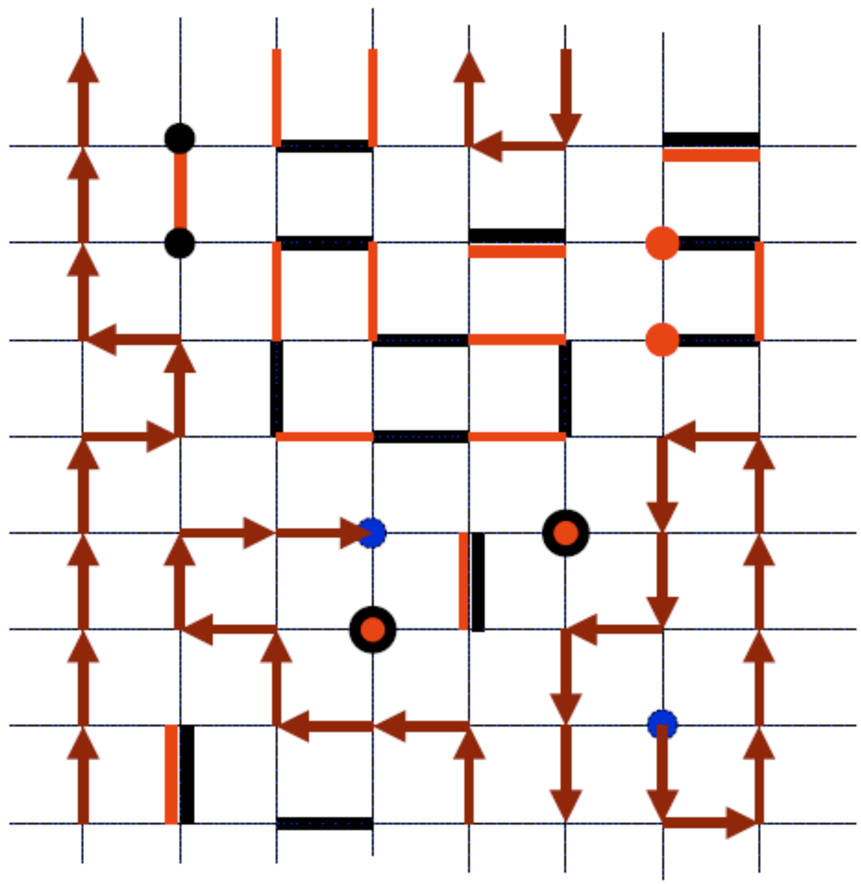
Example of a "Directed Loop update for charged pions"

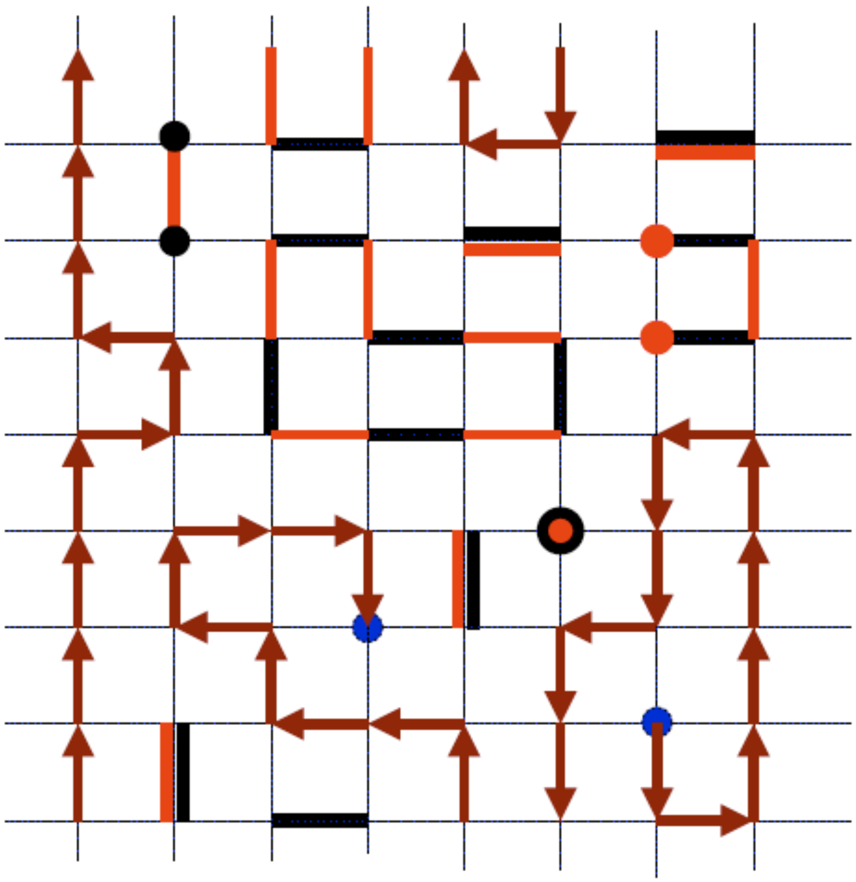
First configuration

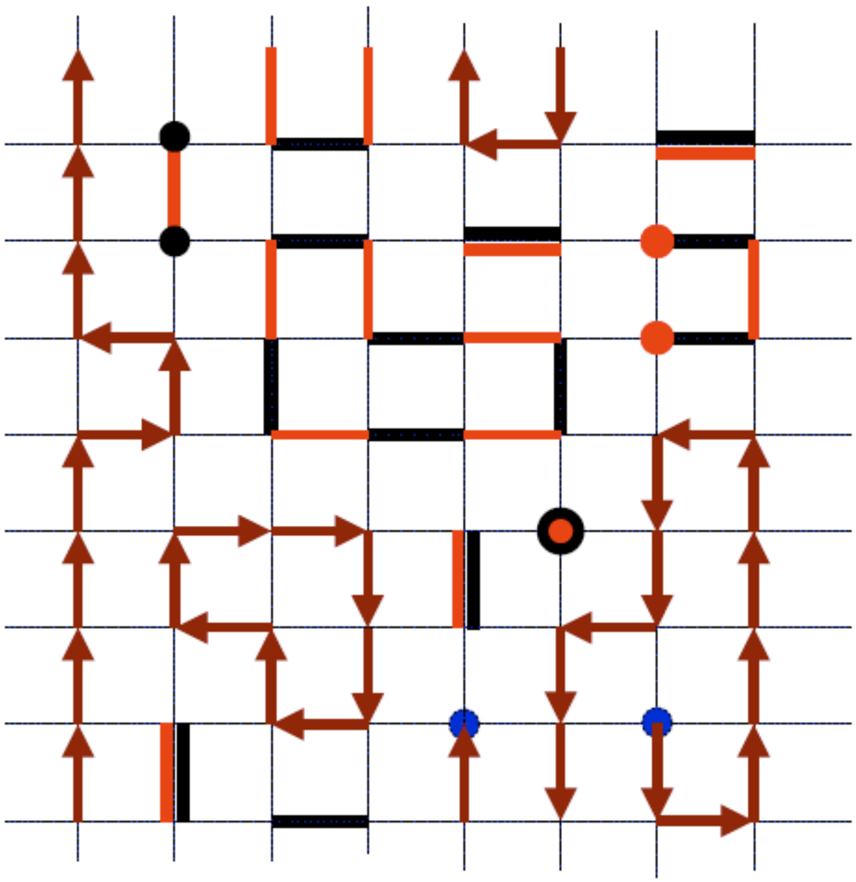


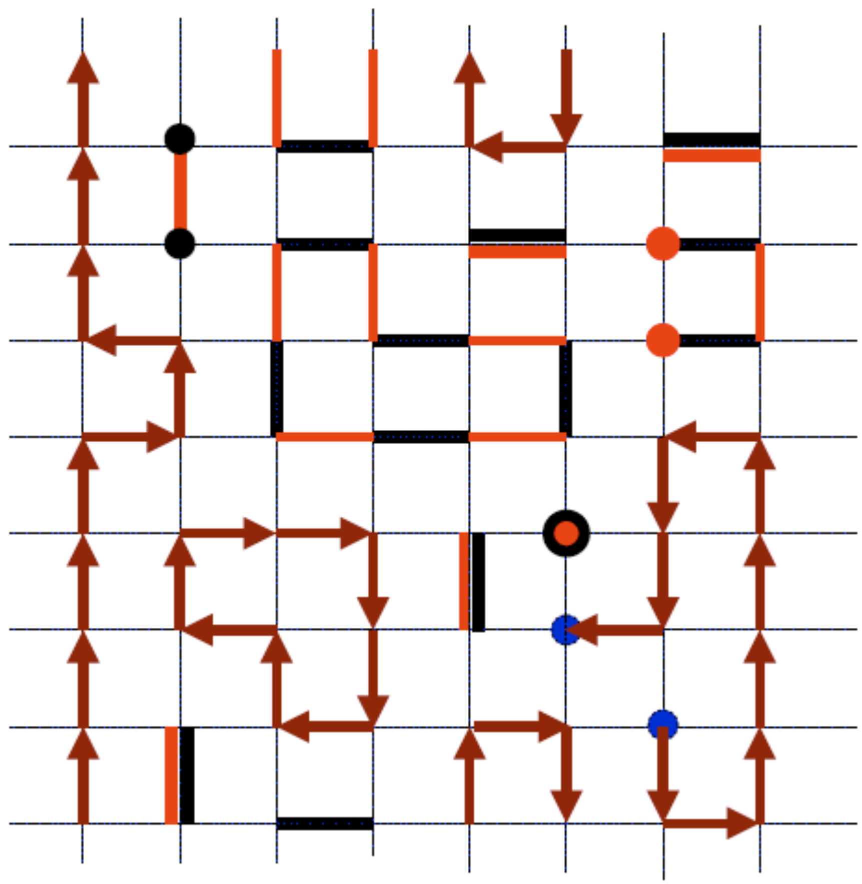


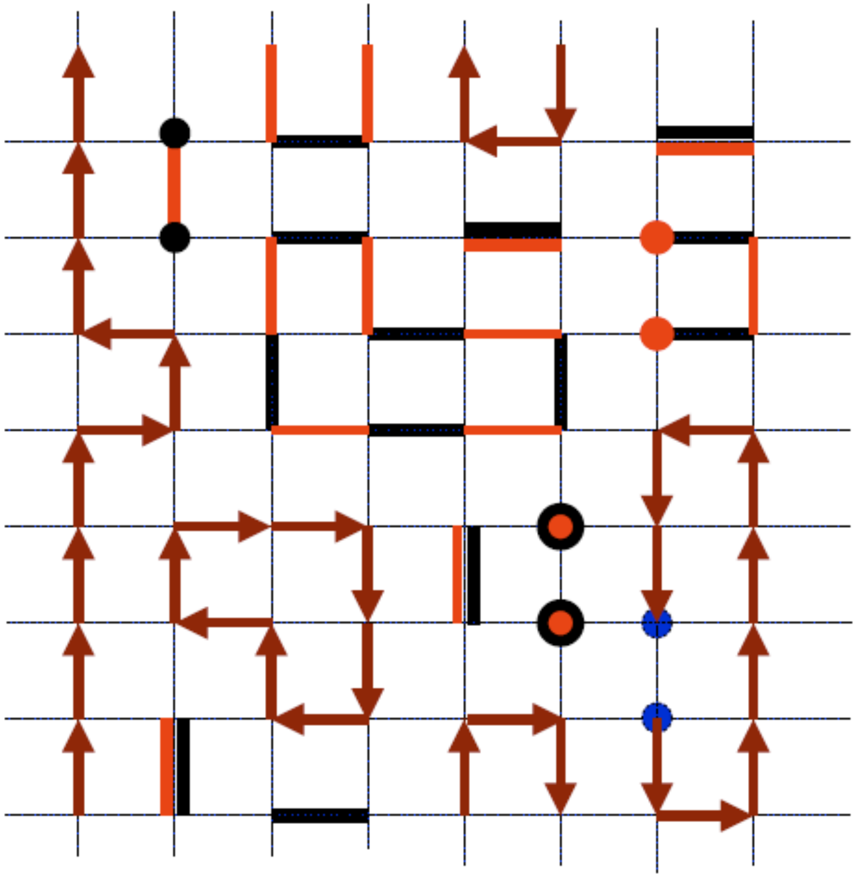




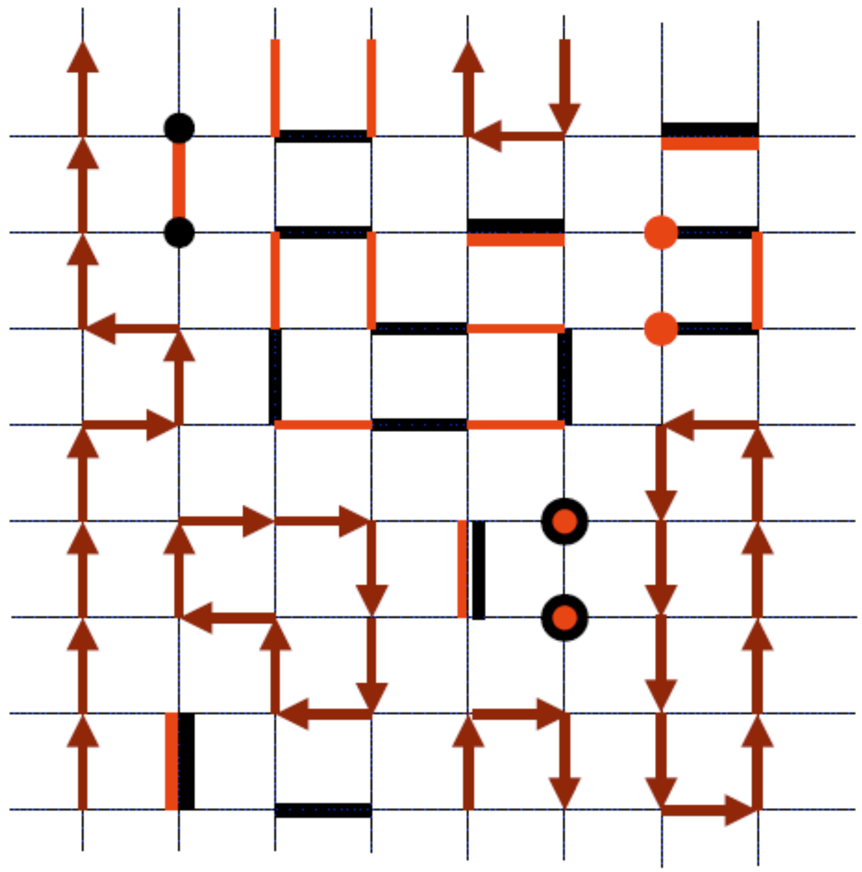








New configuration

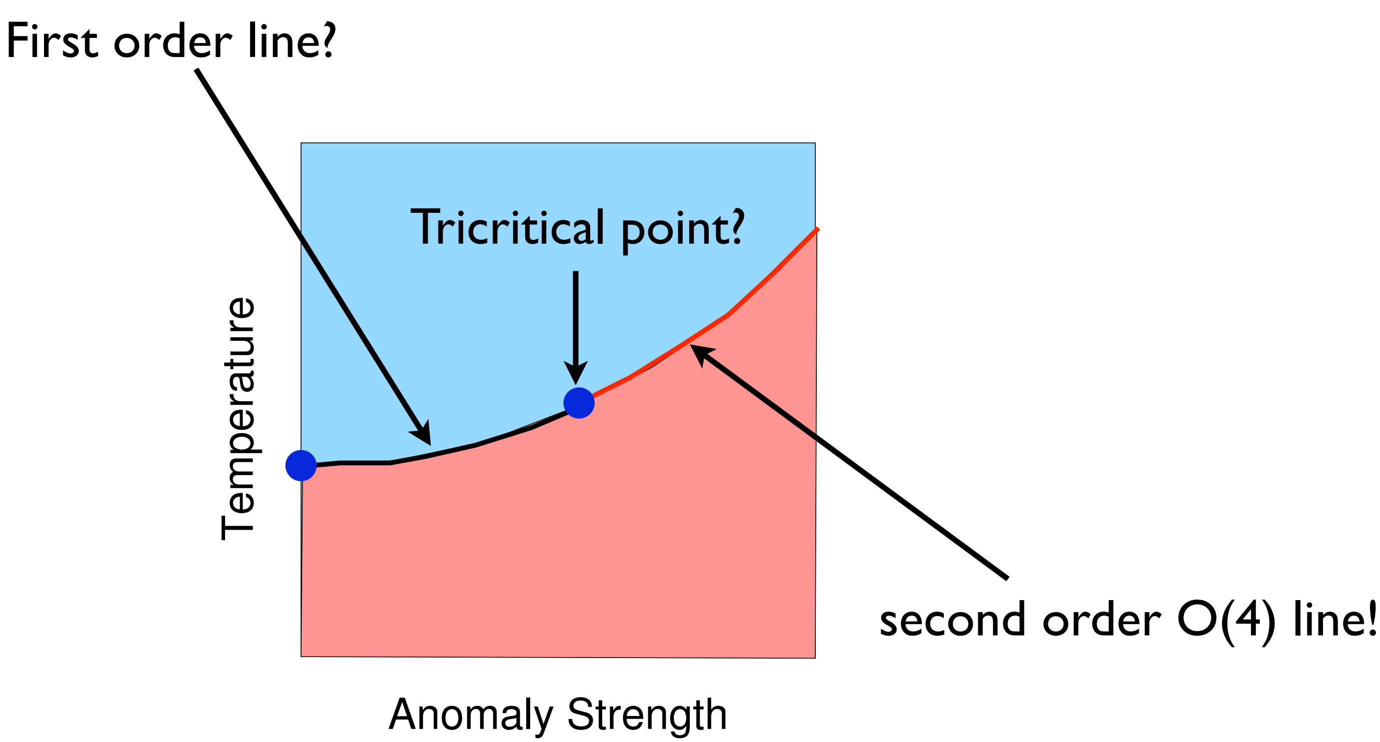


Results: Expected Phase Diagram

Fixed Running Parameters

$L_t = 4$ $m=0$

Study the physics as a function of T and c and box size L x L x L



Observables

current-current susceptibility

$$Y_i = \frac{1}{dL^d}\left\langle \sum_{\mu=1}^{d}\left(\sum_x J_\mu^i(x)\right)^2\right\rangle$$

continuum notation

Vector Current: $J_\mu^v(x) \longrightarrow Y_v \sim \overline{u}_x\gamma_\mu u_x - \overline{d}_x\gamma_\mu d_x$

Chiral Current: $J_\mu^c(x) \longrightarrow Y_c \sim \overline{u}_x\gamma_\mu\gamma_5 u_x - \overline{d}_x\gamma_\mu\gamma_5 d_x$

Axial Current: $J_\mu^a(x) \longrightarrow Y_a \sim \overline{u}_x\gamma_\mu\gamma_5 u_x + \overline{d}_x\gamma_\mu\gamma_5 d_x$

Chiral condensate susceptibility: $\chi_\sigma = \frac{1}{L^d}\,\frac{1}{Z}\,\frac{\partial^2 Z}{\partial m^2}$

How do these behave as a function of c, T, L?

Critical finite size scaling

$$LY_V \;=\; LY_C \;=\; f\left(\frac{T-T_c}{T_c}\,L^{\frac{1}{\nu}}\right)$$

$$LY_A \;=\; g\left(\frac{T-T_c}{T_c}\,L^{\frac{1}{\nu}}\right)$$

$$L^{\eta-2}\chi \;=\; h\left(\frac{T-T_c}{T_c}\,L^{\frac{1}{\nu}}\right)$$

If we have a second order transition, then
plots of the L.H.S as a function of T for different L's
must be functions that pass through a single point at T_c .

First order transition at c=0.

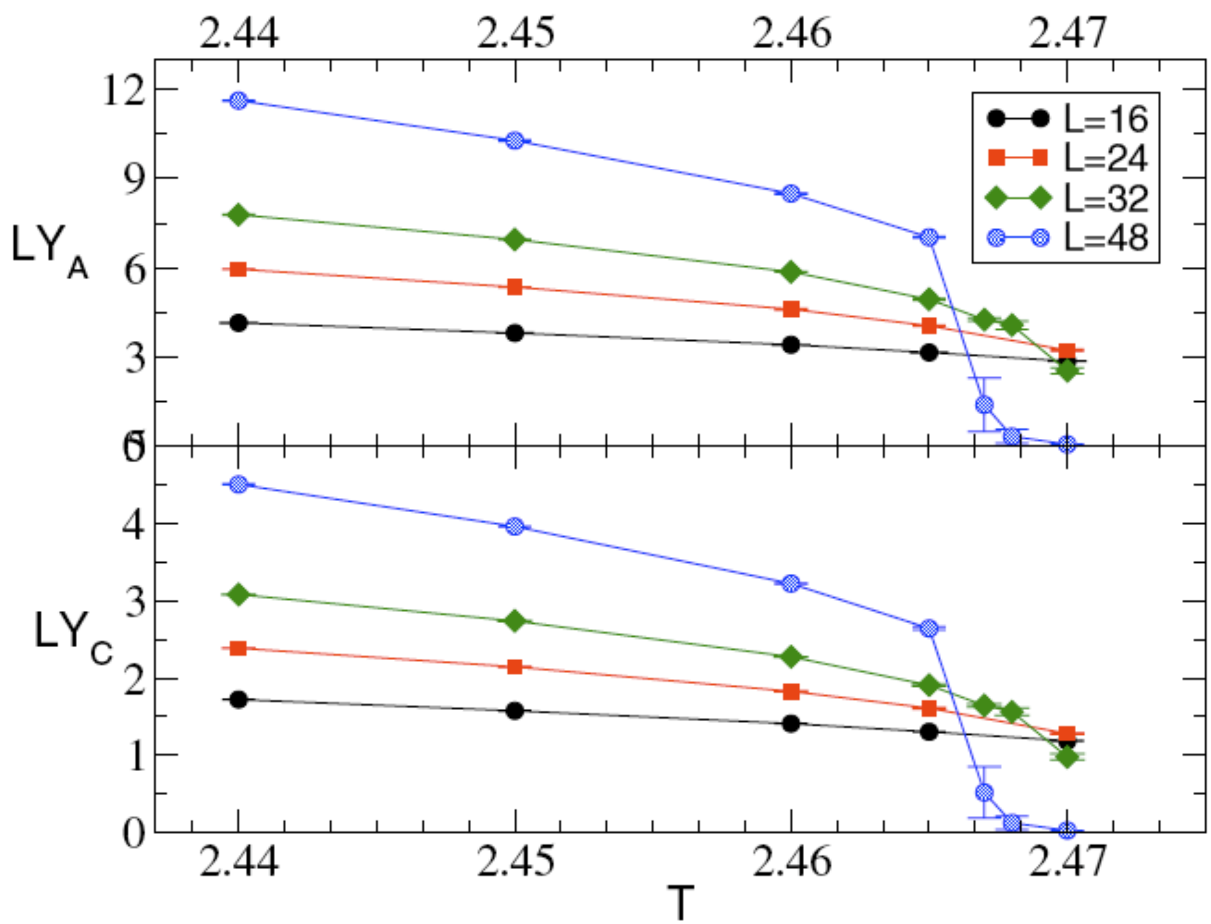
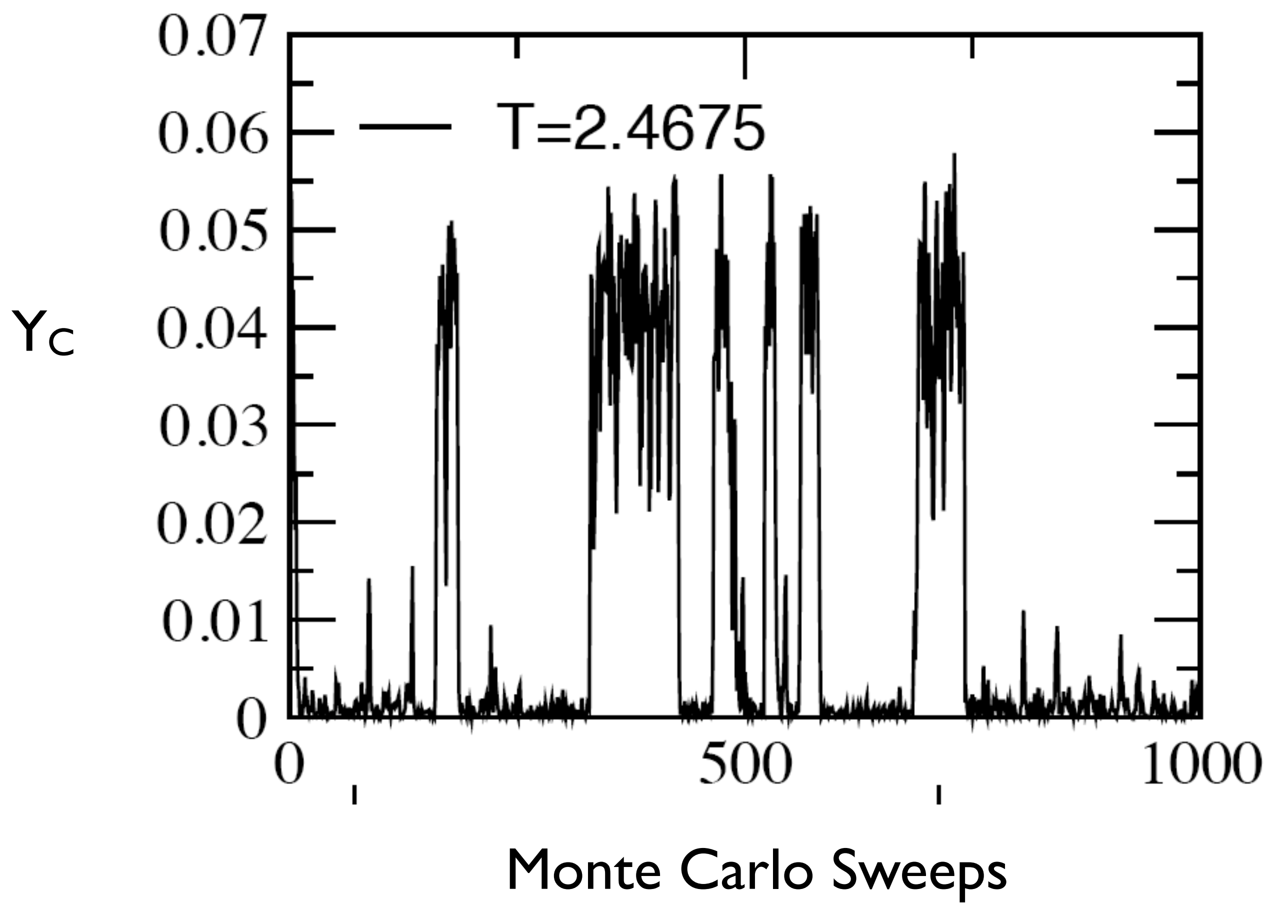


FIG. 2 (color online). Plot of LY_A and LY_C versus T for different values of L for $C = 0$. The lack of a point where all the curves cross shows the absence of a second order transition. We can estimate $T_c \sim 2.466(1)$.

Two state signal at c=0



O(4) Scaling at C=0.3

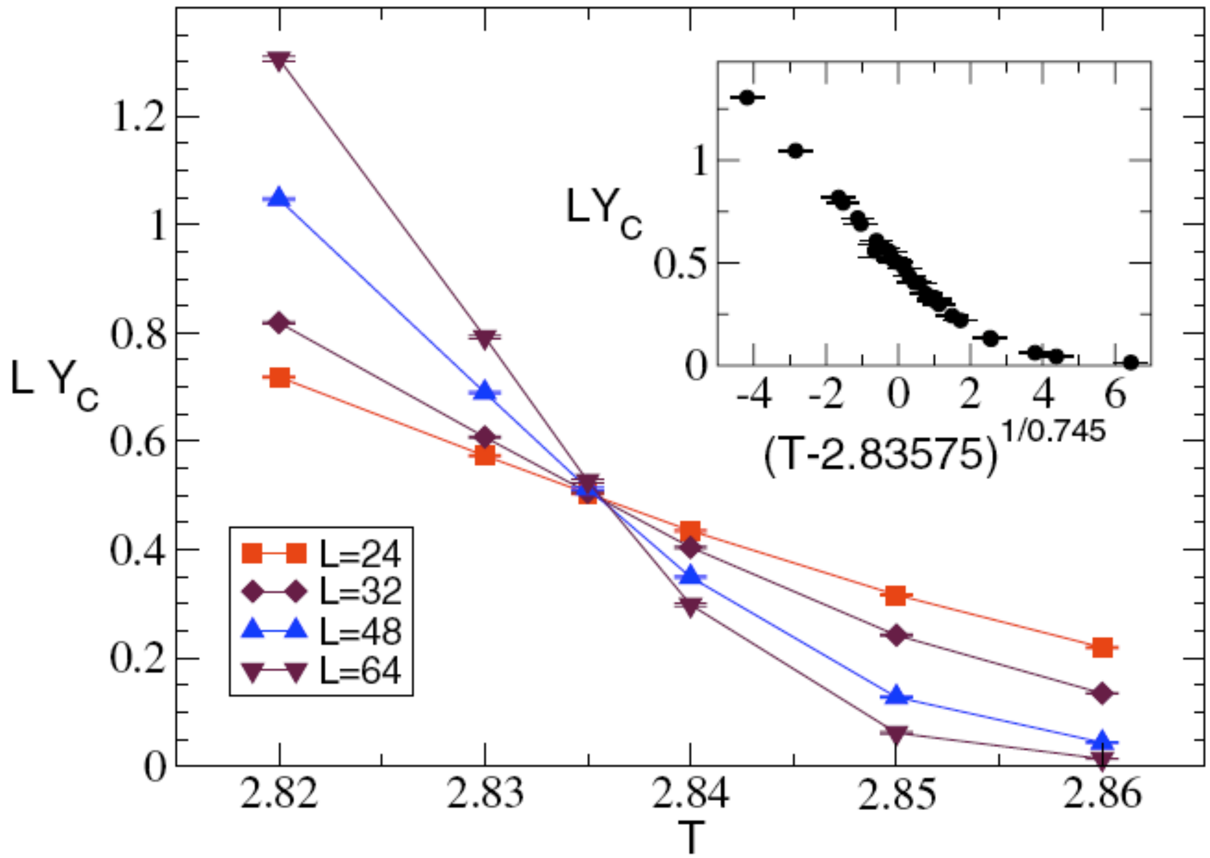
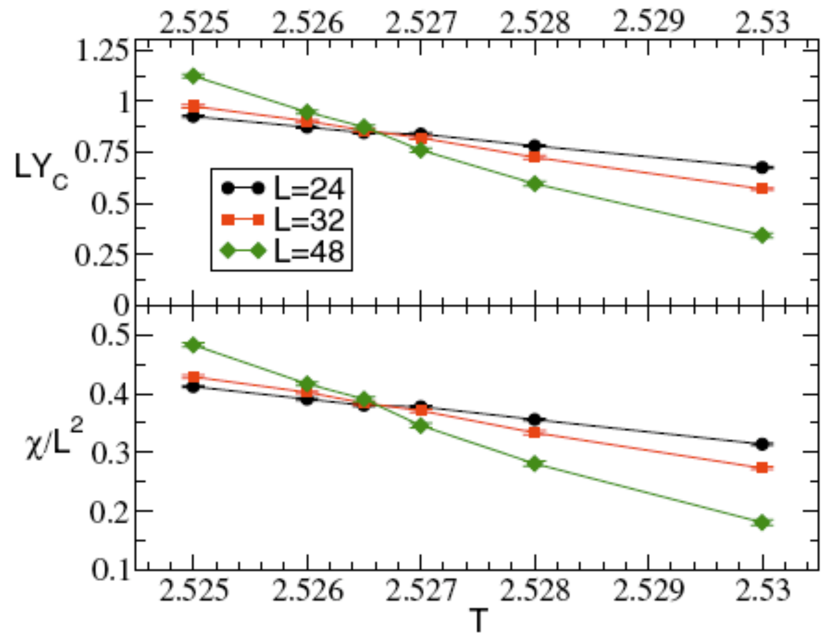
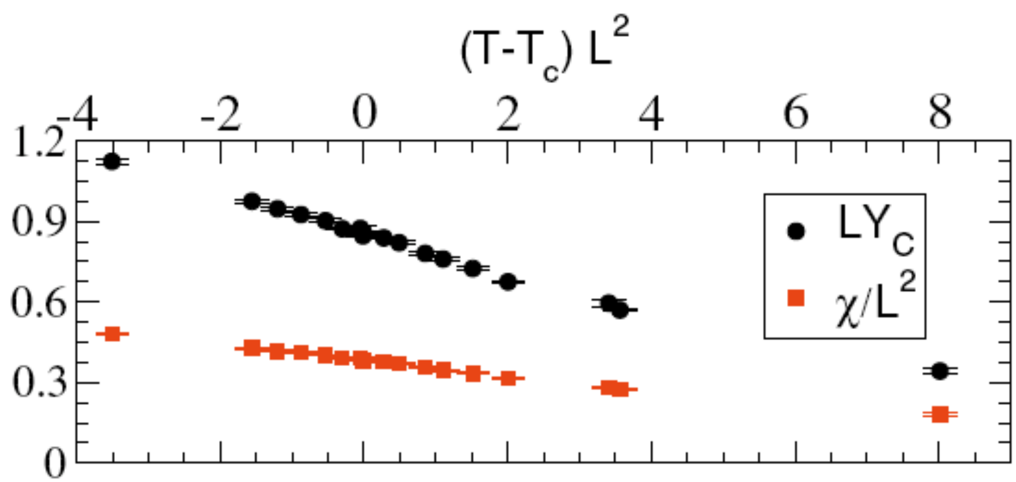


FIG. 3 (color online). Plot of LY_C versus T for different values of L for $C = 0.3$. The presence of a point where all the curves cross shows the presence of a second order transition. We can estimate $T_c = 2.83555(10)$. The inset shows that all our data fit $O(4)$ scaling extremely well.

Tricritical point at C=0.03 (mean field scaling)





Can we find a dimensionless scale that characterizes the strength of the anomaly the tricritical point?

Back to the Sigma Model

Order Parameter: $\left\langle \psi_i(x)\overline{\psi}_j(x) \right\rangle = \Sigma_{ij}(x)$ 2 x 2 complex matrix

There are 8 fields that are important at the phase transition

$$\begin{array}{ccccc} & \overline{\psi}\vec{\tau}\gamma_5\psi \to M_\pi & \longleftrightarrow & \overline{\psi}\psi \to M_\sigma \\ \text{O(2)} \;\updownarrow & & \text{O(4)} & \\ & \overline{\psi}\vec{\tau}\psi \to M_\delta & \longleftrightarrow & \overline{\psi}\gamma_5\psi \to M_\eta' \end{array}$$

The physics of these "mesons" can be described by
an effective field theory which can tell us about scales in the theory

In the broken phase, in the absence of the anomaly we can characterize the low energy fluctuations with the action

$$S = \int d^3x \left\{ \frac{\rho_\pi}{4} \mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U] + \frac{\rho_{\eta'}}{2} |\partial_\mu u|^2 \right\}$$

with $U(x) \in SU(2)$ and $u(x) = \exp(i\eta')$.

ρ_π and $\rho_{\eta'}$ are mass scales in the broken phase

They are similar to F_π of QCD!

In the presence of an anomaly,
these scales get shifted a bit but continue to be well defined scales.

What happens in the various phases?

	Absence of Anomaly	Presence of Anomaly
Below T_c	$M_\pi = M_{\eta'} = 0$ $M_\sigma \neq 0 \quad M_\delta \neq 0$ $\rho_\pi \neq 0 \quad \rho_{\eta'} \neq 0$	$M_\pi = 0 \quad M_{\eta'} \neq 0$ $M_\sigma \neq 0 \quad M_\delta \neq 0$ $\rho_\pi \neq 0 \quad \rho_{\eta'} \neq 0$
Above T_c	$M_\pi = M_\sigma = M_{\eta'} = M_\delta \neq 0$ $\rho_\pi = 0 \quad \rho_{\eta'} = 0$	$M_\pi = M_\sigma \neq M_{\eta'} = M_\delta \neq 0$ $\rho_\pi = 0 \quad \rho_{\eta'} \neq 0$
	$M_{\eta'} - M_\pi = 0$	$M_{\eta'} - M_\pi \neq 0$

Anomaly Strength $\propto$ $M_{\eta'} - M_\pi$

Dimensionless Parameter characterizing the anomaly strength

Define a scale: $\rho_{\eta'} \equiv [(\rho_{\eta'})_{T_c^-} + (\rho_{\eta'})_{T_c^+}]/2,$

Properties of this scale

Characterize the fluctuations of the anomalous field

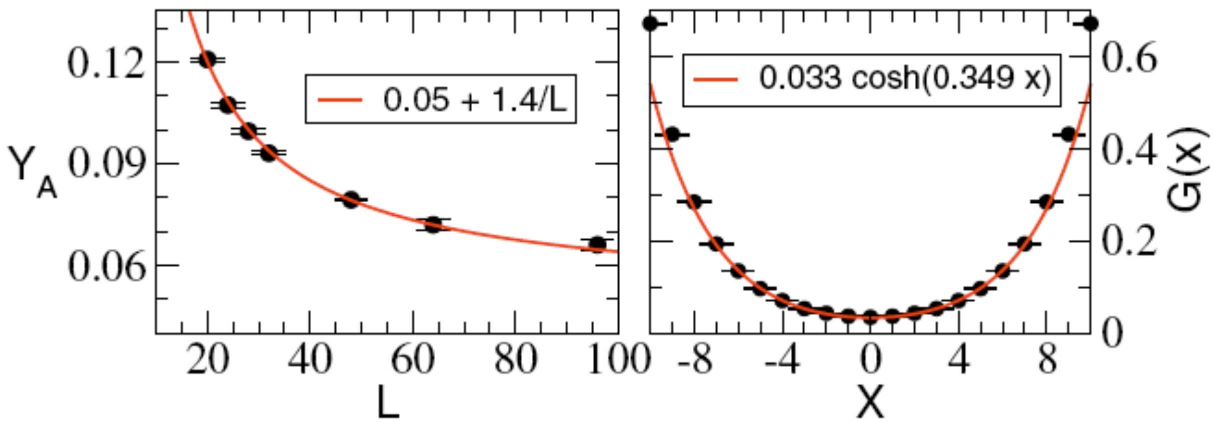
Does not change much along the critical line

At the tri-critical point it is continuous and well defined.

Dimensionless parameter for the Anomaly Strength

$$r = (M_{\eta'} - M_{\pi})/\rho_{\eta'}$$

Strength of the anomaly at the tricritical point



$M_{\eta'} = 0.35$, $M_\pi = 0$, $\rho_{\eta'} = 0.05$

$$r = (M_{\eta'} - M_\pi)/\rho_{\eta'} \sim 7$$

Compare with chiral symmetry breaking in QCD $2M_\pi/F_\pi \approx 3.0$

A strong anomaly may be necessary before O(4) scaling sets in!

Conclusions

- Anomaly plays a central role in the physics of the chiral phase transition.
- Clear evidence that O(4) universality may arise only with a strong anomaly.
- Anomaly is known to be suppressed at finite temperatures.
- The anomaly is known to depend strongly on the lattice formulations (specially badly distorted with staggered fermions).
- Correct anomaly is reproduced only in the continuum limit.

Perhaps the last word on the nature of the chiral phase transition has not yet been spoken?