Searching for the QCD critical point on the lattice

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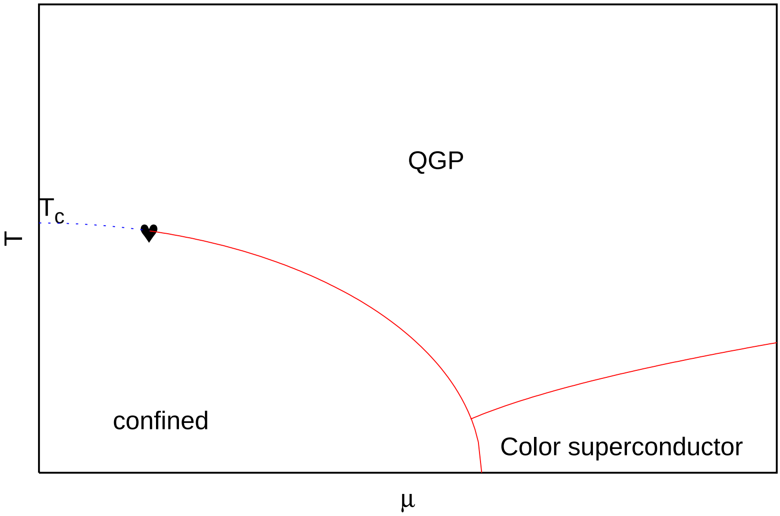
Schematic QCD phase diagram



Can one locate the critical point (μ_E, T_E) ? **3 approaches**

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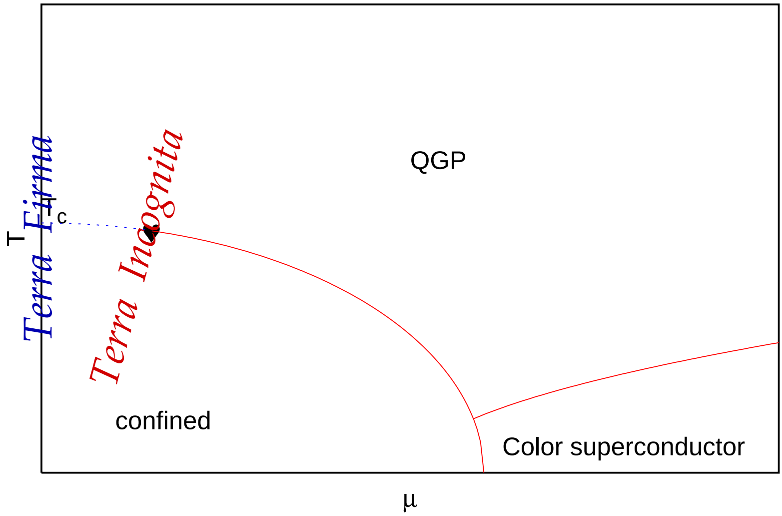

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The sign and overlap problems

- Integrate over fermions: $\det(\not{D} + m + \mu\gamma_0)$ complex *unless* $\mu = 0$ *or* $\mu = i\mu_i$

  $\longrightarrow$ standard importance sampling $\Leftrightarrow$ $\langle \mathrm{Re(baryon\ density)} \rangle = 0$

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- **Reweighting**: - simulate theory with no sign pb., eg. $|\det(\mu)|$
  - reweight each measurement with $\rho(U) = \frac{\det(U,\mu)}{|\det(U,\mu)|}$ complex phase
  - $\langle \rho(U) \rangle = \frac{Z(\mu,\det)}{Z(\mu,|\det|)} \sim \exp(-V\frac{\Delta f(\mu)}{T})$ $\longrightarrow$ large V ?, large μ ?

1. maintain statistical accuracy on $\langle \rho \rangle$: **sign** pb.
2. ensure that $Z(\mu,\det)$ is properly sampled: **overlap** pb.

1 and **2** require statistics $\propto$ $\exp(+V)$

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- Measure **derivatives** w.r.t. μ at $\mu = 0$: $\langle W(\mu) \rangle = \langle W(\mu = 0) \rangle + \sum_k \mathbf{c_k} \left(\frac{\mu}{\pi T}\right)^k$
  - directly at $\mu = 0$ MILC, TARO, Bielefeld-Swansea, Gavai-Gupta,..
  - by fitting polynomial to $\mu = i\mu_i$ results D'Elia-Lombardo, PdF-Philipsen,..

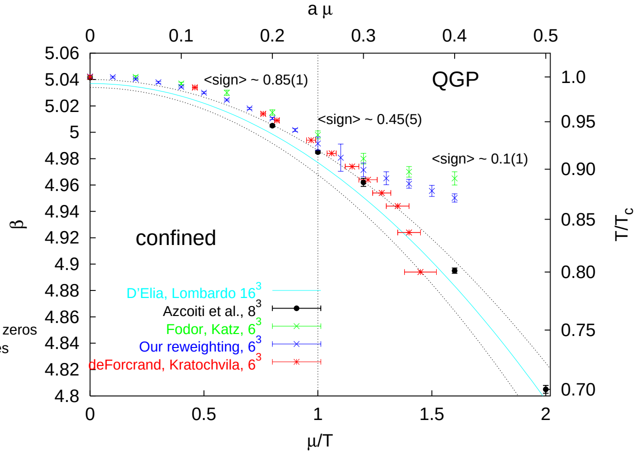
Controlled thermodynamics and continuum limits $\Rightarrow$ **derivatives** only

The good news: curvature of the pseudo-critical line

All with $N_f = 4$ staggered fermions, $am_q = 0.05, N_t = 4$ $(a \sim 0.3$ fm)

PdF & Kratochvila LAT05

imaginary μ
2 param. imag. μ
dble reweighting, LY zeros
Same, susceptibilities
canonical



Agreement for $\mu/T \lesssim 1$

The good news: curvature of the pseudo-critical line

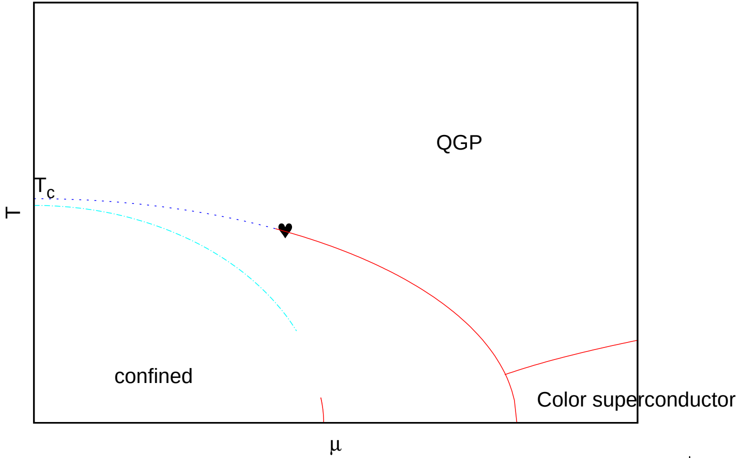
$$\frac{T_c(\mu)}{T_c(\mu=0)} = 1 - \mathbf{t_2}\left(\frac{\mu}{\pi T}\right)^2 + \dots$$

Lattice, all with $N_t = 4$:

N_f	am	N_s	t_2	Action	β-Function	Method
2	0.1	16	0.69(35)	p4	non-pert.	Taylor+Rew.
	0.025	6,8	0.500(34)	stag.	2-loop pert.	Imag.
3	0.1	16	0.247(59)	p4	non-pert.	Taylor+Rew.
	0.026	8,12,16	0.667(6)	stag.	2-loop pert.	Imag.
	0.005	16	1.13(45)	p4	non-pert.	Taylor+Rew.
4	0.05	16	0.93(9)	stag.	2-loop pert.	Imag.
2+1	0.0092,0.25	6-12	0.284(9)	stag.	non-pert.	Rew.

- Compare with **freeze-out**: $t_2 \approx$ **2.05** Cleymans et al.
- Extrapolation $m_q \to m_{\rm phys}, \; a \to 0$ feasible
- Indications (PdF& OP $N_t = 6$; Fodor et al. LAT08): t_2 *decreases* as $a \to 0$

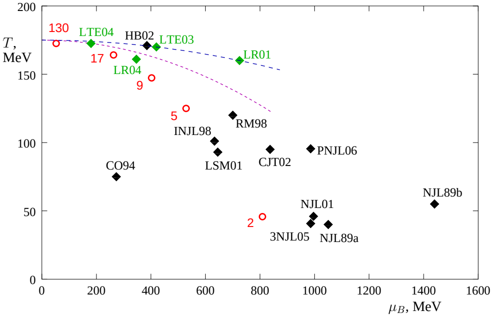
Comparison experiment ↔ lattice: $N_t = 4$



$T_c(\mu)$ considerably flatter than **freeze-out** curve (factor ~ 3 in $\left.\frac{d^2 T_c}{d\mu^2}\right|_{\mu=0}$)

$\Longrightarrow$ signal from critical pt. washed out by evolution until freeze-out

The bad news: locating the critical point



M. Stephanov, hep-lat/0701002

- Challenging task:

  detect divergent correlation length (2nd order)

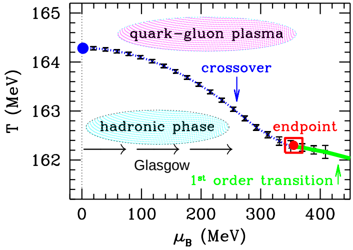
  vs finite but large (crossover, 1rst order)

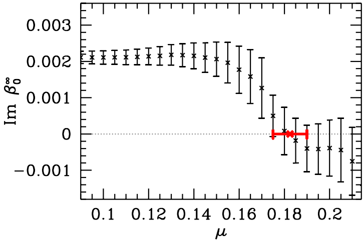
  on small lattice

  Mission impossible?

1. CEP already determined, but...

Fodor & Katz: hep-lat/0402006 ($\sim$ physical quark masses)





$$(\mu_E^q, T_E) = \ (120(13), 162(2))\ \text{MeV}$$

Strategy: reweight from $(\mu = 0, T_c)$ along pseudo-critical line

Legitimate **concerns**:

- Discretization error? $N_t = 4 \implies a \sim 0.3$ fm
- Abrupt qualitative change near μ_E:

  abrupt change of physics or breakdown of algorithm (Splittorff)?

$\longrightarrow$ repeat with conservative approach (**derivative**)

2. CEP from radius of convergence? ratio method

$$\frac{p}{T^4} = \sum_{n=0}^{\infty} c_{2n}(T) \left(\frac{\mu}{T}\right)^{2n}$$

Singularity $(\mu_E, T_E) \Rightarrow \boxed{\frac{\mu_E}{T_E} = \lim_{n\to\infty} \sqrt{\left|\frac{c_{2n}}{c_{2n+2}}\right|(T_E)}}$ Karsch et al.

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- T_E ?

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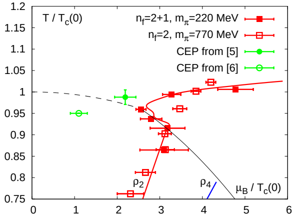
- Equally good: $\frac{\chi_q}{T^2} = \sum_{n=1}^{\infty} 2n(2n-1)\, c_{2n} \left(\frac{\mu}{T}\right)^{2n-2}$

$$\rightarrow \frac{\mu_E}{T_E} = \lim_{n\to\infty} \sqrt{\left|\frac{2n(2n-1)c_{2n}}{(2n+2)(2n+1)c_{2n+2}}\right|}$$

$n = 1 \rightarrow$ factor $1/\sqrt{6}$ Gavai & Gupta



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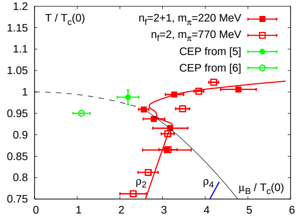
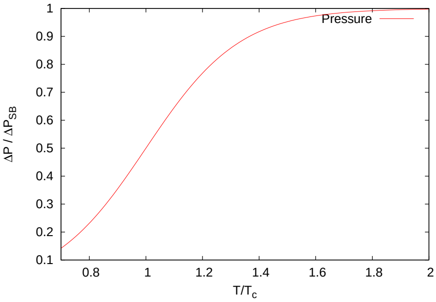


Can systematic error be controlled ?

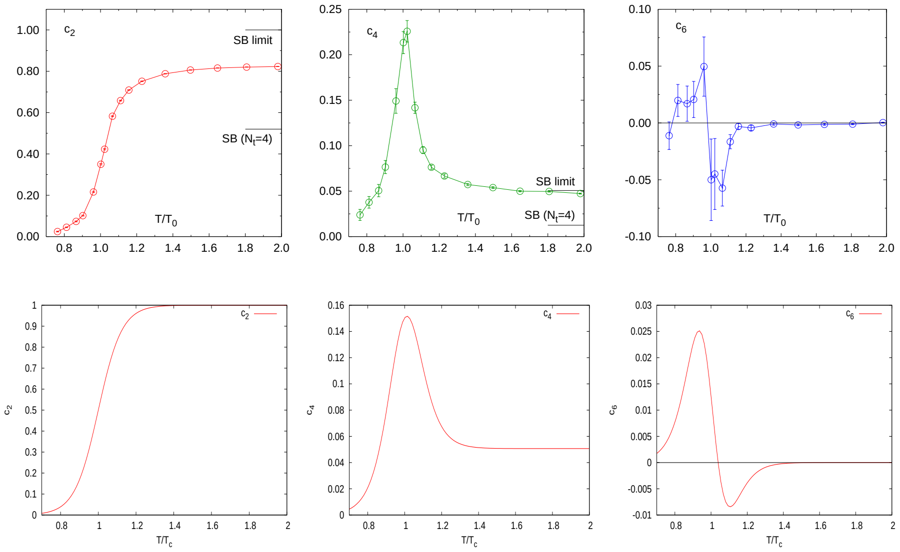
First goal: distinguish between CEP and crossover (no CEP)

Case study on toy model, with Roger Herrigel

- **Idea**: study Taylor coeffs of $\frac{\Delta P}{T^4}\left(\frac{\mu}{T}\right) \equiv \Delta\hat{P}(\hat{\mu})$
  and "effective radius of convergence" in **controlled** situation
- Ansatz: $\Delta\hat{P}(\hat{\mu}) = \frac{\Delta\hat{P}_{SB}}{2} + \log(\cosh(\lambda(\hat{T}-1) + \frac{\Delta\hat{P}_{SB}}{2}))$. Why ?
  - correct limits high-T, low-T
  - $c_2 \approx (1+\tanh(\lambda(\hat{T}-1))$, *sigmoid*, no phase trans., no CEP
  - single parameter λ controls width of crossover (rescaling of $\hat{T}$)
- no singularity on real or imaginary $\hat{\mu}$ axis $\longrightarrow$ test for spurious signals



Toy ansatz: compare with Bielefeld et al.



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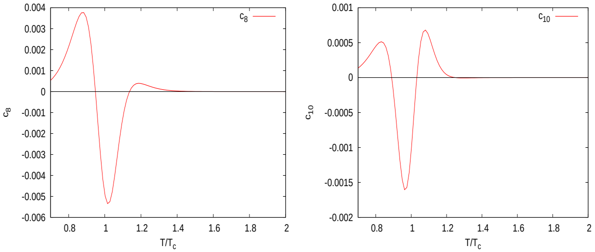
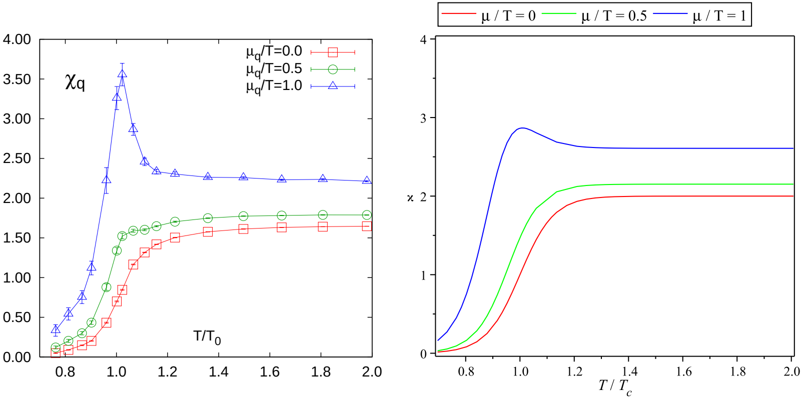


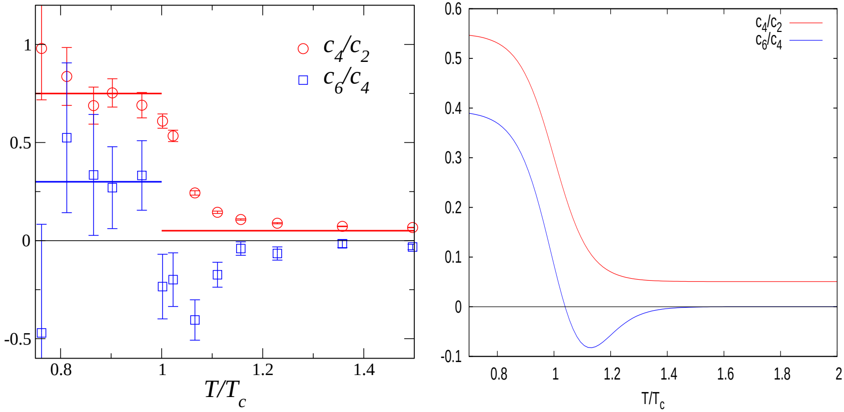
Oscillatory pattern $\longrightarrow c_k = 0$ at lower $\hat{T}$ as k increases

Toy ansatz: compare with Bielefeld et al.



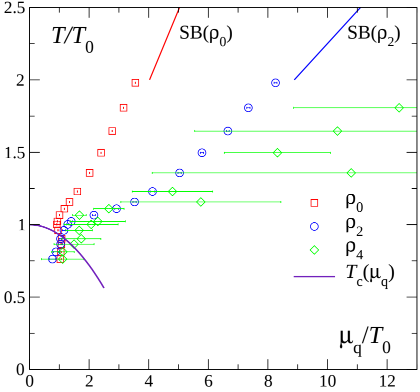
- Quark susceptibility rises, even without phase transition

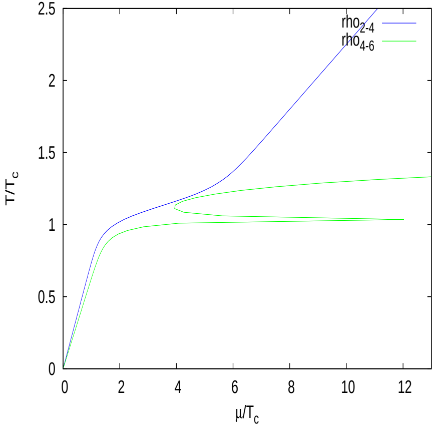
Toy ansatz: compare with Bielefeld et al.



- Qualitative agreement below T_c although HRG not built-in

Toy ansatz: compare with Bielefeld et al.





- Qualitative agreement

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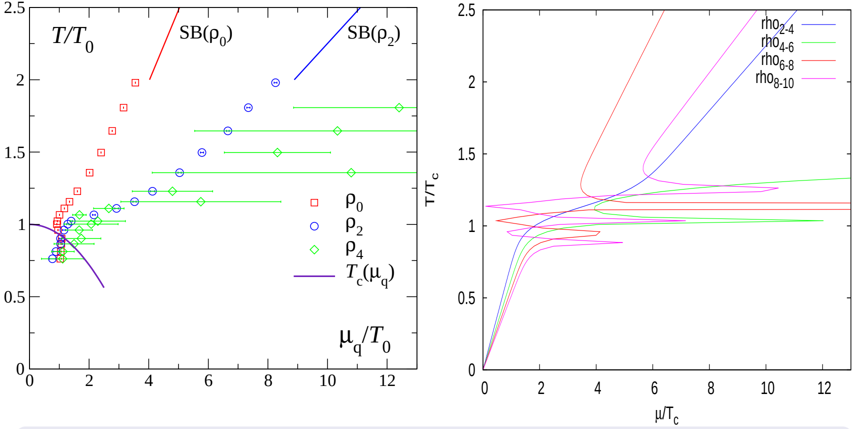
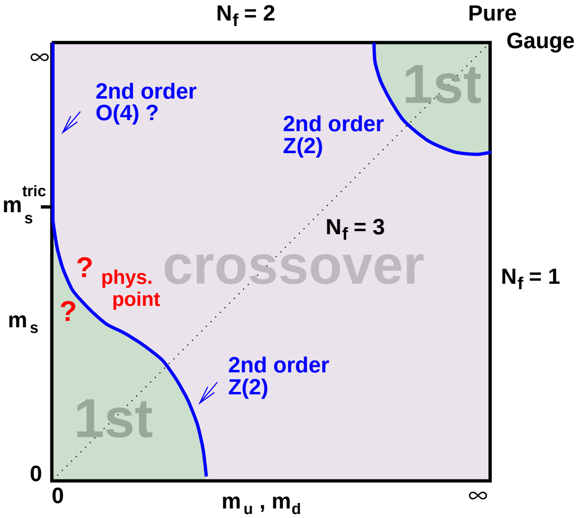


- **IF** we know that there is a critical point: how to choose T_E ??
- **Can we predict whether or not there is a CEP ??**

3. Better? generalize QCD to arbitrary $(m_{u,d}, m_s), T$

$\mu = 0$



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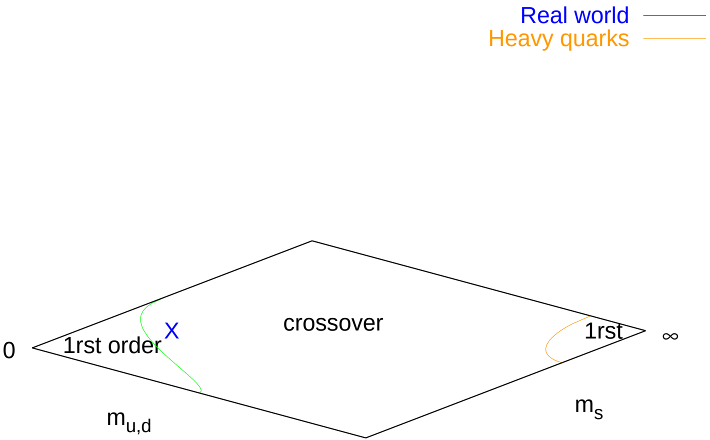
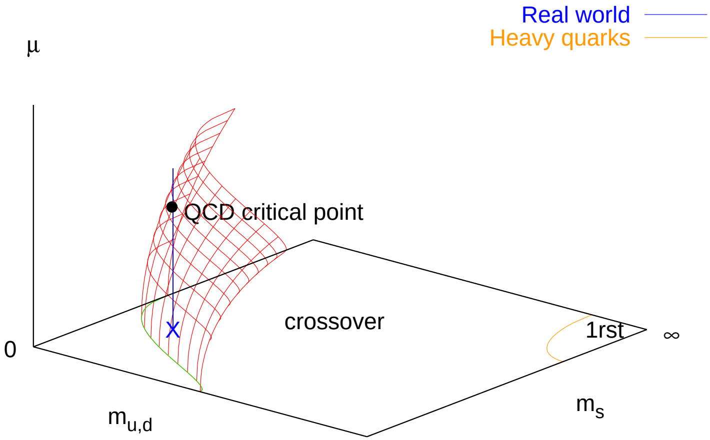


Now turn on μ

3. Better? generalize QCD to arbitrary $(m_{u,d}, m_s)$, T

$\mu \neq 0$



Locate critical point from intersection with *critical surface* $(m_{u,d}, m_s)^{\mathrm{crit}}(\mu)$

3. Better? generalize QCD to arbitrary $(m_{u,d}, m_s)$, T

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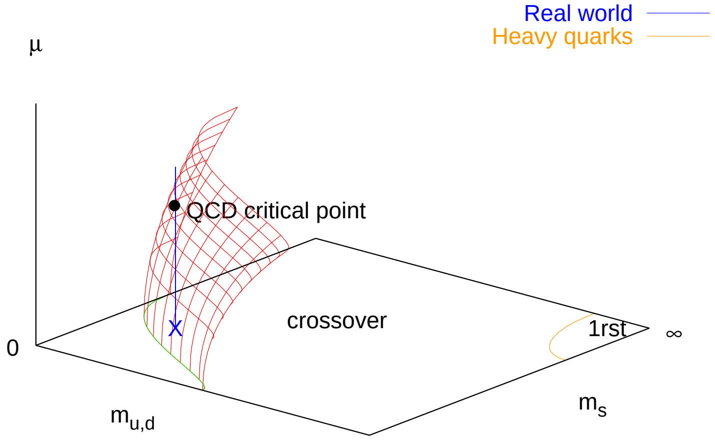
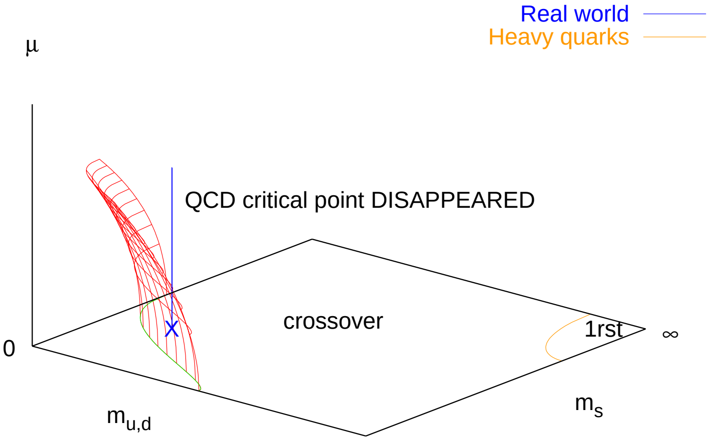


Follow vertical (cf. Glasgow) or critical surface (cf. Fodor & Katz)

3. Better? generalize QCD to arbitrary $(m_{u,d}, m_s)$, T

$\mu \neq 0$



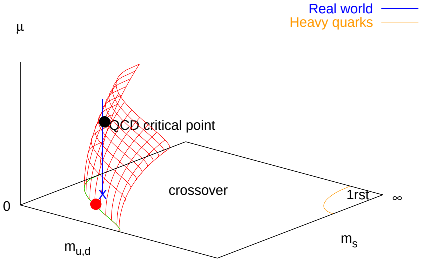
Exotic possibility: first-order region shrinks with real $|\mu|$: $\frac{d\ m_c}{d\mu^2}|_{\mu=0} < 0$

Strategy, with Owe Philipsen

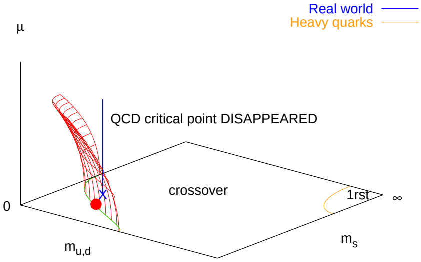
1. Tune quark mass(es) to $m_c(0)$: 2nd order transition at $\mu = 0, T = T_c$
   known universality class: 3d Ising

2. Measure derivatives $\frac{d^k m_c}{d\mu^{2k}}|_{\mu=0}$:

$$\frac{m_c(\mu)}{m_c(0)} = 1 + \sum_{k=1} \mathbf{c_k} \left(\frac{\mu}{\pi T}\right)^{2k}$$



$c_1 > 0$



$c_1 < 0$

Observable: Binder cumulant

- Probability distribution of order parameter
  - distinguishes crossover (Gaussian) vs 1rst order (2 peaks)
  - 2nd order: scale-invariant distribution with known Ising exponents
  - encoded in Binder cumulant

- Measure $B_4(\bar\psi\psi) \equiv \frac{\langle(\delta\bar\psi\psi)^4\rangle}{\langle(\delta\bar\psi\psi)^2\rangle^2}\Big|_{\langle(\delta\bar\psi\psi)^3\rangle=0} = \begin{cases} 3 & \text{crossover} \\ 1 & \text{first-order} \quad \text{for } V\to\infty \\ 1.604 & \text{3d Ising} \end{cases}$

Finite volume: $B_4(am, a\mu) = 1.604 + \sum_{k,l=1} \mathbf{b_{kl}}\,(am - am_0^c)^k (a\mu)^{2l}$

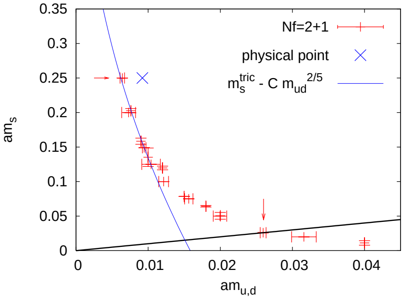
$$\frac{d\,am^c}{d(a\mu)^2} = -\frac{\partial B_4}{\partial(a\mu)^2} \Big/ \frac{\partial B_4}{\partial am} = -b_{01}/b_{10}, \quad \text{hard / easy}$$

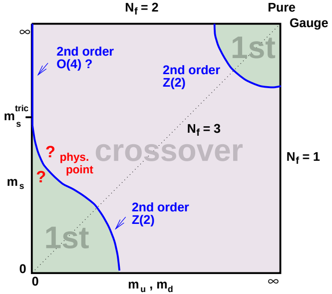
State of the art: measure b_{02}, ie. $4+4=$ **8 derivatives** of pressure

Results: hep-lat/0607017, 0808.1096

1. $\mu = 0$:

Line of second-order phase transitions in the quark mass plane $(m_{u,d}, m_s)$ via Binder cumulant $B_4 = \langle(\delta\bar{\psi}\psi)^4\rangle / \langle(\delta\bar{\psi}\psi)^2\rangle^2$





- data consistent with tricritical point at $m_{u,d} = 0,\ m_s \sim 2.8T_c$
- physical point in crossover region cf. Fodor & Katz

2. $\mu \neq 0$: best method to measure change in B_4, ie. $\frac{\partial B_4}{\partial (a\mu)^2}$

- Monte Carlo at $\mu = 0$, reweight to small $\mu = i\mu_i$, measure $\frac{\Delta B_4}{\Delta \mu^2}$

  Advantages: - fluctuations cancel in ΔB_4

  - reweighting done stochastically $\longrightarrow$ cheap

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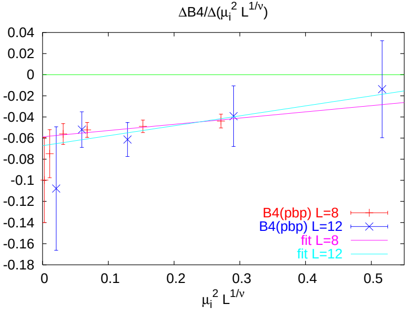
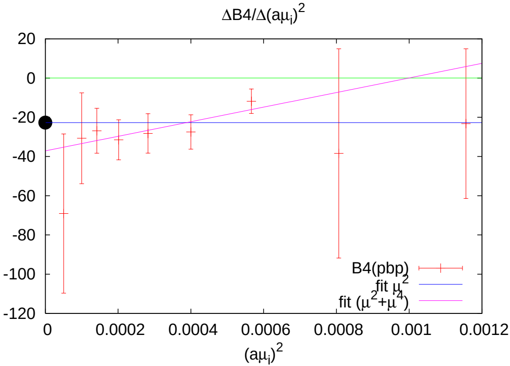
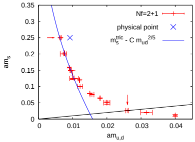
- Scaling test: $\frac{d\,am^c}{d(a\mu)^2} = -\frac{\partial B_4}{\partial (a\mu)^2} / \frac{\partial B_4}{\partial am}$; each factor $\propto L^{1/\nu}$, $\nu = 0.63$



$N_t = 4, N_f = 3$: $\frac{m_c(\mu)}{m_c(0)} = 1 - \mathbf{3.3(3)} \left(\frac{\mu}{\pi T}\right)^2 - \mathbf{47(20)} \left(\frac{\mu}{\pi T}\right)^4 - \ldots$ no CEP ??

$N_t = 4, N_f = 2+1$: moving along the critical line



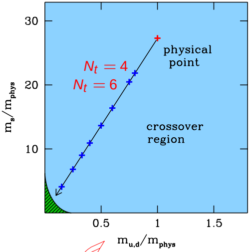


- $16^3 \times 4, am_s = 0.25, am_{u,d} = 0.005$, *lighter than in nature* $(m_\pi L = 3.4)$
- $\mathcal{O}(10^6)$ trajectories, 300 CPU-years by **Grid computing**
  $\mathcal{O}(10^3)$ independent single-CPU threads of $\mathcal{O}(10^3)$ trajectories each

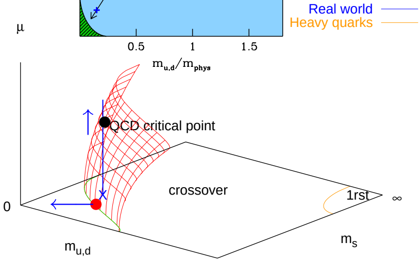
$N_t = 6, N_f = 3$: towards the continuum limit

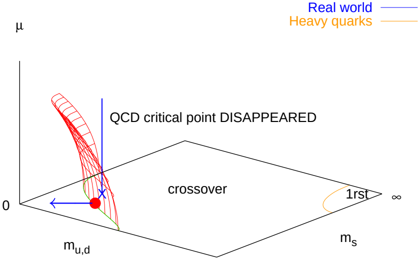
1. $\mu = 0$: re-tune the quark mass for 2nd-order transition at $T = T_c$

$\longrightarrow$ At $T = 0$, $\frac{m_\pi}{T_c} = 0.954(12)$ instead of $1.680(4)$ ($N_t = 4$)



cf. Endrodi, Fodor et al., arXiv:0710.0998



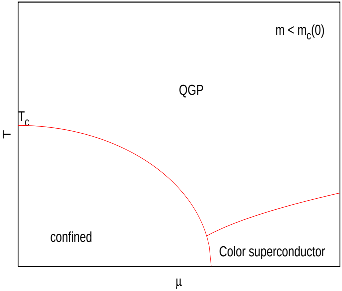


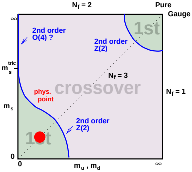
Critical surface recedes from physical (m_{ud}, m_s) point $\longrightarrow$ CEP at larger μ

2. $\mu \neq 0$: measure $\frac{\Delta B_4}{\Delta \mu^2}$ —- in progress

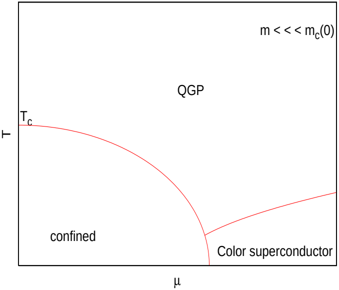
Resulting phase diagram (simplest possibility)

Standard scenario





Exotic scenario



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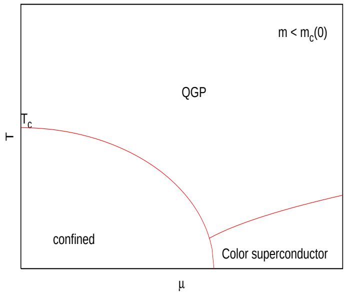

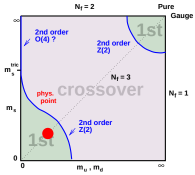

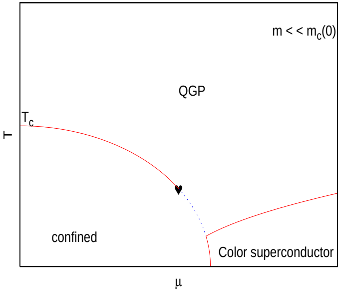

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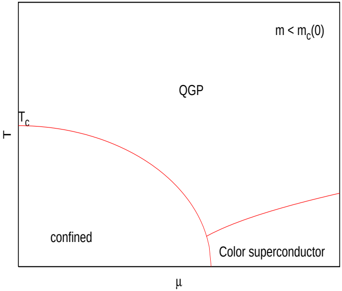

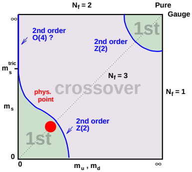

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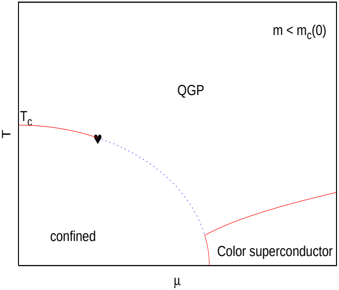

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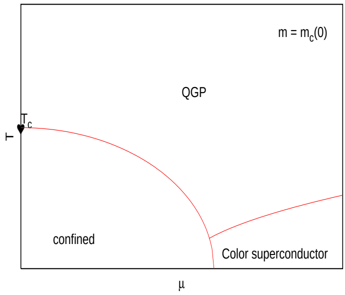

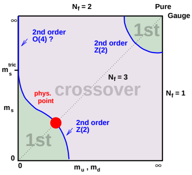

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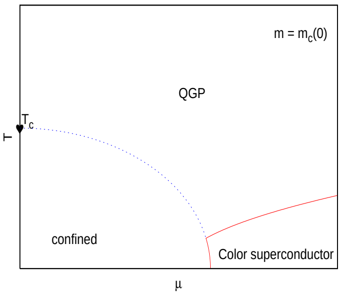

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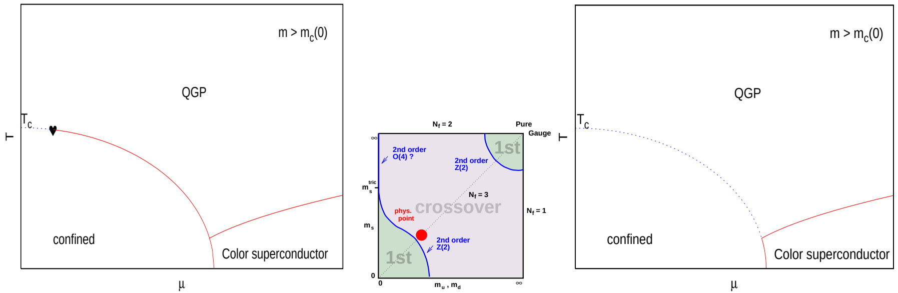

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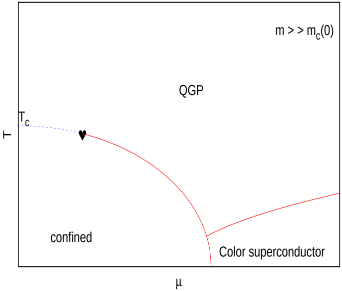

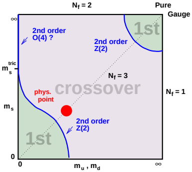

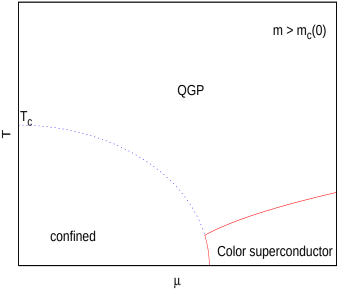

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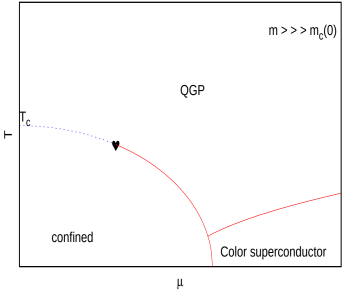

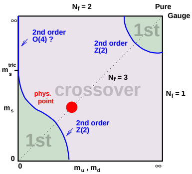

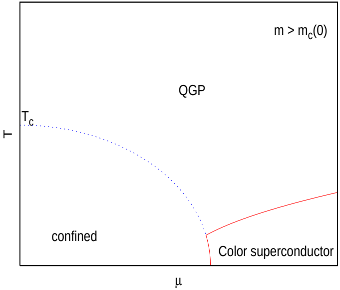

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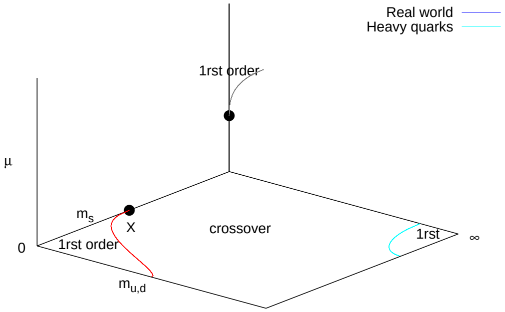
Arguments for standard wisdom?

- $O(4)$ transition for 2 massless flavors Pisarski & Wilczek

$\Rightarrow$ tricritical points $(m_{u,d} = 0, m_s = \infty, \mu = \mu^*)$ and $(m_{u,d} = 0, m_s = m_s^*, \mu = 0)$



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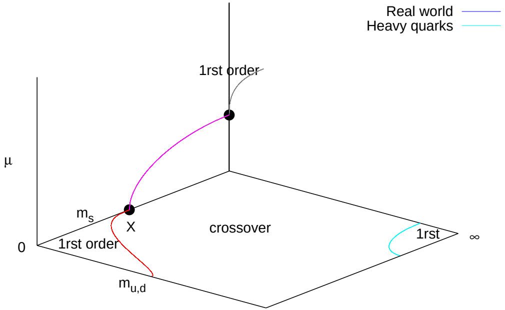

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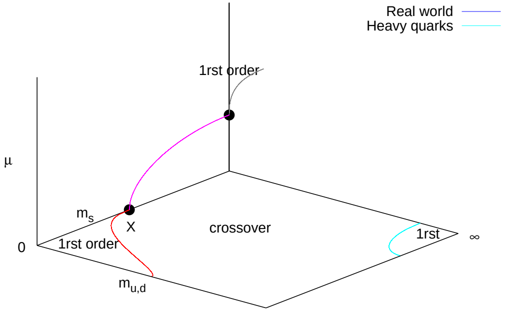

Critique:

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Chandrasekharan & Mehta

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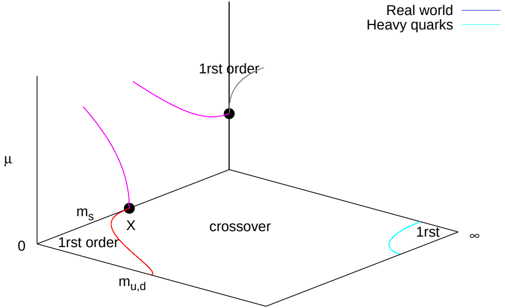


Critique:

- $O(4)$ if strong enough $U_A(1)$ anomaly, otherwise first-order — Chandrasekharan & Mehta
- $N_f = 2$ and $N_f = 2+1$ need not be connected

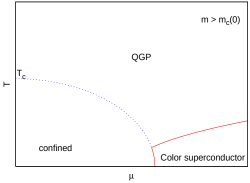
Conclusions

- Pseudo-critical temperature $\frac{T_c(\mu)}{T_c(\mu=0)} = 1 - t_2 \left(\frac{\mu}{\pi T}\right)^2 + \ldots$ *soon*
  $t_2 < t_2^{\rm freeze-out}$, factor $\gtrsim 3$?

- $\frac{m_c(\mu)}{m_c(0)} = 1 + \mathbf{c_1} \left(\frac{\mu}{\pi T}\right)^2 + \ldots$: *can control systematics*
  Non-standard scenario $c_1 < 0$ for $N_t = 4$

- $a \to 0$: critical surface *far* from physical point
  $\Longrightarrow$ need $c_1 > 0$ *and large* for $\frac{\mu_E}{T_E} \lesssim 1$, disfavored by data

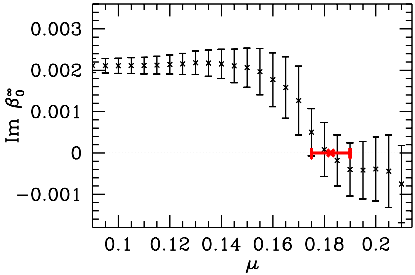
- **QCD critical point?**

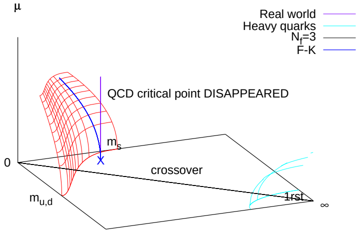


$\mu_E^B \lesssim$ 500 MeV unlikely, or non-chiral

Contradiction with other lattice studies?

- Fodor & Katz: $(T_E, \mu_E) = (162(2), 120(13))$ MeV ?

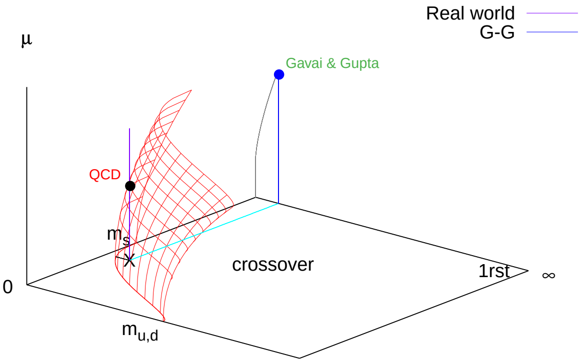




- Very little μ-dependence until $\mu \sim \mu_E \longrightarrow$ need high-degree Taylor expansion
- $m_q a$, ie. $\frac{m_q}{T_c}$ fixed, while $T_c(\mu)$ decreases for $\mu \neq 0 \Rightarrow$ non-const. physics
  Lighter quarks at larger μ favor first-order transition

Contradiction with other lattice studies?

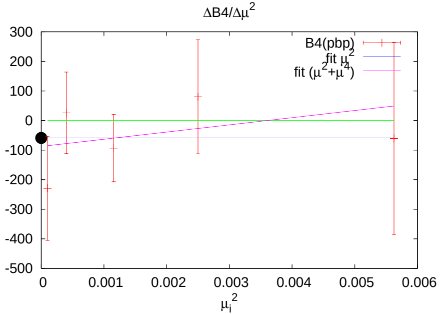
- Gavai & Gupta: $\mu_E / T_E \lesssim 1$?
  different theory $N_f = 2$



- Agreement with isospin μ Kogut-Sinclair, PdF-Stephanov-Wenger

$N_t = 6, N_f = 3$: towards the continuum limit

2. Measure $\frac{\partial B_4}{\partial (am)}$ (easy) and $b_1 \equiv \frac{\partial B_4}{\partial (a\mu)^2}$ (hard)



- $18^3 \times 6, am = 0.003, m_\pi = 0.95 T_c \sim$ 170 MeV $(m_\pi L = 2.9)$
  120k trajectories, 6 months of SX-8
- $b_{01} = -58(49)$ (μ^2 fit) $\longrightarrow c_1 = -28(23)$, ie. $\frac{m_c(\mu)}{m_c(0)} = 1 - \mathbf{28(23)} \left(\frac{\mu}{\pi T}\right)^2$

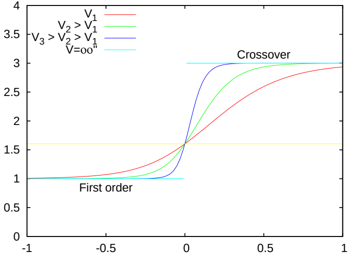
[or $b_{01} = -88(75)$ ($\mu^2 + \mu^4$ fit)]

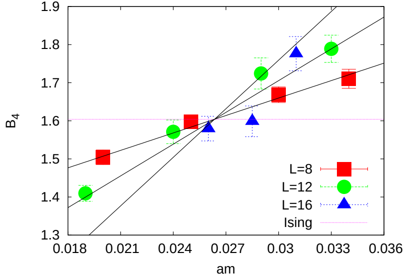
- Assume $c_1 = +18$, ie. 2 sigmas away; then $\frac{\mu_E}{T_E} = 1 \Rightarrow \frac{m_c(\mu_E)}{m_c(0)} \sim 3$,

insufficient to reach physical point

Observable: Binder cumulant

- Probability distribution of order parameter
  - distinguishes crossover (Gaussian) vs 1rst order (2 peaks)
  - 2nd order: scale-invariant distribution with known Ising exponents
  - encoded in Binder cumulant
- Measure $B_4(\bar\psi\psi) \equiv \frac{\langle(\delta\bar\psi\psi)^4\rangle}{\langle(\delta\bar\psi\psi)^2\rangle^2}\Big|_{\langle(\delta\bar\psi\psi)^3\rangle=0} = \begin{cases} 3 & \text{crossover} \\ 1 & \text{first}-\text{order} \\ 1.604 & \text{3d Ising} \end{cases}$ for $V \to \infty$





- Finite volume, $\mu = 0$: $B_4(am) = 1.604 + c(L)(am - am_0^c) + \ldots,\ c(L) \propto L^{1/\nu}$