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NON-LINEAR DYNAMICS PRINCIPLES UNDERLYING SLOW RESONANT EXTRACTION

Armando Bazzani

Physics and Astronomy Department, Bologna University
INFN-Bologna

Adiabatic Invariance Framework

$$H_0(q,p,\lambda) \qquad \lambda = \delta t \qquad (q,p) \in R^2$$

$H(q,p,\lambda)$ is integrable and one defines the action-angle variables

$$I = I(E,\lambda) = \frac{1}{2\pi}\oint_{H_0=E} p(E,q,\lambda)\,dq$$
$$\theta = \left.\frac{\partial F}{\partial I}\right|_q$$

where $F(I,q,\lambda)$ is the generatring function

$$F(q,E,\lambda) = \int_{H_0=E}^{q} p(q,E,\lambda)dq$$

The frozen dynamics as an evolution time scale

$$T(E,\lambda) = \frac{2\pi}{\Omega(E,\lambda)} \qquad \frac{\partial H_0}{\partial I}$$

the adiabatic conditions are defined

$$\frac{\partial \Omega}{\partial \lambda}\frac{\delta}{\Omega} \ll \Omega \quad \Rightarrow \quad \frac{\delta}{\Omega^2} \ll 1$$

and $\varepsilon = \delta/\Omega^2$ is the *adiabatic parameter*. The Hamilton-Jacobi theory implies

$$H(I,\theta,\lambda) = H_0(I,\lambda) + \delta \left.\frac{\partial F}{\partial \lambda}\right|_{q,I}(I,\theta,\lambda)$$

where

$$\left.\frac{\partial F}{\partial \lambda}\right|_{q,I} = -\frac{1}{\Omega(E,\lambda)}\int^{\theta}\left(\left.\frac{\partial H_0}{\partial \lambda}\right|_{q,p} - \left\langle \left.\frac{\partial H_0}{\partial \lambda}\right|_{q,p}\right\rangle\right) d\theta$$

The Action variable is an adiabatic invariant

In 1-d system if $\Omega(I,\lambda) \simeq O(1)$

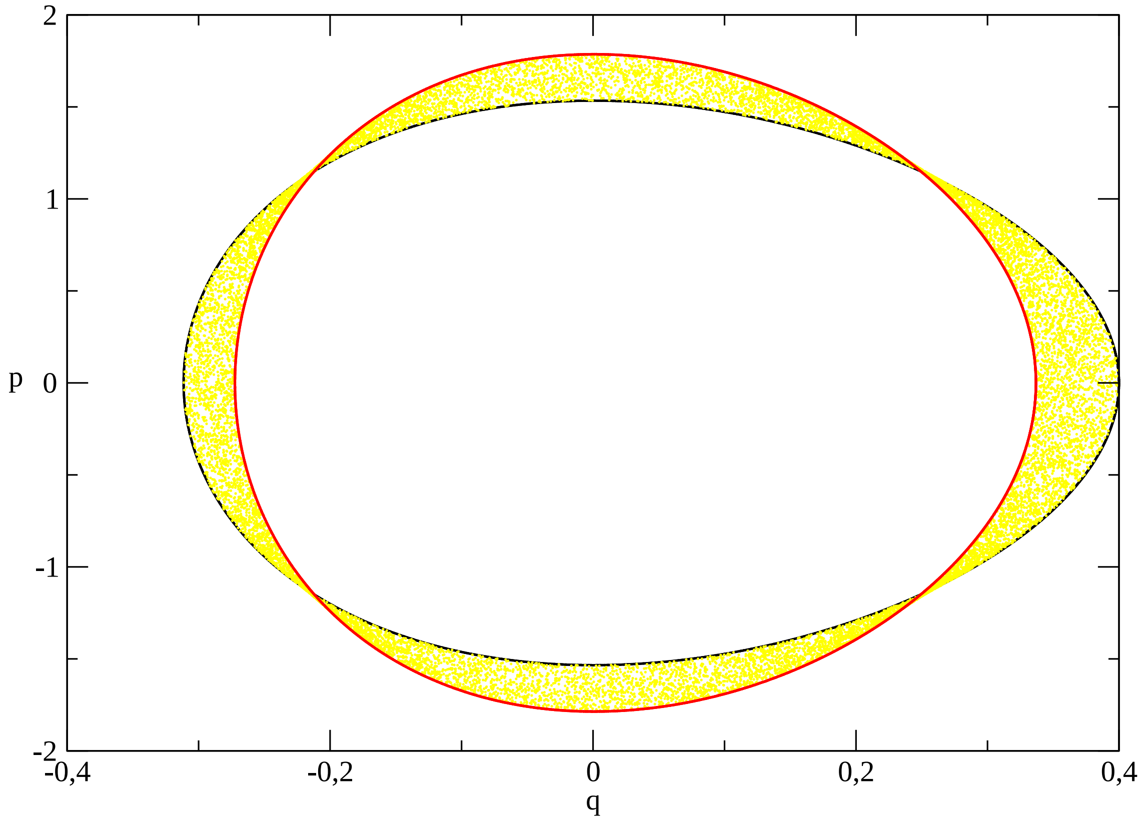
$$|I(p(t),q(t),\delta t) - I_0| \leq O(\delta) \qquad \text{for} \quad t \leq \frac{1}{\delta}$$

This a consequence of the relation

$$I(t) = I_0 + \frac{\delta}{\Omega(I_0,\lambda)} \int^t \left(\left.\frac{\partial H_0}{\partial \lambda}\right|_{p,q} - \left\langle \left.\frac{\partial H_0}{\partial \lambda}\right|_{p,q} \right\rangle \right) dt + O(\delta^2)$$

and of the proportionality $t = \theta/\Omega$.

Example: nonlinear oscillator Hamiltonian $\varepsilon \simeq 10^{-4}$



Limits of the adiabatic theory

1) Local character of the Action-angle variables due to the presence of separatrix curves (resonant regions)

$$\Omega(E, \lambda) \simeq 1/\ln|E - E_s|$$

2) Passage through resonances: the frequency $\Omega(I, \lambda)$ assume resonant values in the phase space.

3) Loss of the adiabaticity condition $\varepsilon \ll 1$.

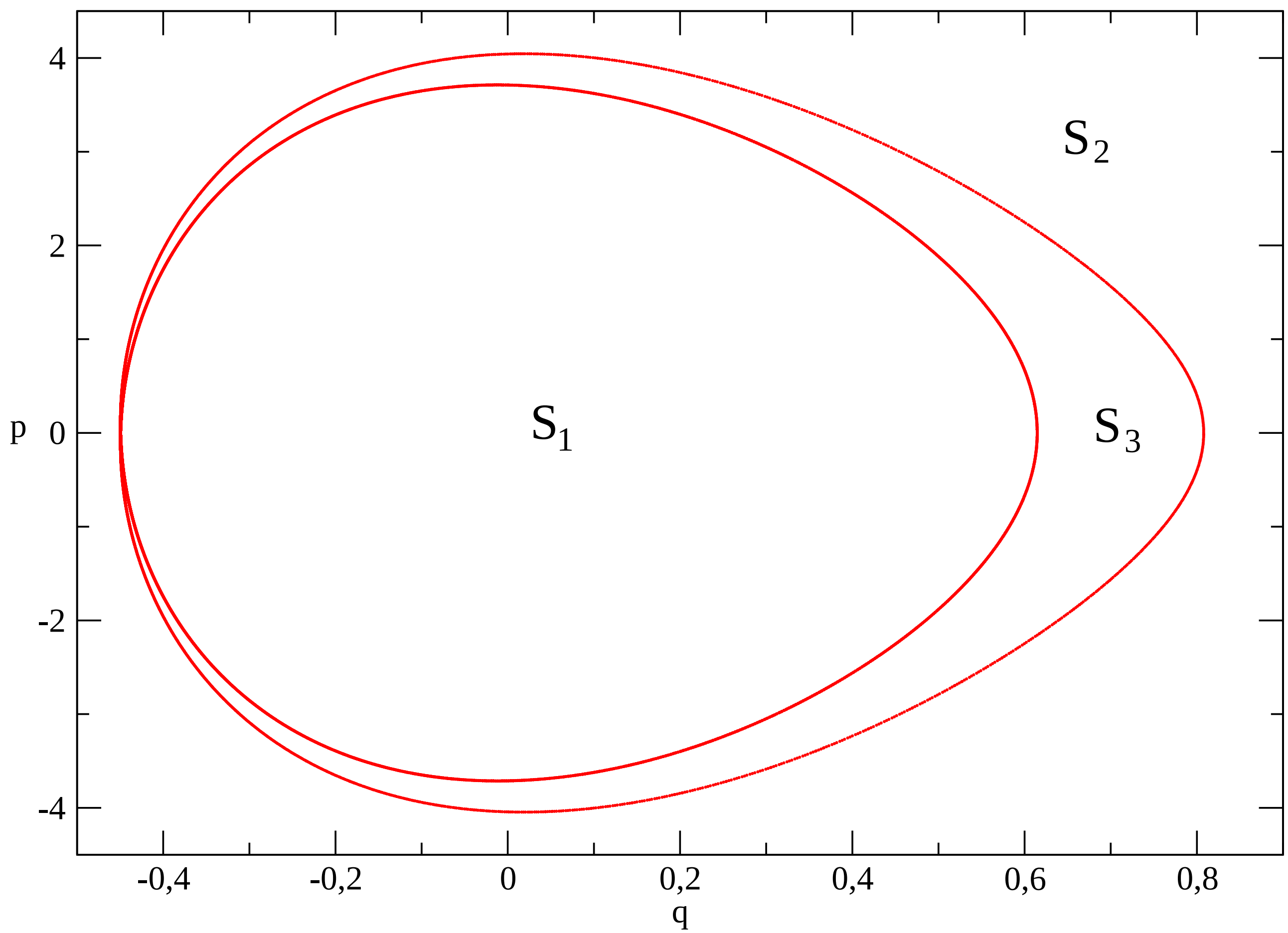
4) Non-integrable character of the frozen dynamics: existence of chaotic layers.

Trapping into resonance

We consider explicitly the Hamiltonian

$$H(p,q,t) = \frac{p^2}{2} + \omega_0^2(\lambda)\frac{q^2}{2} + \sum_{n\geq 3} K_n \frac{q^n}{n} + \epsilon(\lambda)\frac{q^{n_r}}{n_r}\cos\psi(t)$$

where $\psi(t) = \psi_0 + \omega_r t$ and $\lambda = \delta t$. We have a resonance condition $n_r\Omega(I,\lambda) = \omega_r$.


S$_2$
S$_1$
S$_3$
p
4
2
0
-2
-4
-0,4
-0,2
0
0,2
0,4
0,6
0,8
q

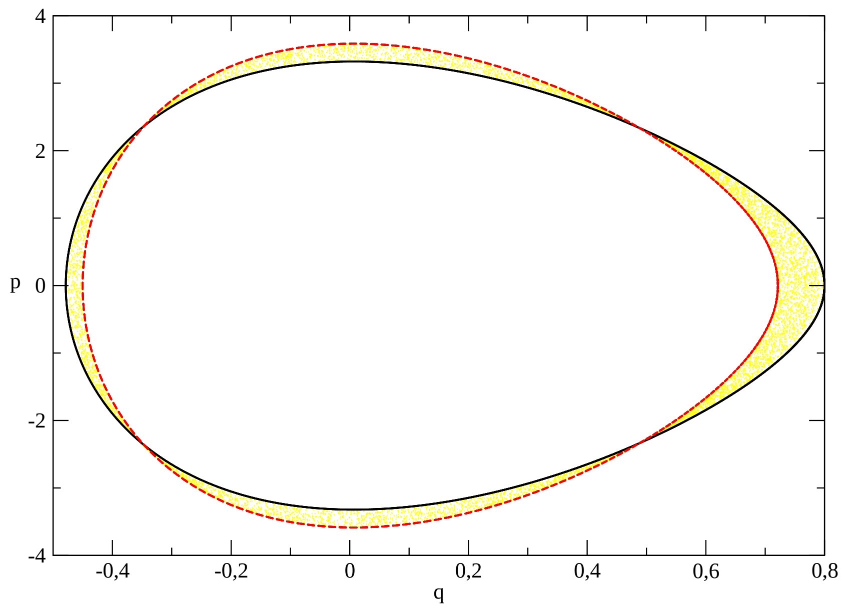
We define the quantities: the resonance action curves $I_r(\lambda)$

$$n_r\Omega(I_r(\lambda),\lambda) = \omega_r$$

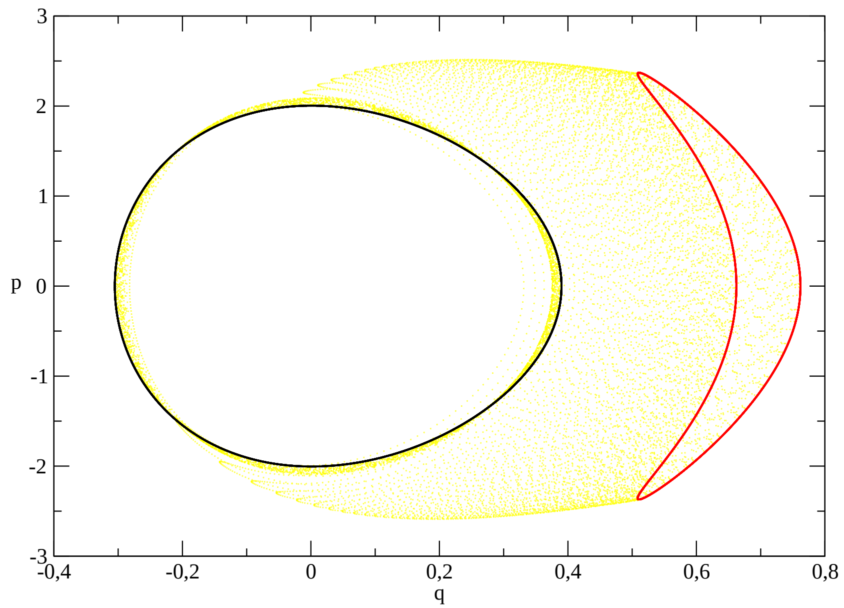
the crossing parameter $\lambda_*(I_0)$ (if it exists)

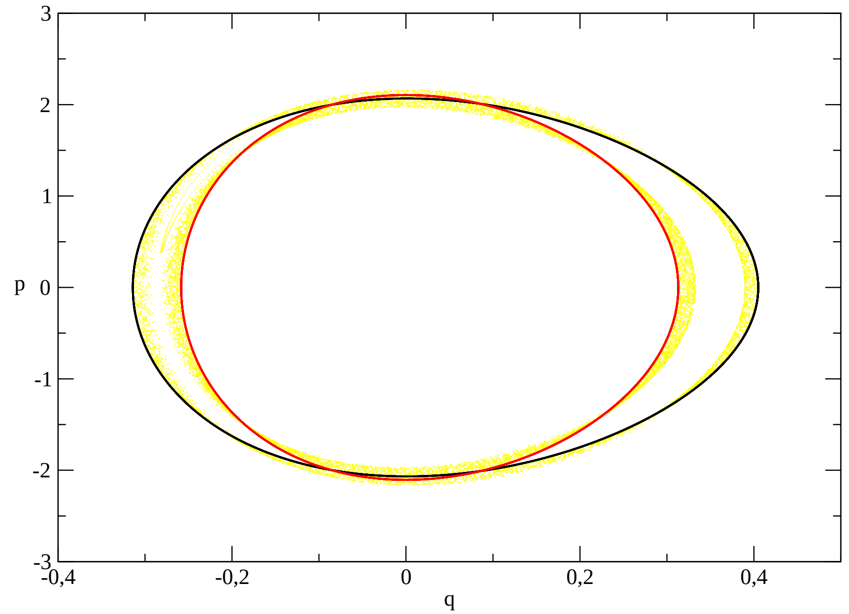
$$n_r\Omega(I_0,\lambda_*) = \omega_r$$

If we have no crossing parameter, the adiabatic invariance holds



The existence of crossing parameter allows the trapping into resonance of an orbit





The description of trapping phenomenon is due to A.Neishtadt (1986) (see also J.L. Tennyson, J.R. Gary, D.F. Escande (1986)).

Time-energy map

By introducing a Poincaré section $\theta = 0$ in the phase space, one defines the symplectic map

$$\lambda_{n+1} = \lambda_n + 2\pi\epsilon \, \frac{\partial I}{\partial E}(E,\lambda)\Big|_{\lambda} + O(\epsilon^2)$$

$$E_{n+1} = E_n - 2\pi\epsilon \, \frac{\partial I}{\partial \lambda}(E,\lambda)\Big|_{E} + O(\epsilon^2)$$

Remark: the map is singular at the separatrix and we have a different actions $I_{1,2,3}$ in the different regions.
The time-energy map describes the change of the adiabatic invariant during separatrix crossing.

Let $\Sigma_{1,2,3}$ the area enclosed by the separatrices, we define

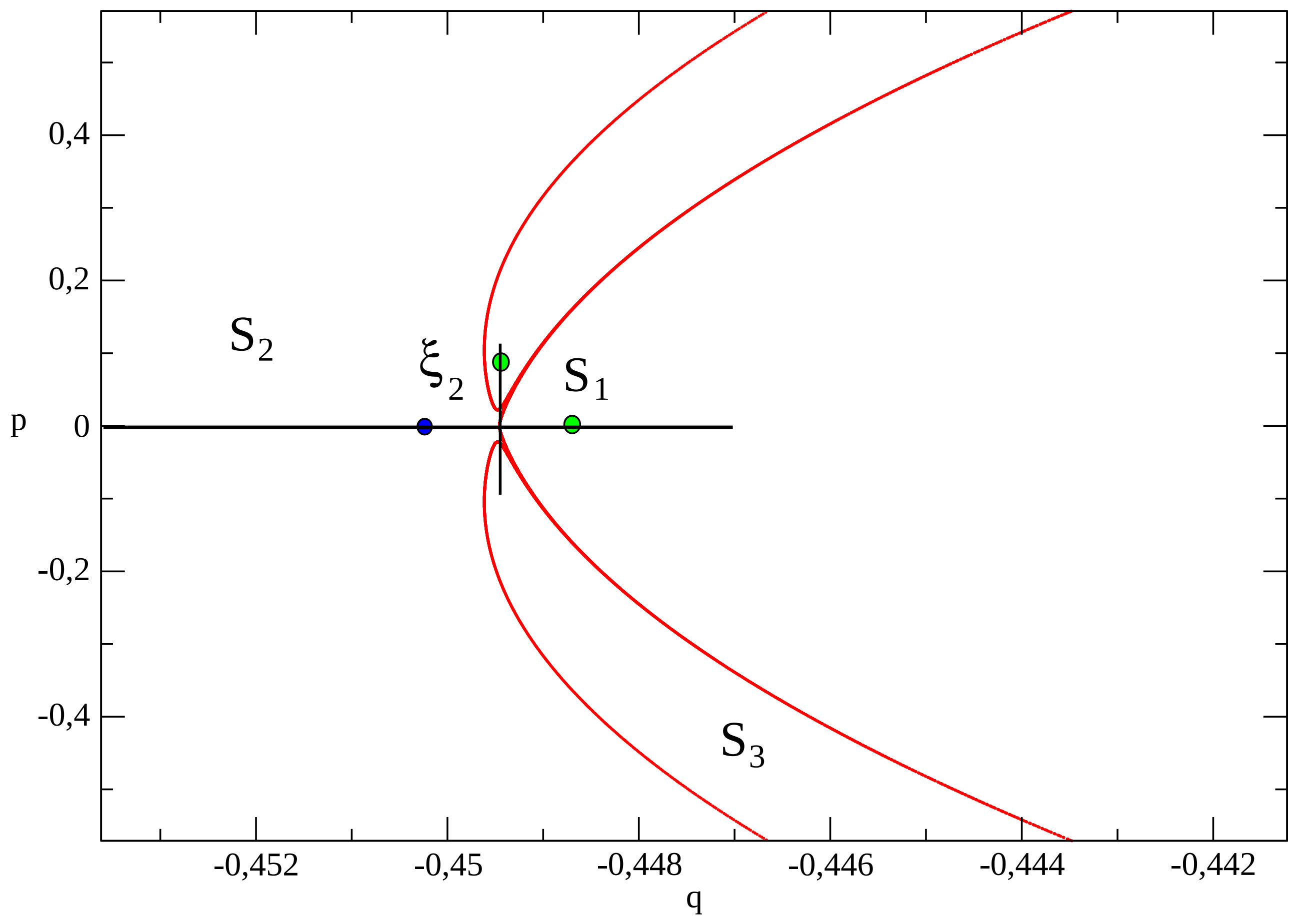
$$\Sigma_2 = \Sigma_1 + \Sigma_3 \quad , \quad \frac{d\Sigma_i}{d\lambda} = \Theta_i(\lambda) > 0 \quad i = 1, 2, 3$$

The trapping into resonance of an orbit starting in the region S_2 is a *random event* depending on the stochastic variables

$$\xi_2 = |E_0^{(2)}|/(\delta\Theta_2) \in \ [0, 1]$$

ξ_2 is uniformly distributed and quite sensitive dependence on initial condition. The adiabatic description implies

$$\begin{aligned} 2 \to 3 \quad \text{if} \quad \xi_2 \in [0, \Theta_3/\Theta_2] \\ 2 \to 1 \quad \text{if} \quad \xi_2 \in [\Theta_3/\Theta_2, 1] \end{aligned}$$



0,4
0,2
p
0
-0,2
-0,4
S_2
ξ_2
S_1
S_3
-0,452
-0,45
-0,448
-0,446
-0,444
-0,442
q

Trapping Efficiency

The trapping into resonance is described as

I) $2 \to 1$ with probability

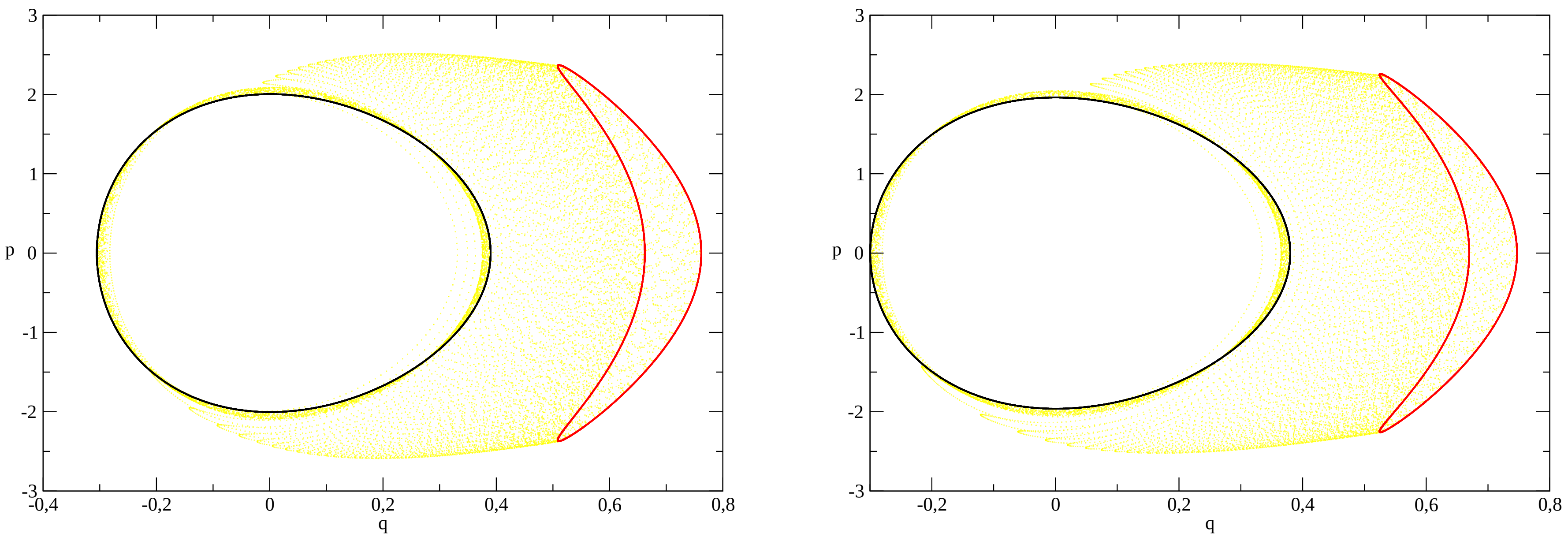
$$P(2 \to 1) = \frac{\Theta_1}{\Theta_2}$$

and a new action $I_1 = S_1(\lambda_*)$.

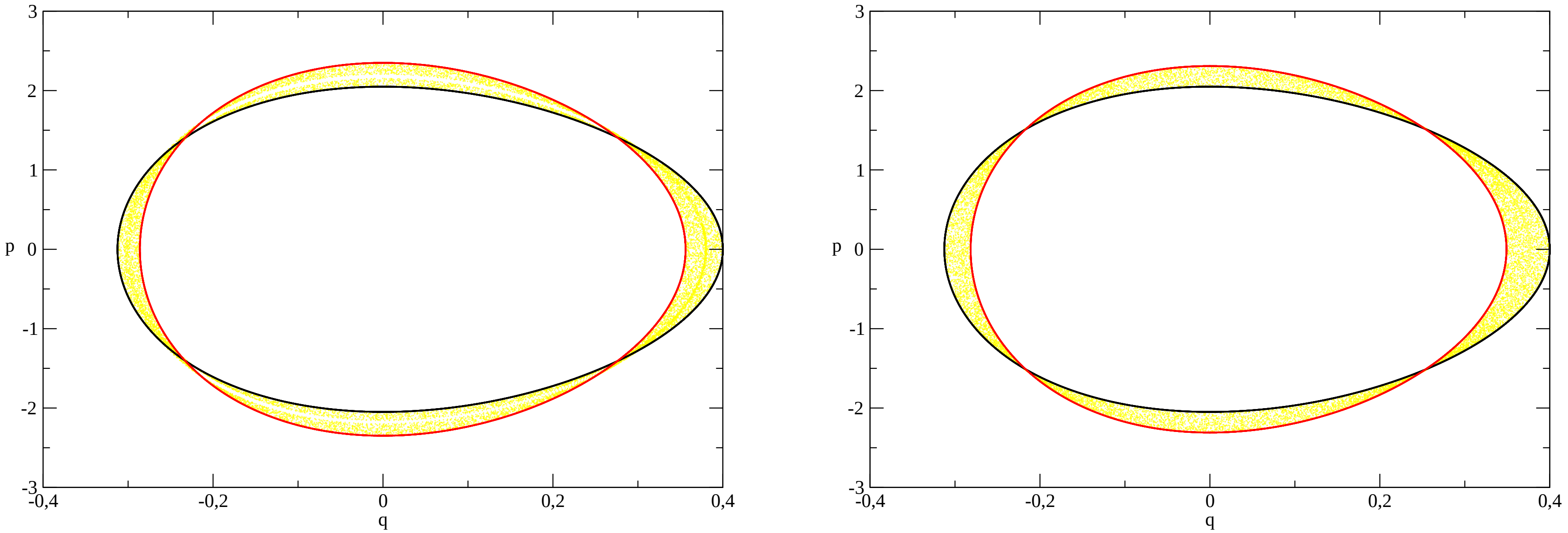
II) $2 \to 3$ with probability

$$P(2 \to 3) = \frac{\Theta_3}{\Theta_2}$$

and a new action $I_3 = S_3(\lambda_*)$.
These results hold for all values of ξ_2 except a set of measure $O(\sqrt{\delta})$ and an error in the action value $O(\delta \ln \delta)$.

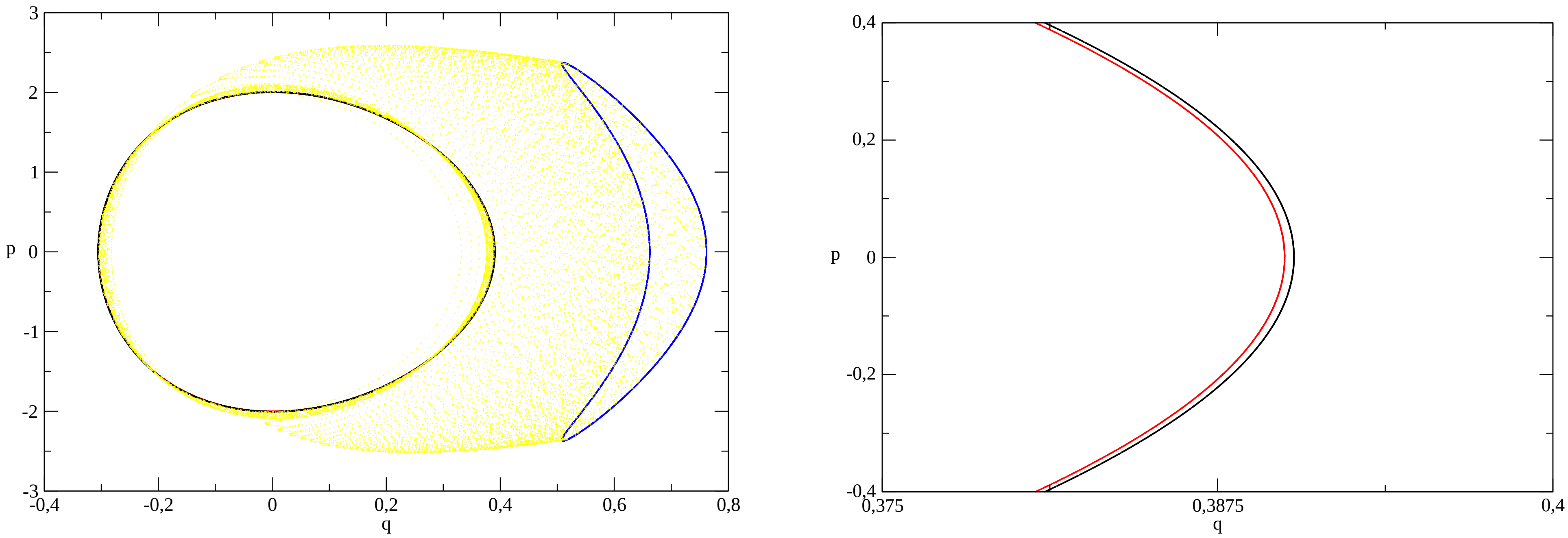


Adiabatic trapping at different values of the resonance amplitude ϵ.



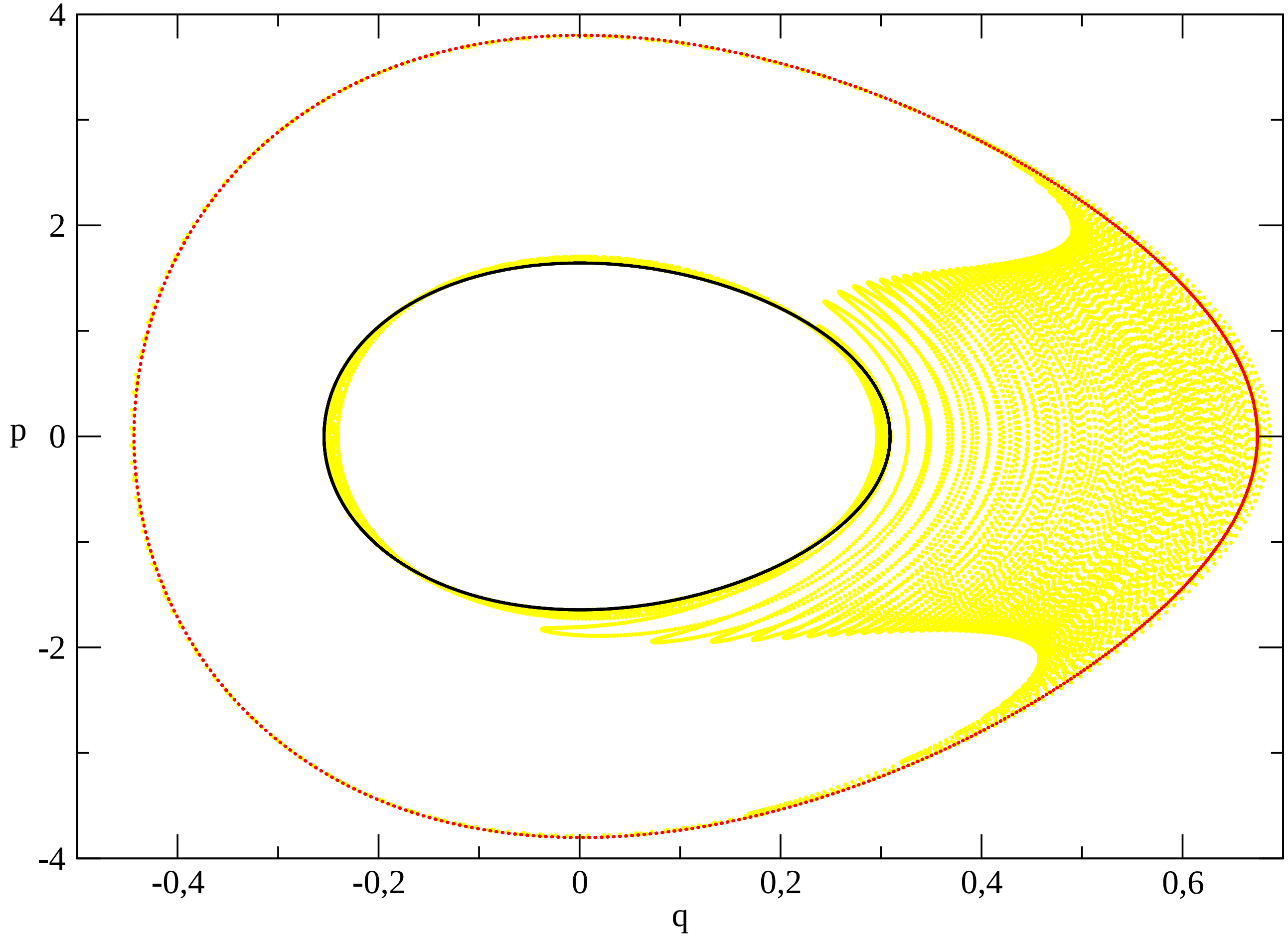
Adiabatic trapping at different values of the resonance amplitude ϵ.

The trapping is a random event depending on initial condition but the process can be reverted



Trapping and detrapping of an orbit by inverting the tune modulation $\delta = 10^{-4}$.

Modulating the resonance amplitude ϵ we can de-trap particles at large emittance



Trapping limit

To study the limit of trapping in resonance we consider the Hamiltonian

$$H(I, \theta, \lambda) = H_0(I, \lambda) + \epsilon h_k(I) \sin(k\theta - \omega_r t)$$

Introducing a rotating phase $\gamma = k\theta - \omega_r t$ we reduce $(kJ = I)$

$$H(J, \phi, \lambda) = H_0(kJ, \lambda) - \omega_r J + \epsilon h_k(kJ) \sin(\gamma)$$

and we introduce the crossing parameter and the resonance position

$$\frac{\partial H_0}{\partial J}(I_0, \lambda_*) = \omega_r \qquad \frac{\partial H_0}{\partial J}(J_r(\lambda), \lambda)) = \omega_r$$

Near the resonance the dynamics can be approximated by

$$H(J,\phi,\lambda) = \frac{1}{2}\frac{\partial^2 H_0}{\partial J^2}(kJ_r)(J - J_r(\lambda))^2 + \epsilon h_k(kJ_r)\sin(\gamma) + ...$$

with

$$J_r(\lambda) \simeq J_0 + \frac{\omega_r - k\omega_0(\lambda)}{K(\lambda)} \qquad , \qquad \frac{\partial^2 H_0}{\partial J^2}(I_0) = K(\lambda)$$

so that we get a pendulum-like system

$$H(J,\phi,\lambda) = \frac{K(\lambda)}{2}\,(\Delta J - a(\lambda))^2 + \epsilon h_k(kJ_r)\sin(\gamma)$$

We translate the action ΔJ using the generating function

$$G(\tilde{J}, \gamma) = \gamma(\Delta J - a(\lambda))$$

and we get the Hamiltonian

$$H(\tilde{J}, \phi, \delta t) = \frac{K(\lambda)}{2}\tilde{J}^2 + \epsilon h_k(kJ_r)\sin(\gamma) - \delta\frac{da}{d\lambda}\gamma$$

If the following inequality holds

$$\epsilon\left|h_k(kJ_r)\right| < \delta\left|\frac{da}{d\lambda}\right|$$

then we have no oscillation regime in the phase space and no resonance region, so that the adiabatic trapping phenomenon cannot happen. Let $h_k \propto \rho^k$ we get the scaling law for trapping efficiency

$$\rho_{trap} \propto \left(\frac{\delta}{\epsilon}\right)^{1/k}$$

Resonant crossing

What happens to the adiabatic invariant during a resonance crossing?
The action evolution reads (leading terms)

$$\Delta J(t) = \Delta J_0 - \epsilon h_k(J_0) \int_0^t \cos(\gamma_0 + K(\lambda)(\Delta J - a(\lambda))s) ds + \ldots\ldots$$

At the resonance crossing we approximate the rotation frequency

$$K(\lambda)(\Delta J - a(\lambda)) \simeq K(\lambda_*) \left(\Delta J - k \frac{d\omega_0}{d\lambda}(\lambda - \lambda_*) \right)$$

Then we get the an estimate for the action variation

$$\Delta J(t) = \Delta J_0 - \frac{\epsilon}{\sqrt{\delta \Delta(J_0, \lambda_*)}} h_k(J_0) \int_{-t_0/\sqrt{\delta\Delta(J_0,\lambda_*)}}^{t_0/\sqrt{\delta\Delta(J_0,\lambda_*)}} \cos\left(\phi_* + \frac{u^2}{2}\right) du$$

i.e.

$$J(t) = J_0 + O\left(\frac{\epsilon}{\sqrt{\delta\Delta(J_0, \lambda_*)}}\right) |h_k(J_0)|$$

Conclusions and Perspectives

I) Adiabatic Theory gives quite robust results provided that the adiabaticity condition δ/Ω_{typ}^2 holds.

II) The results can be applied to many dimensional quasi integrable systems (resonance regions are separated).

III) Analytical estimates of the trapping threshold, trapping efficiency and transport efficiency are possible.

IV) Develop diagnostic tools to measure beam dynamics perturbations.

V) Slow adiabatic extraction using coupled resonance between horizontal and vertical tunes.

VI) Study the Adiabatic Theory in the framework of Vlasov problem to understand space charge effects.