Momentum-space treatment of the Coulomb force: Screening and renormalization

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In collaboration with A. C. Fonseca and P. U. Sauer

Outline

- Momentum-space description of few-body scattering: screening and renormalization for Coulomb
  [Taylor, Alt, Sandhas, ...]
- S&R variations and other methods
- Applications: 3N, 4N, nuclear reactions, ...

Screened Coulomb

$$w_R(r) = w_C(r)e^{-\left(\frac{r}{R}\right)^n}$$

- standard scattering theory

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- standard scattering theory
- nature: Coulomb is screened at large distances
- large R:
  physical observables insensitive to screening,
  screened and full Coulomb physically indistinguishable

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- nature: Coulomb is screened at large distances
- large R:
  physical observables insensitive to screening,
  screened and full Coulomb physically indistinguishable
- in the $R \to \infty$ limit physical results are recovered

Screened and full Coulomb physically indistinguishable

$$\langle \mathbf{p}' | T_R | \mathbf{p} \rangle \quad \overset{?}{\underset{R\to\infty}{\longrightarrow}} \quad \langle \mathbf{p}' | T_C | \mathbf{p} \rangle$$

Screened and full Coulomb physically indistinguishable

$$e^{2i\phi_R} \langle \mathbf{p}' | T_R | \mathbf{p} \rangle \quad \overset{?}{\underset{R\to\infty}{\longrightarrow}} \quad \langle \mathbf{p}' | T_C | \mathbf{p} \rangle$$

Screened and full Coulomb physically indistinguishable

initial physical state: wave packet $\varphi_{\text{in}}(\mathbf{p})$

outgoing wave packet

$$\varphi_{\text{out}}(\mathbf{p}') = \int d^3\mathbf{p}\,\langle\mathbf{p}'|S|\mathbf{p}\rangle\varphi_{\text{in}}(\mathbf{p})$$

$$\sim \int d^2\hat{\mathbf{p}}\, e^{2i\phi_R}\langle\mathbf{p}'|T_R|\mathbf{p}\rangle\varphi_{\text{in}}(\mathbf{p}) \xrightarrow[R\to\infty]{} \int d^2\hat{\mathbf{p}}\,\langle\mathbf{p}'|T_C|\mathbf{p}\rangle\varphi_{\text{in}}(\mathbf{p})$$

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$$p' = p: \qquad e^{2i\phi_R} \langle \mathbf{p}' | T_R | \mathbf{p} \rangle \xrightarrow[R\to\infty]{} \langle \mathbf{p}' | T_C | \mathbf{p} \rangle \quad \text{as distribution}$$

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$$\phi_R \xrightarrow[R\to\infty]{} [\sigma_L - \eta_{LR}] \xrightarrow[R\to\infty]{} \alpha_e M/p\,[\ln(2pR) - C/n]$$

[*J. R. Taylor, Nuovo Cimento* **B23**, *313 (1974)*]

Screened and full Coulomb wave functions

$$r < R: \qquad w_R(r) \approx w_C(r)$$

$$\Downarrow$$

$$e^{i\phi_{LR}} \langle r | \psi_{LR}^{(+)}(p) \rangle \approx \langle r | \psi_{LC}^{(+)}(p) \rangle$$

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$$e^{i\phi_R} | \psi_R^{(+)}(\mathbf{p}) \rangle \xrightarrow[R \to \infty]{} | \psi_C^{(+)}(\mathbf{p}) \rangle$$

[*V. G. Gorshkov, Sov. Phys.-JETP* **13**, *1037 (1961)*]

Screening and renormalization

Renormalization of the on-shell screened Coulomb transition matrix $T_R = w_R + w_R G_0 T_R$ and wave function in the limit $R \to \infty$ yields Coulomb amplitude

and Coulomb wave function

$$T_R \, z_R^{-1} \xrightarrow[R\to\infty]{} T_C \qquad \text{as distribution}$$

$$(1 + G_0 T_R)|\mathbf{p}\rangle z_R^{-1/2} \xrightarrow[R\to\infty]{} |\psi_C^{(+)}(\mathbf{p})\rangle$$

$$z_R = e^{-2i\phi_R}$$

Two-particle scattering

transition matrix

$$T^{(R)} = v + w_R + (v + w_R) G_0 T^{(R)}$$

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$$T^{(R)} = v + w_R + (v + w_R) G_0 T^{(R)}$$

with long-range and Coulomb-distorted short-range parts

$$T^{(R)} = T_R + (1 + T_R G_0) \tilde{T}^{(R)} (1 + G_0 T_R)$$

$$\tilde{T}^{(R)} = v + v G_R \tilde{T}^{(R)}$$

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Renormalized amplitude:

$$T^{(R)} z_R^{-1} \xrightarrow[R \to \infty]{} T = T_C + \langle \psi_C^{(-)} | \tilde{T}^{(C)} | \psi_C^{(+)} \rangle$$

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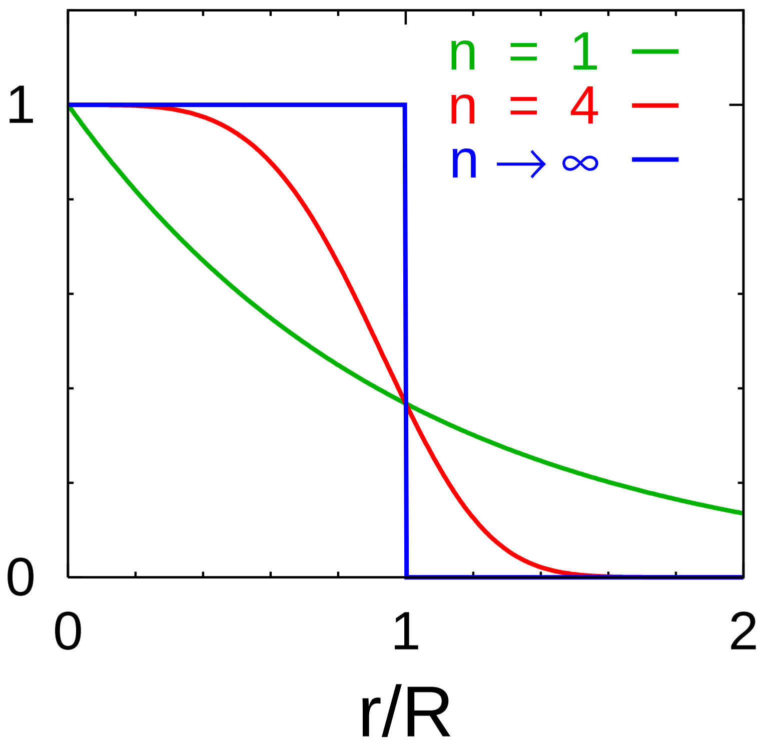
$$= T_C + \lim_{R \to \infty} z_R^{-\frac{1}{2}} [T^{(R)} - T_R] z_R^{-\frac{1}{2}}$$

short-range part: fast convergence with R

Test: convergence with R in pp scattering

$$\frac{w_R(r)}{w_C(r)} = e^{-\left(\frac{r}{R}\right)^{\mathrm{n}}}$$



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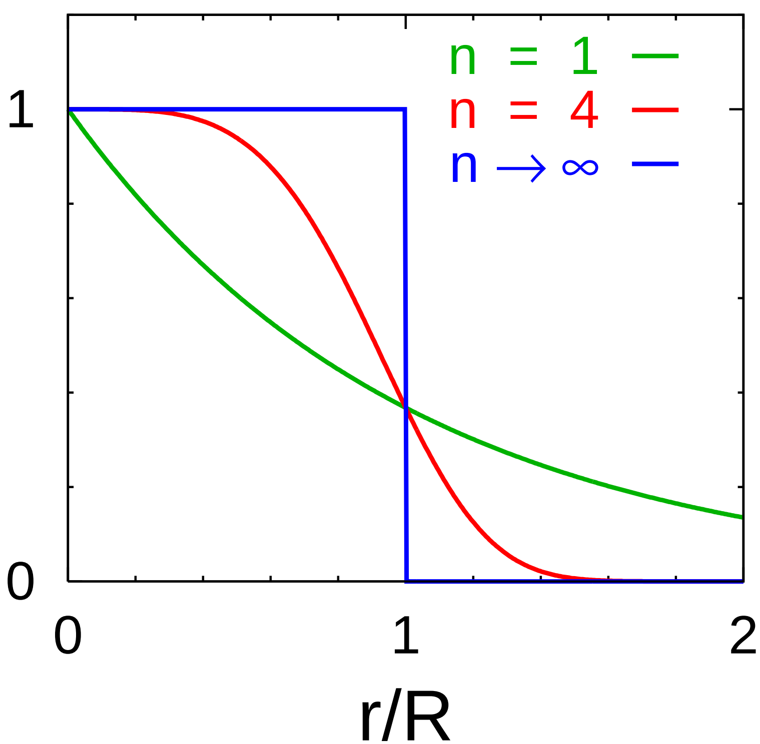

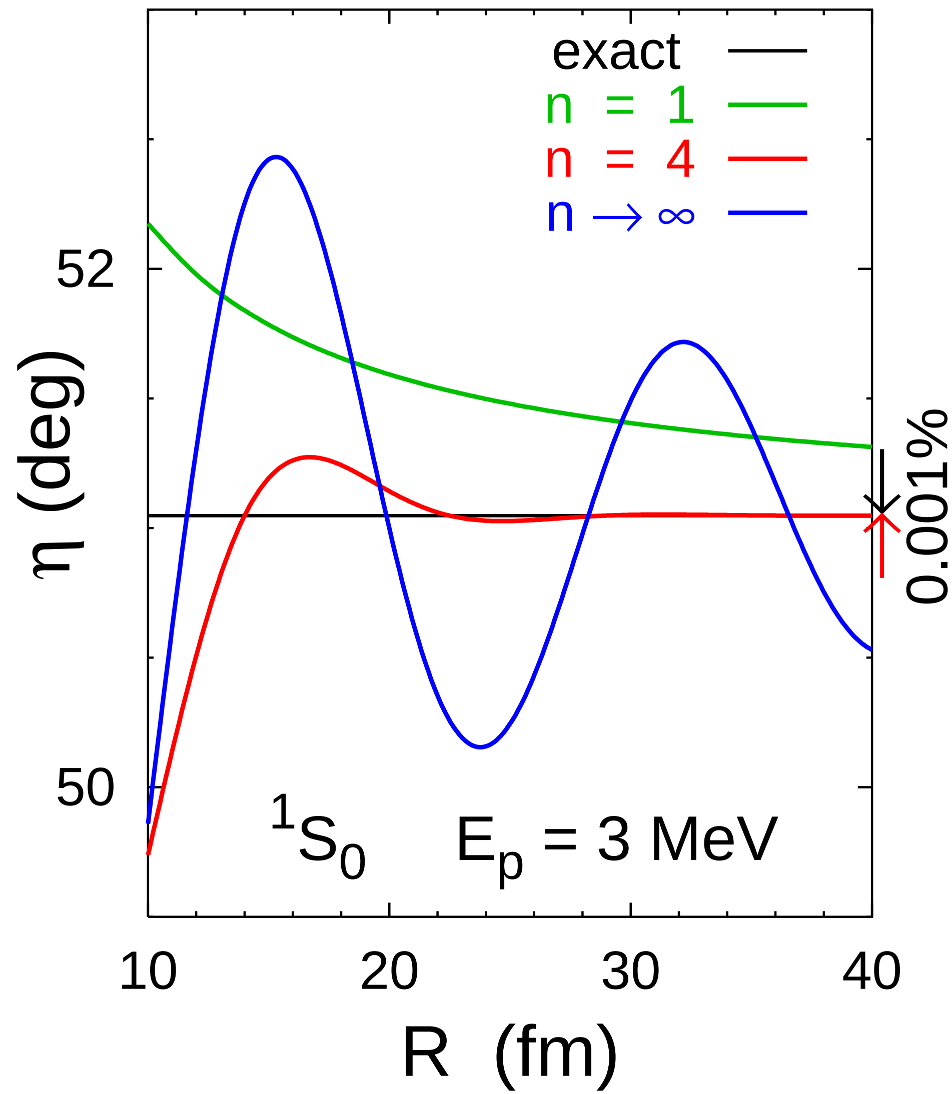

optimal choice: $3 \leq \mathrm{n} \leq 8$

Limits of practical applicability

$p \to 0$:

$\kappa = \alpha M/p$, $\sigma_L = \arg\Gamma(1 + L + i\kappa)$, and z_R diverge,

renormalization procedure ill-defined

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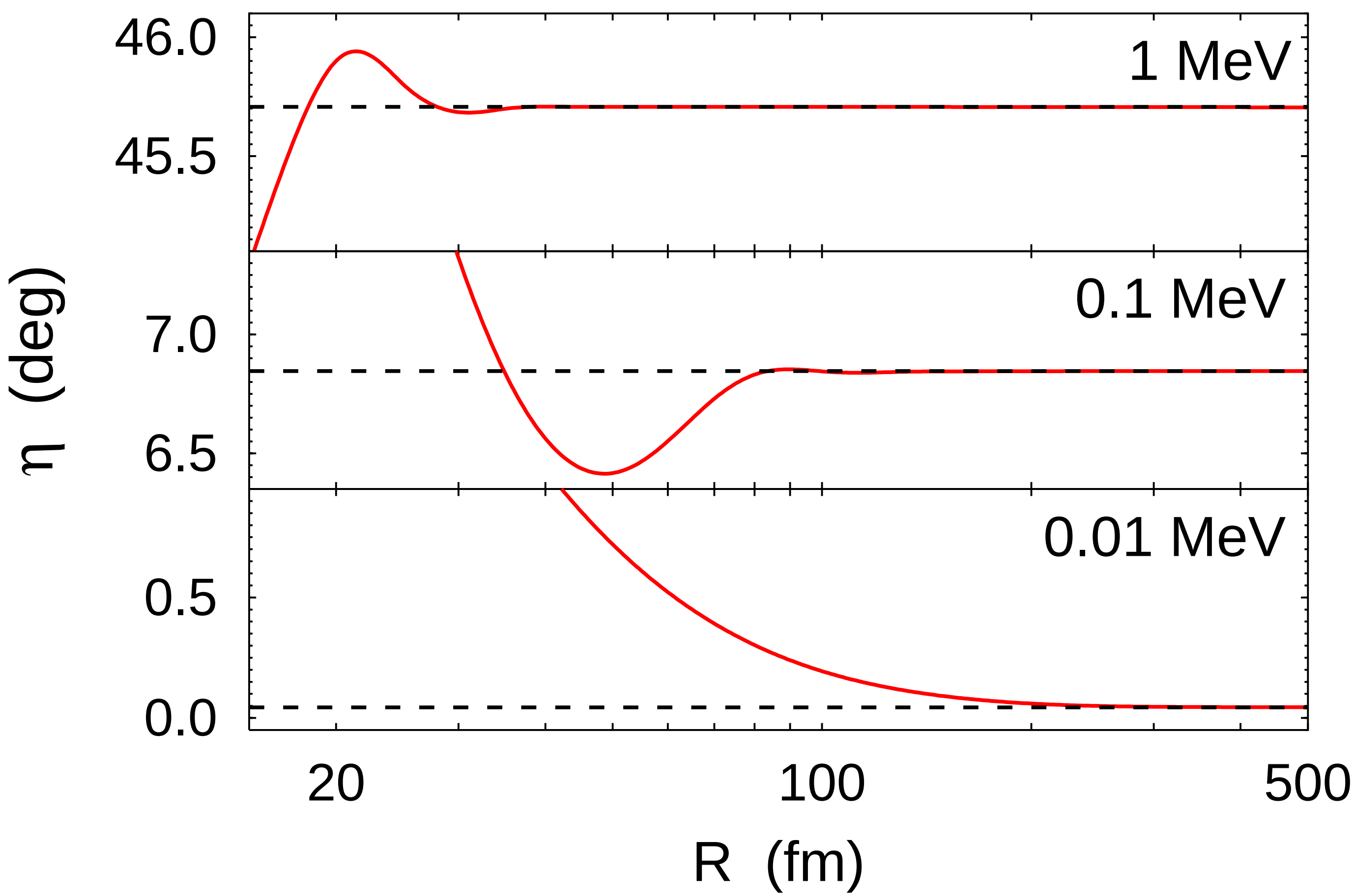
$\Rightarrow$ slow convergence with R at low relative energies



Three-particle scattering: short-range forces

- Faddeev / Alt, Grassberger, and Sandhas equations

$$U_{\beta\alpha} = \bar{\delta}_{\beta\alpha} G_0^{-1} + \sum_{\sigma} \bar{\delta}_{\beta\sigma} T_{\sigma} G_0 U_{\sigma\alpha}$$

$$U_{0\alpha} = G_0^{-1} + \sum_{\sigma} T_{\sigma} G_0 U_{\sigma\alpha}$$

$$T_{\sigma} = v_{\sigma} + v_{\sigma} G_0 T_{\sigma}$$

$$G_0 = (E + i0 - H_0)^{-1}$$

- momentum-space partial-wave representation

AGS equations with 3BF

$$V_{3BF} = \sum_{\alpha=1}^{3} u_\alpha$$

$$U_{\beta\alpha} = \bar{\delta}_{\beta\alpha} G_0^{-1} + \sum_\gamma \bar{\delta}_{\beta\gamma} T_\gamma G_0 U_{\gamma\alpha}$$

$$+ u_\alpha + \sum_\gamma u_\gamma G_0 (1 + T_\gamma G_0) U_{\gamma\alpha}$$

Three-particle scattering: including screened Coulomb

- Faddeev / Alt, Grassberger, and Sandhas equations

$$U^{(R)}_{\beta\alpha} = \bar{\delta}_{\beta\alpha} G_0^{-1} + \sum_{\sigma} \bar{\delta}_{\beta\sigma} T^{(R)}_{\sigma} G_0 U^{(R)}_{\sigma\alpha}$$

$$U^{(R)}_{0\alpha} = G_0^{-1} + \sum_{\sigma} T^{(R)}_{\sigma} G_0 U^{(R)}_{\sigma\alpha}$$

$$T^{(R)}_{\sigma} = v_{\sigma} + w_{\sigma_R} + (v_{\sigma} + w_{\sigma_R}) G_0 T^{(R)}_{\sigma}$$

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- momentum-space partial-wave representation

Three-particle scattering: including screened Coulomb

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- momentum-space partial-wave representation
- Additional difficulties:
  quasi-singular nature of screened Coulomb potential
  slow partial-wave convergence

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- momentum-space partial-wave representation
- Additional difficulties:
  quasi-singular nature of screened Coulomb potential
  slow partial-wave convergence
- $R \to \infty$ limit?

Three-particle scattering: $R \to \infty$ limit

long-range part

$$T_{\alpha R}^{\text{c.m.}} = W_{\alpha R}^{\text{c.m.}} + W_{\alpha R}^{\text{c.m.}} G_{\alpha}^{(R)} T_{\alpha R}^{\text{c.m.}}$$

Three-particle scattering: $R \to \infty$ limit

Split into long-range part

$$T^{\text{c.m.}}_{\alpha R} = W^{\text{c.m.}}_{\alpha R} + W^{\text{c.m.}}_{\alpha R} G^{(R)}_{\alpha} T^{\text{c.m.}}_{\alpha R}$$

and Coulomb-distorted short-range part

$$U^{(R)}_{\beta\alpha} = \delta_{\beta\alpha} T^{\text{c.m.}}_{\alpha R} + [1 + T^{\text{c.m.}}_{\beta R} G^{(R)}_{\beta}] \tilde{U}^{(R)}_{\beta\alpha} [1 + G^{(R)}_{\alpha} T^{\text{c.m.}}_{\alpha R}]$$

$$U^{(R)}_{0\alpha} = [1 + T_{\rho R} G_0] \tilde{U}^{(R)}_{0\alpha} [1 + G^{(R)}_{\alpha} T^{\text{c.m.}}_{\alpha R}] \quad [\rho \text{ is neutral}]$$

Three-particle scattering: $R \to \infty$ limit

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Renormalized amplitudes:

$$U_{\beta\alpha} = \delta_{\beta\alpha} T^{\text{c.m.}}_{\alpha C} + \lim_{R\to\infty} Z_{R\,f}^{-\frac{1}{2}} [U^{(R)}_{\beta\alpha} - \delta_{\beta\alpha} T^{\text{c.m.}}_{\alpha R}] Z_{R\,i}^{-\frac{1}{2}}$$

$$U_{0\alpha} = \lim_{R\to\infty} z_{R}^{-\frac{1}{2}} U^{(R)}_{0\alpha} Z_{R\,i}^{-\frac{1}{2}}$$

Three-particle scattering: $R \to \infty$ limit

Split into long-range part

$$T^{\text{c.m.}}_{\alpha R} = W^{\text{c.m.}}_{\alpha R} + W^{\text{c.m.}}_{\alpha R} G^{(R)}_{\alpha} T^{\text{c.m.}}_{\alpha R}$$

and Coulomb-distorted short-range part

$$U^{(R)}_{\beta\alpha} = \delta_{\beta\alpha} T^{\text{c.m.}}_{\alpha R} + [1 + T^{\text{c.m.}}_{\beta R} G^{(R)}_{\beta}] \tilde{U}^{(R)}_{\beta\alpha} [1 + G^{(R)}_{\alpha} T^{\text{c.m.}}_{\alpha R}]$$

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Renormalized amplitudes:

$$U_{\beta\alpha} = \delta_{\beta\alpha} T^{\text{c.m.}}_{\alpha C} + \lim_{R\to\infty} Z^{-\frac{1}{2}}_{R\,f} [U^{(R)}_{\beta\alpha} - \delta_{\beta\alpha} T^{\text{c.m.}}_{\alpha R}] Z^{-\frac{1}{2}}_{R\,i}$$

$$U_{0\alpha} = \lim_{R\to\infty} z^{-\frac{1}{2}}_{R} U^{(R)}_{0\alpha} Z^{-\frac{1}{2}}_{R\,i}$$

short-range part: fast convergence with R

r-space methods

- Kohn VP + HH
  [Kievsky *et al*]
- differential Faddeev equations
  [Payne *et al*, Lazauskas *et al*, Suslov *et al*]
- integral Faddeev equations
  [Ishikawa]

Screening and renormalization: variations

- separable potentials, quasiparticle equations, effective two-body potentials, Coulomb distorted ffs, ...
  [Alt *et al*]
- "rigorous Coulomb treatment"
  [Oryu *et al*]
- no/different renormalization
  [Witała *et al*]
- in progress:
  separable potentials, unscreened Coulomb representation
  [Mukhamedzhanov *et al*, TORUS]

Proton-deuteron scattering

- Symmetrized Faddeev / AGS equations

$$U^{(R)} = PG_0^{-1} + PT^{(R)}G_0U^{(R)}$$

$$U_0^{(R)} = (1+P)G_0^{-1} + (1+P)T^{(R)}G_0U^{(R)}$$

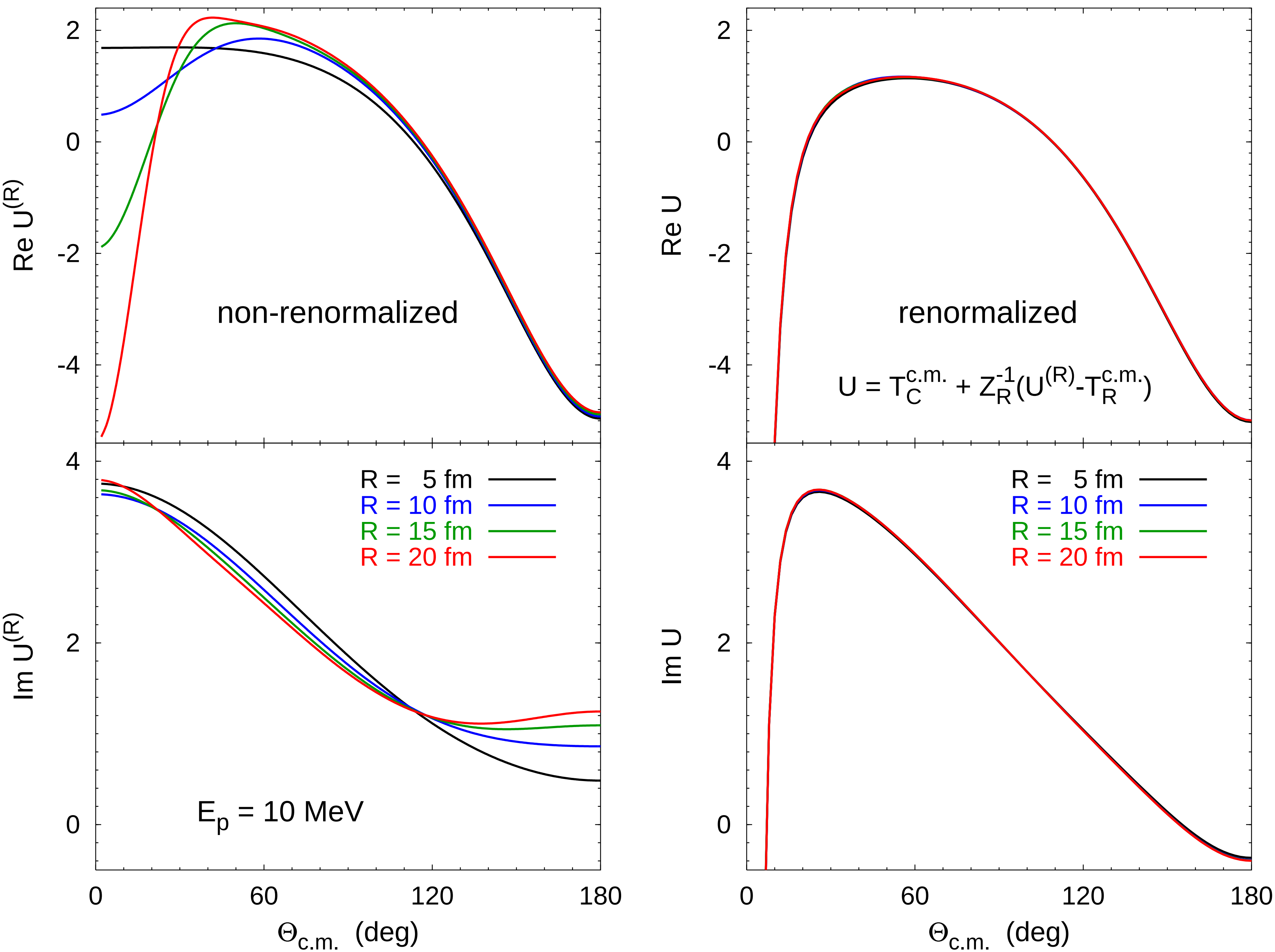
$$P = P_{12}P_{23} + P_{13}P_{23}$$

- Screening function with $n = 4$
- Renormalized amplitudes:

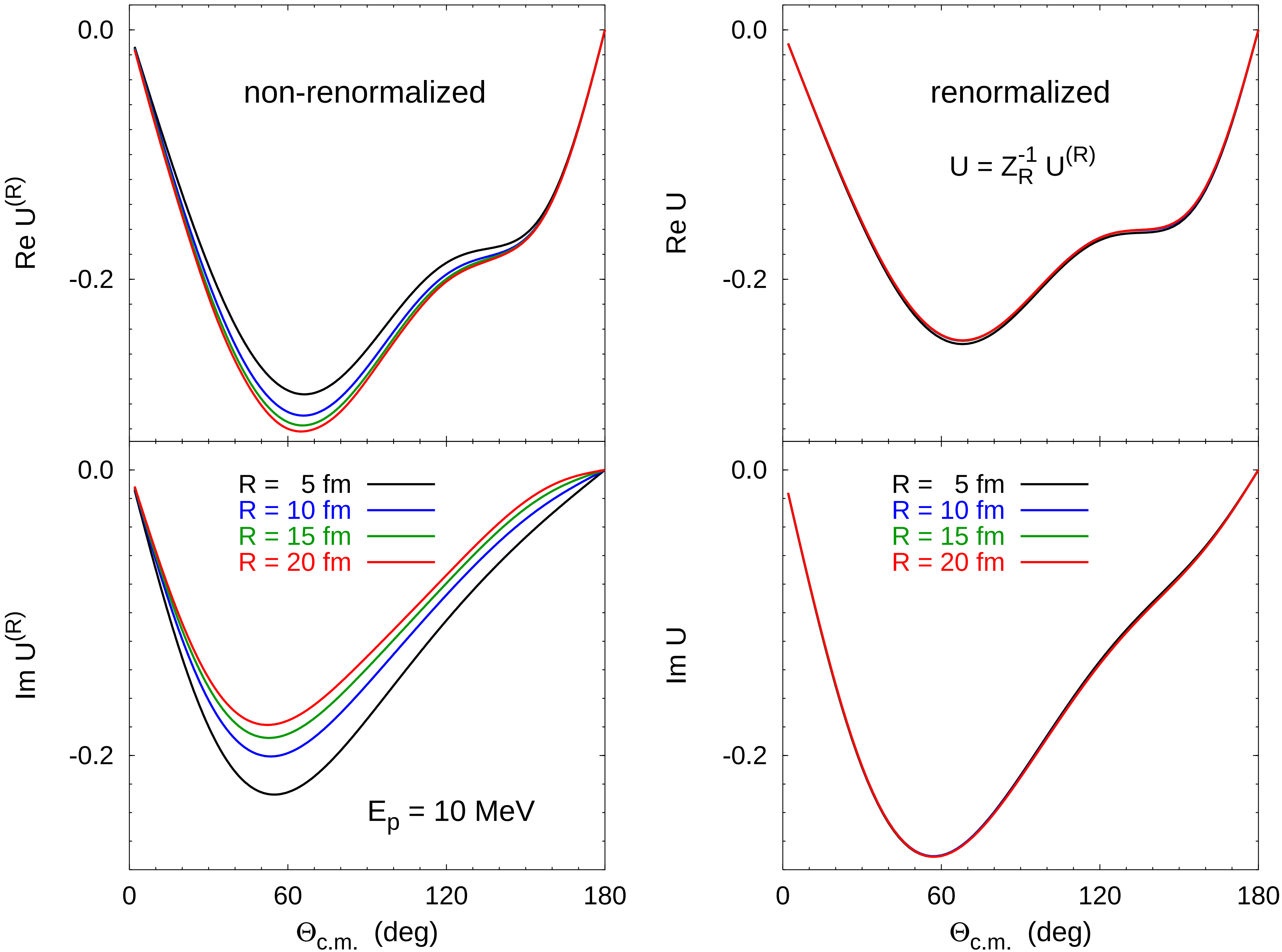
$$U = T_C^{\text{c.m.}} + \lim_{R\to\infty} Z_R^{-1}[U^{(R)} - T_R^{\text{c.m.}}]$$

$$U_0 = \lim_{R\to\infty} z_R^{-\frac{1}{2}} U_0^{(R)} Z_R^{-\frac{1}{2}}$$

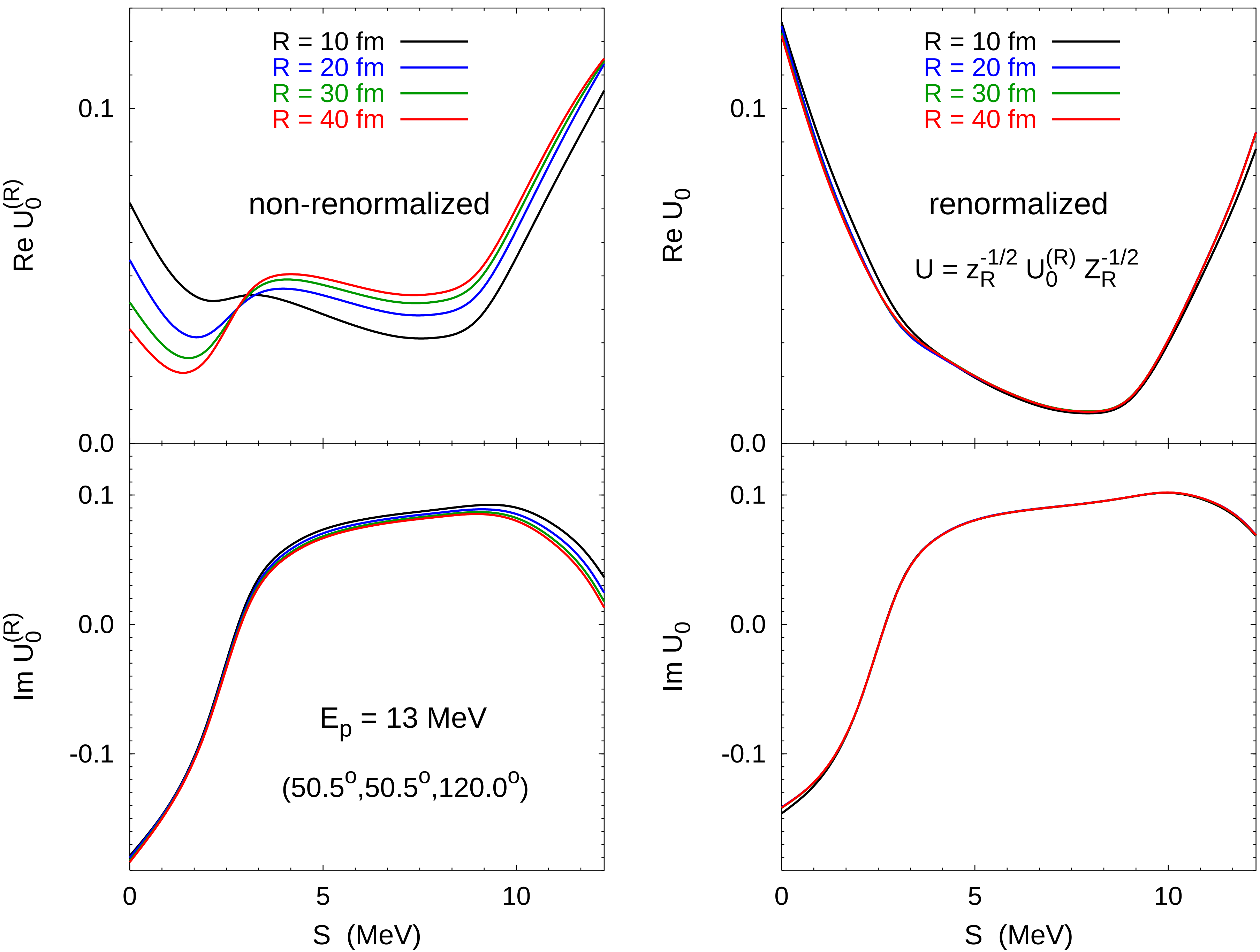
pd elastic amplitude (spin-diagonal)



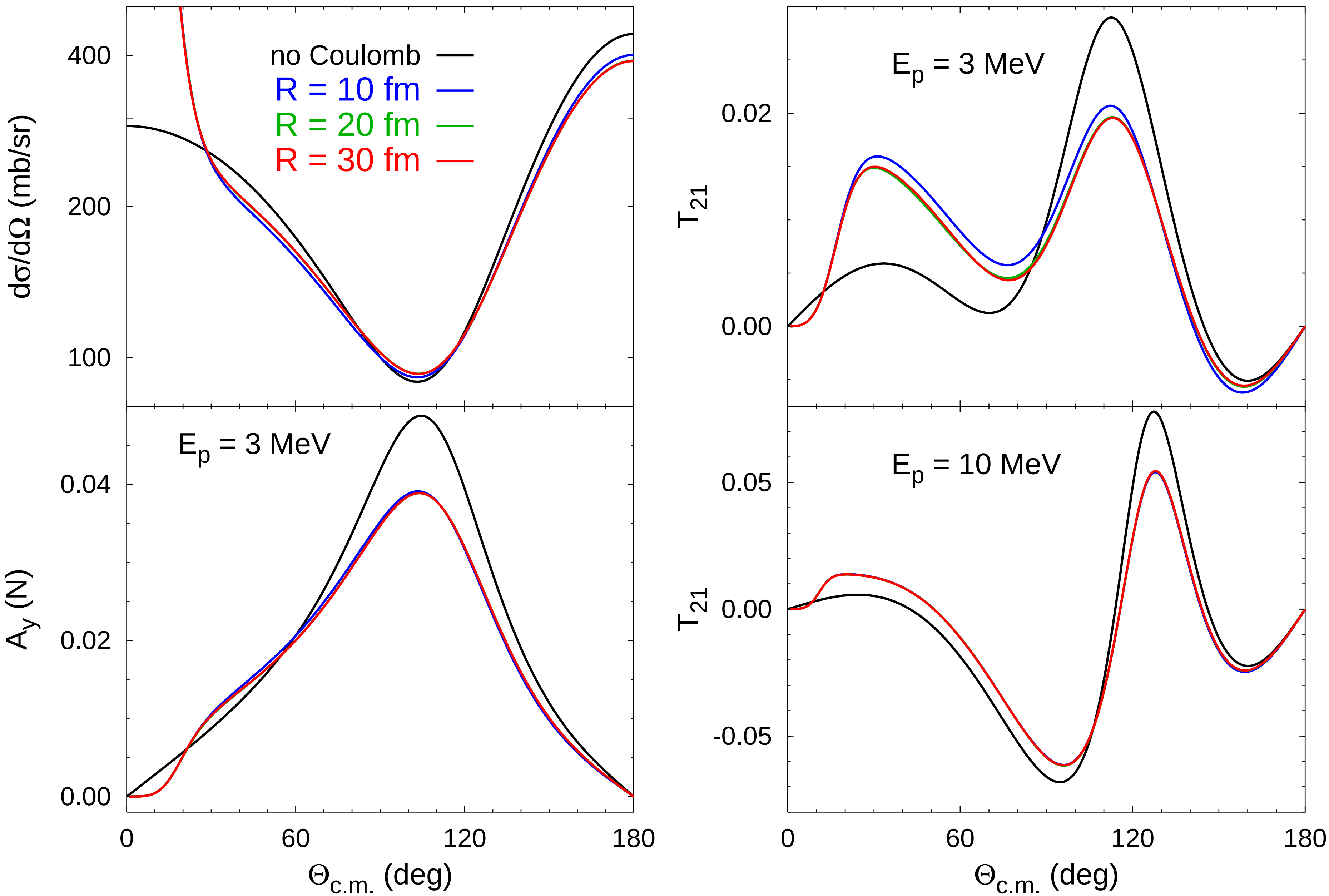
pd elastic amplitude (spin-nondiagonal)



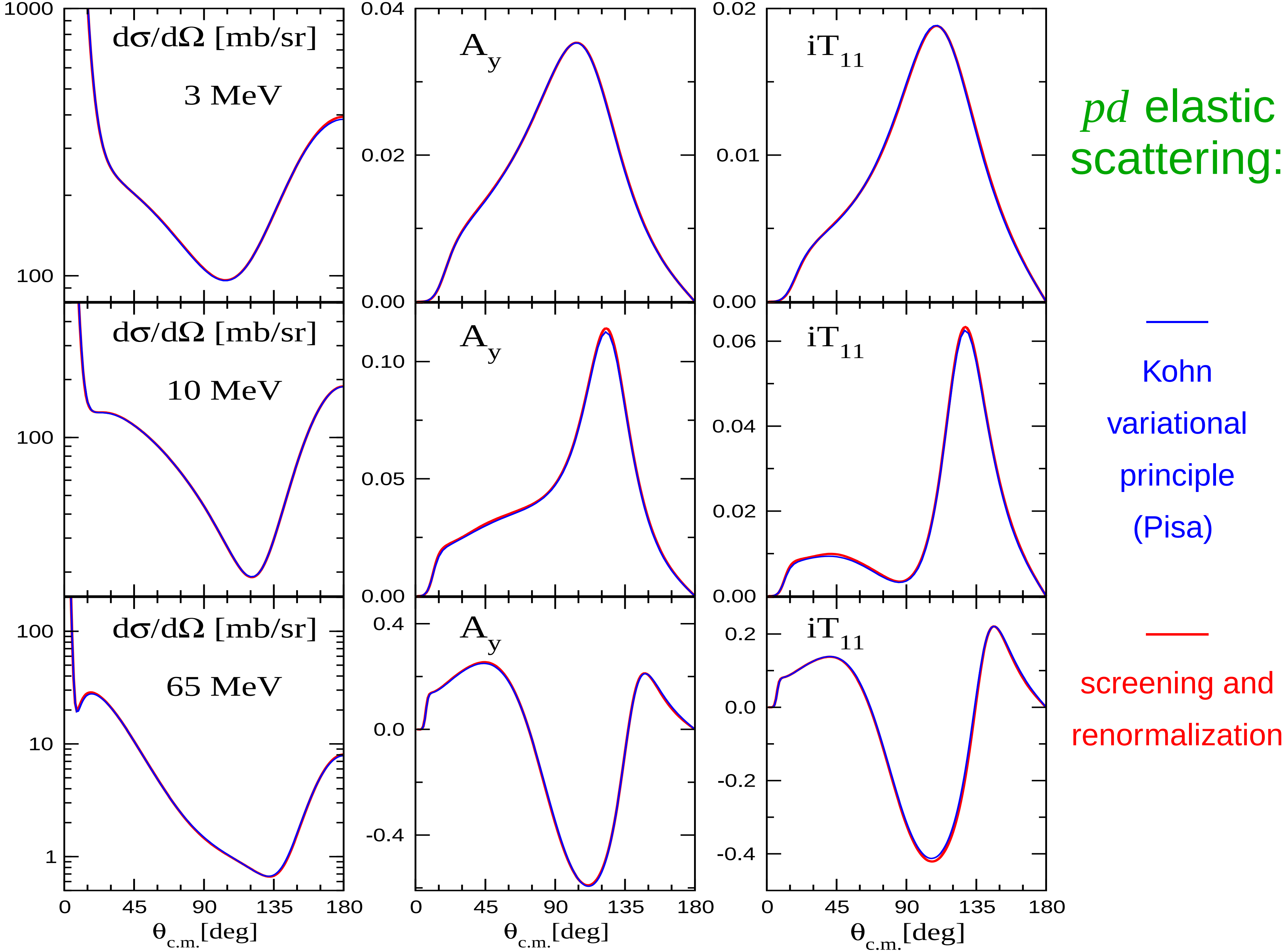
pd breakup amplitude



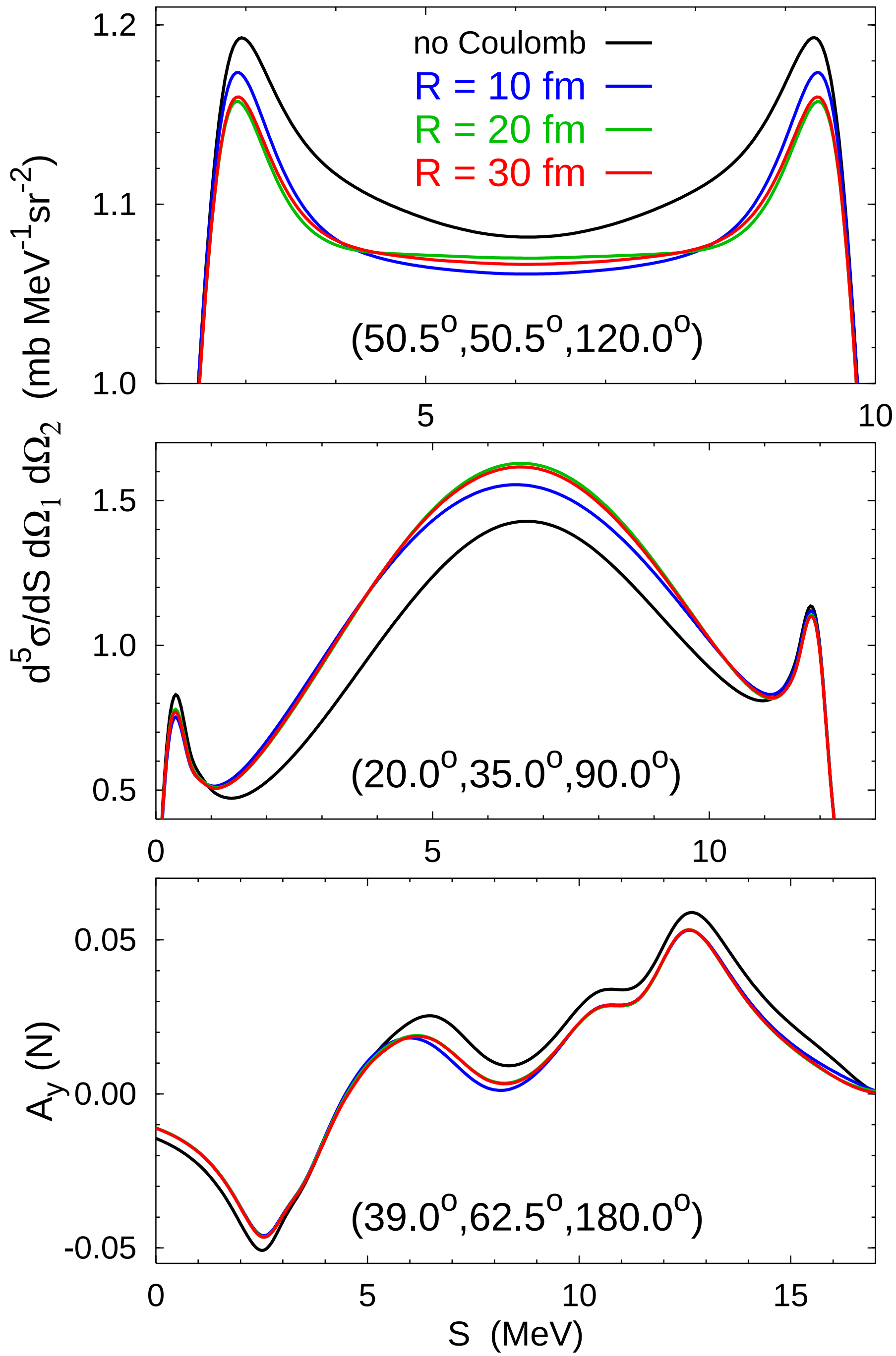
Convergence with *R*: *pd* elastic scattering



Comparison with configuration-space results



Convergence with R: pd breakup at $E_p = 13$ MeV



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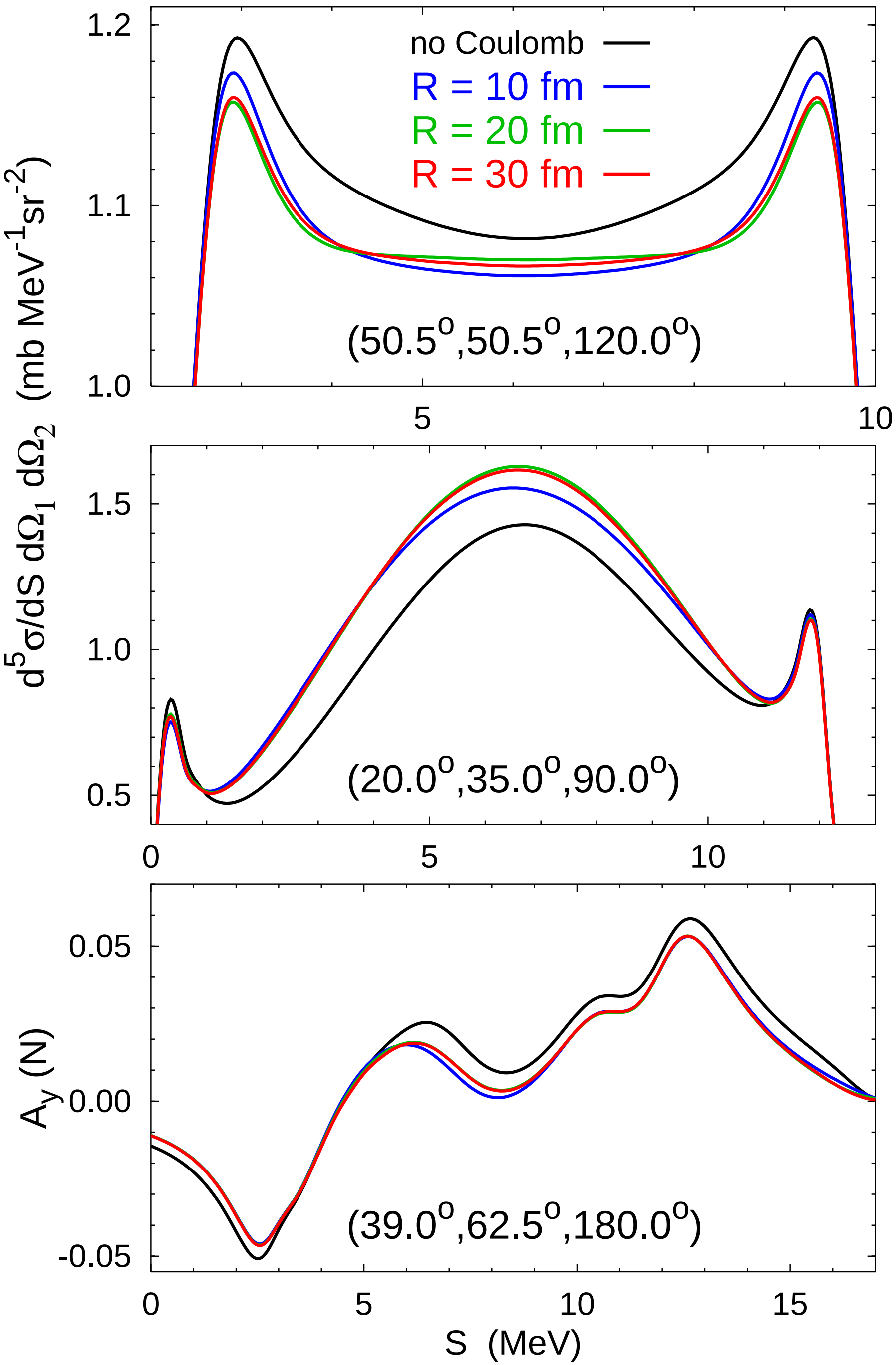
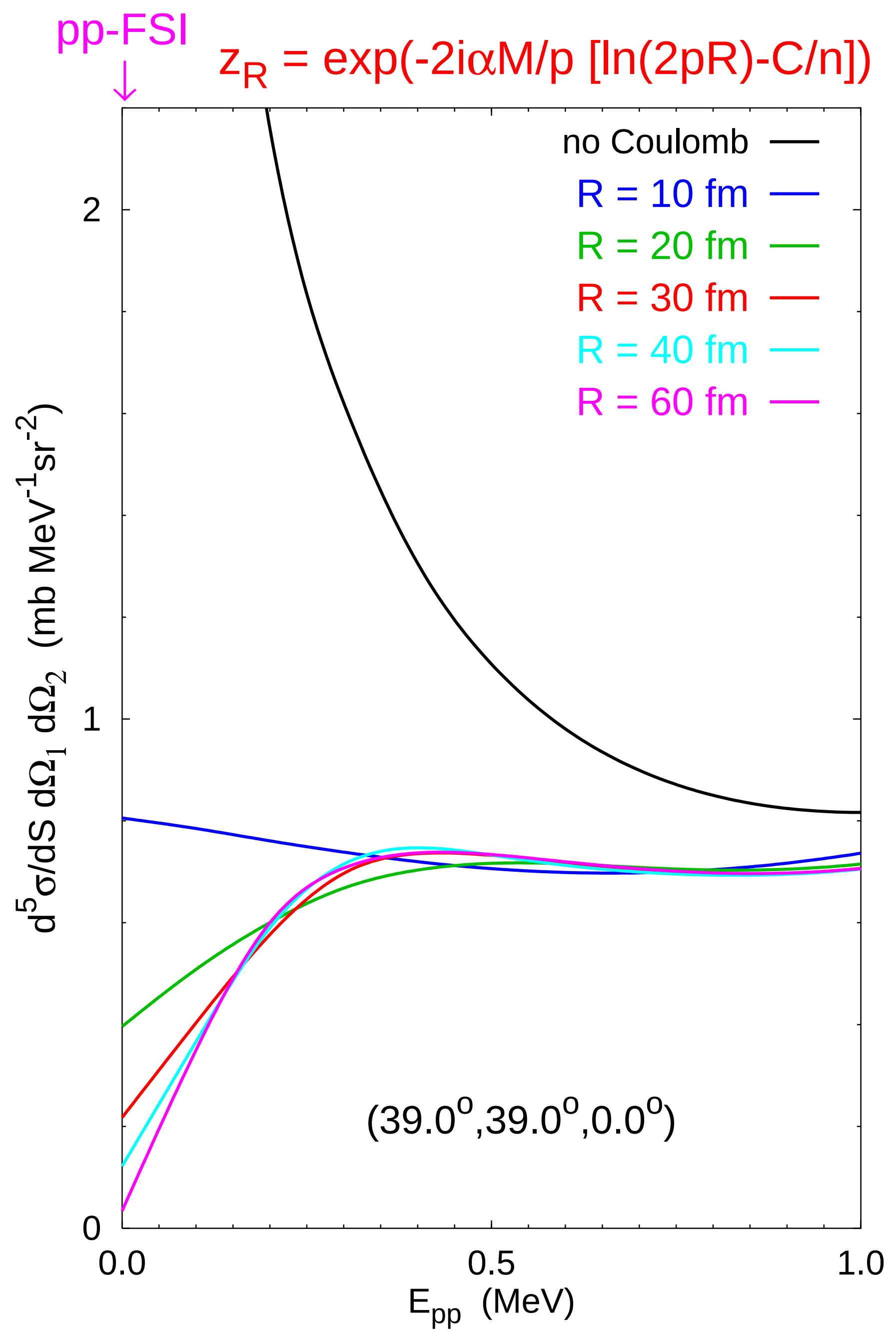
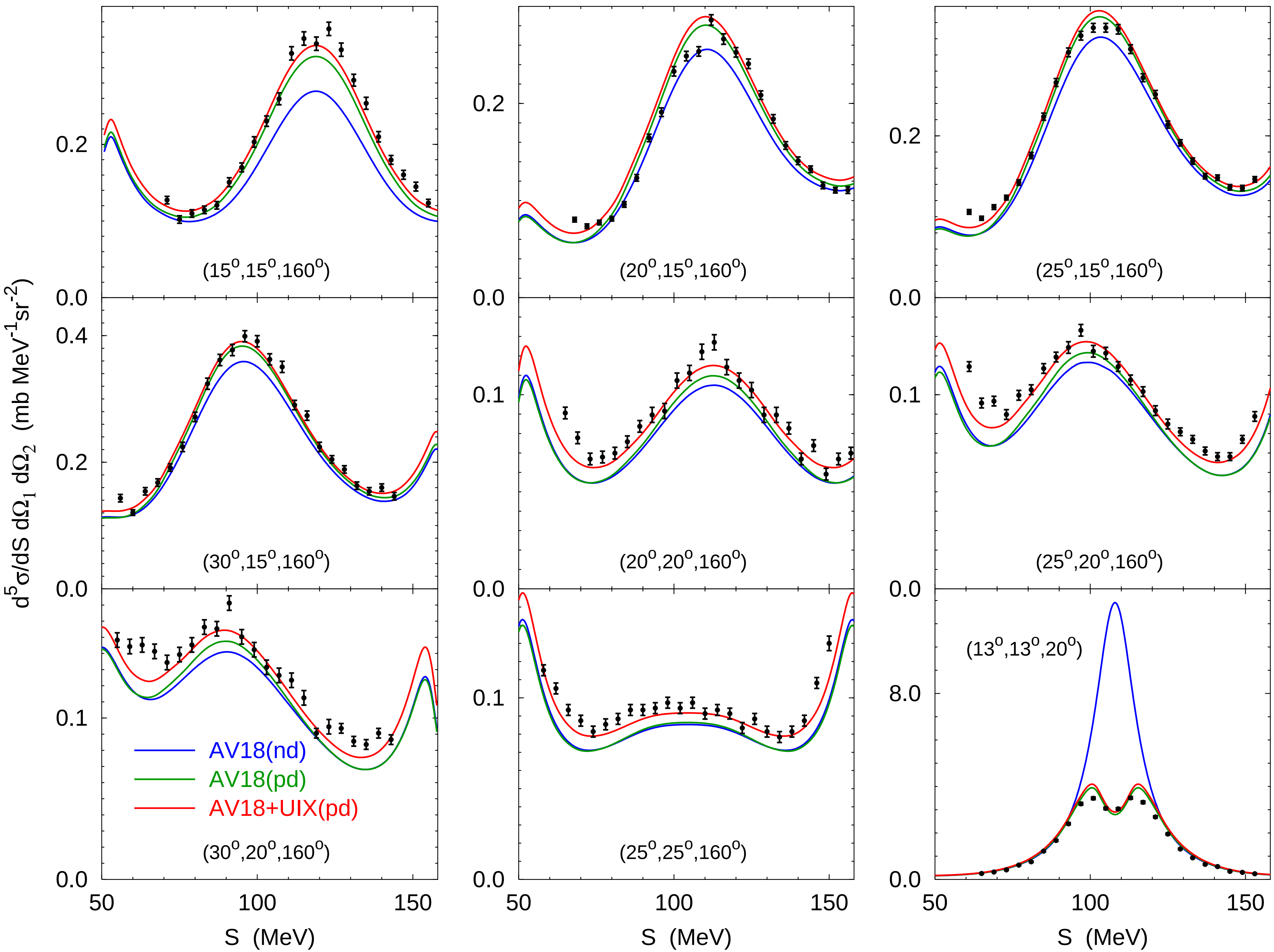
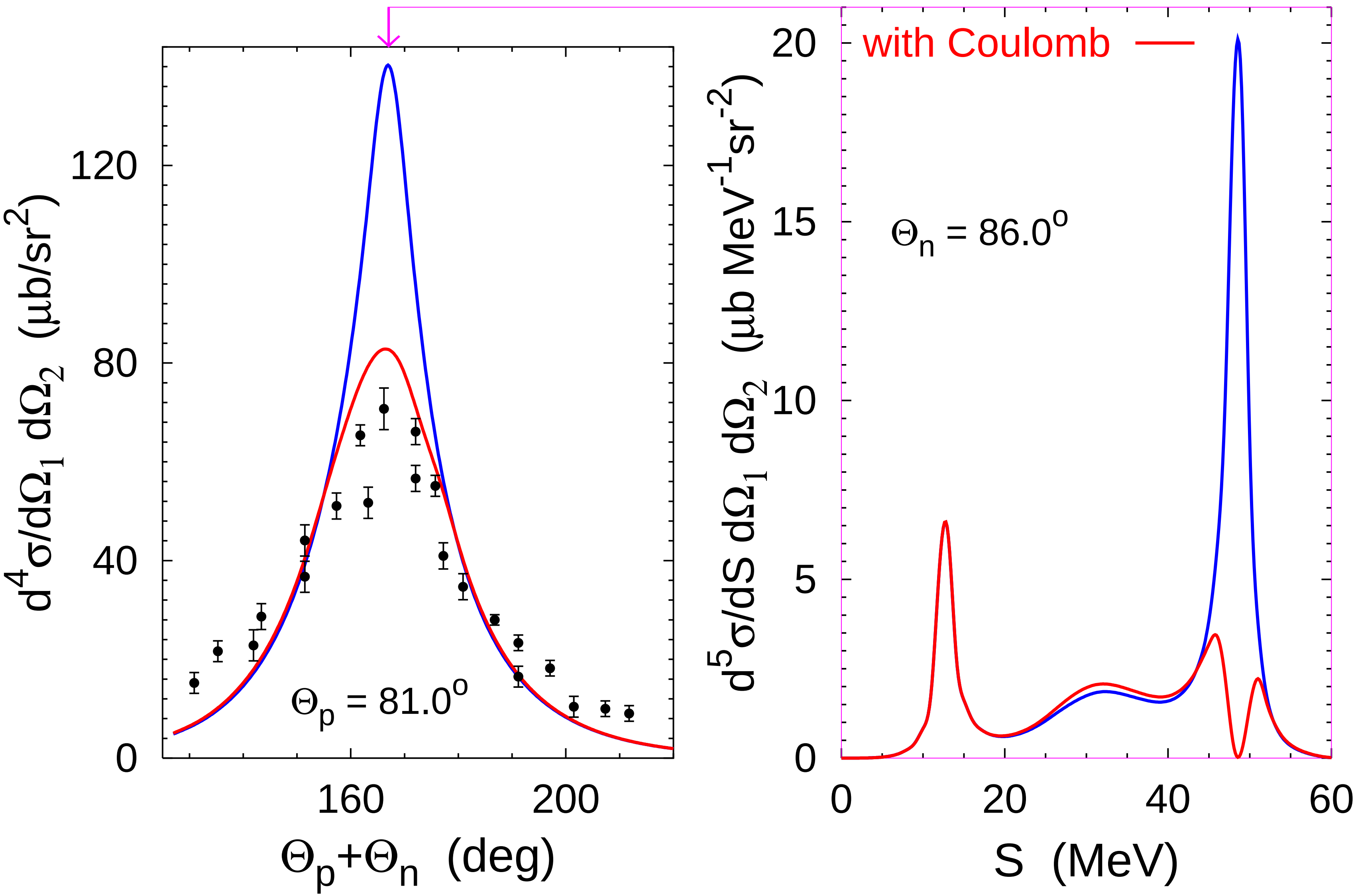




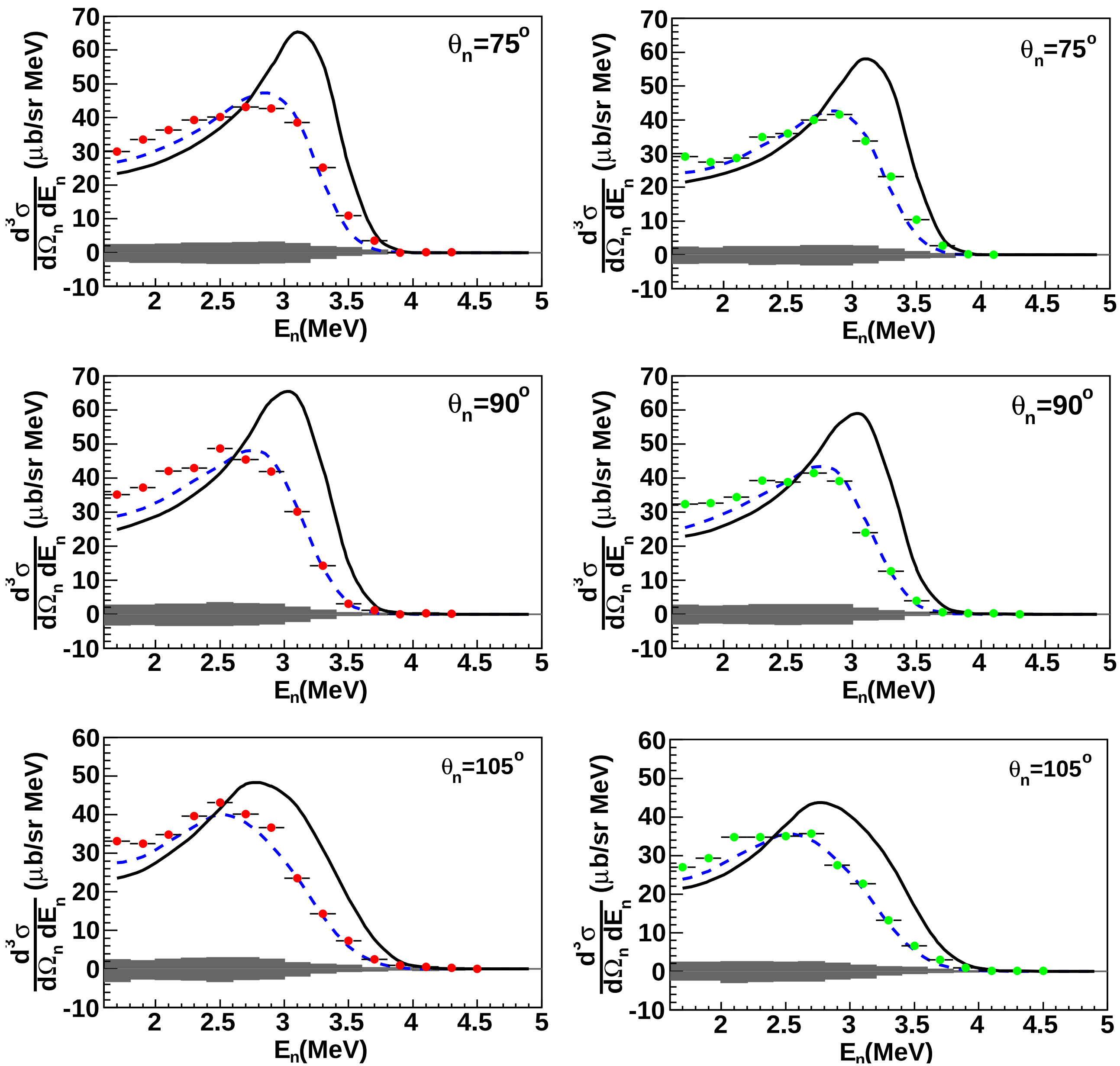
Coulomb vs 3NF: ^{1}H(d,pp)n at $E_d = 130$ MeV



$^3He(\gamma, pn)p$ at $E_\gamma = 55$ MeV



$^3\vec{\mathrm{He}}(\vec{\gamma},n)pp$ at $E_\gamma = 12.8$ MeV



● ● TUNL data

[PRL 110, 202501]

- - - CD Bonn + Δ

(with Coulomb)

(Lisbon)

— AV18 + UIX

(no Coulomb)

(Cracow)

parallel

antiparallel

4N scattering: symmetrized AGS equations

two-cluster 1+3 and 2+2 transition operators

$$\mathcal{U}_{11} = -(G_0 T G_0)^{-1} P_{34} - P_{34} U_1 G_0 T G_0 \, \mathcal{U}_{11} + U_2 G_0 T G_0 \, \mathcal{U}_{21}$$

$$\mathcal{U}_{21} = (G_0 T G_0)^{-1}(1 - P_{34}) + (1 - P_{34}) U_1 G_0 T G_0 \, \mathcal{U}_{11}$$

$$\mathcal{U}_{12} = (G_0 T G_0)^{-1} - P_{34} U_1 G_0 T G_0 \, \mathcal{U}_{12} + U_2 G_0 T G_0 \, \mathcal{U}_{22}$$

$$\mathcal{U}_{22} = (1 - P_{34}) U_1 G_0 T G_0 \, \mathcal{U}_{12}$$

$$U_j = P_j G_0^{-1} + P_j T G_0 U_j$$

$$P_1 = P = P_{12} P_{23} + P_{13} P_{23}$$

$$P_2 = \tilde{P} = P_{13} P_{24}$$

$$T = v + v G_0 T$$

scattering amplitude $\mathcal{T}_{fi} = S_{fi} \langle \mathbf{p}_f \phi_f | \mathcal{U}_{fi} | \mathbf{p}_i \phi_i \rangle$

$$|\phi_j\rangle = G_0 T P_j |\phi_j\rangle$$

Screening and renormalization in 4N scattering

$$v \to v + w_R$$

$$T,\, U_j,\; \mathcal{U}_{fi},\, \mathcal{T}_{fi} \to T^{(R)},\, U_j^{(R)},\; \mathcal{U}_{fi}^{(R)},\, \mathcal{T}_{fi}^{(R)}$$

isolate long-range interaction
and Coulomb distortion between c.m. of two clusters

Screening and renormalization in 4N scattering

$$v \to v + w_R$$

$$T,\ U_j,\ \mathcal{U}_{fi},\ \mathcal{T}_{fi} \to T^{(R)},\ U_j^{(R)},\ \mathcal{U}_{fi}^{(R)},\ \mathcal{T}_{fi}^{(R)}$$

isolate long-range interaction
and Coulomb distortion between c.m. of two clusters

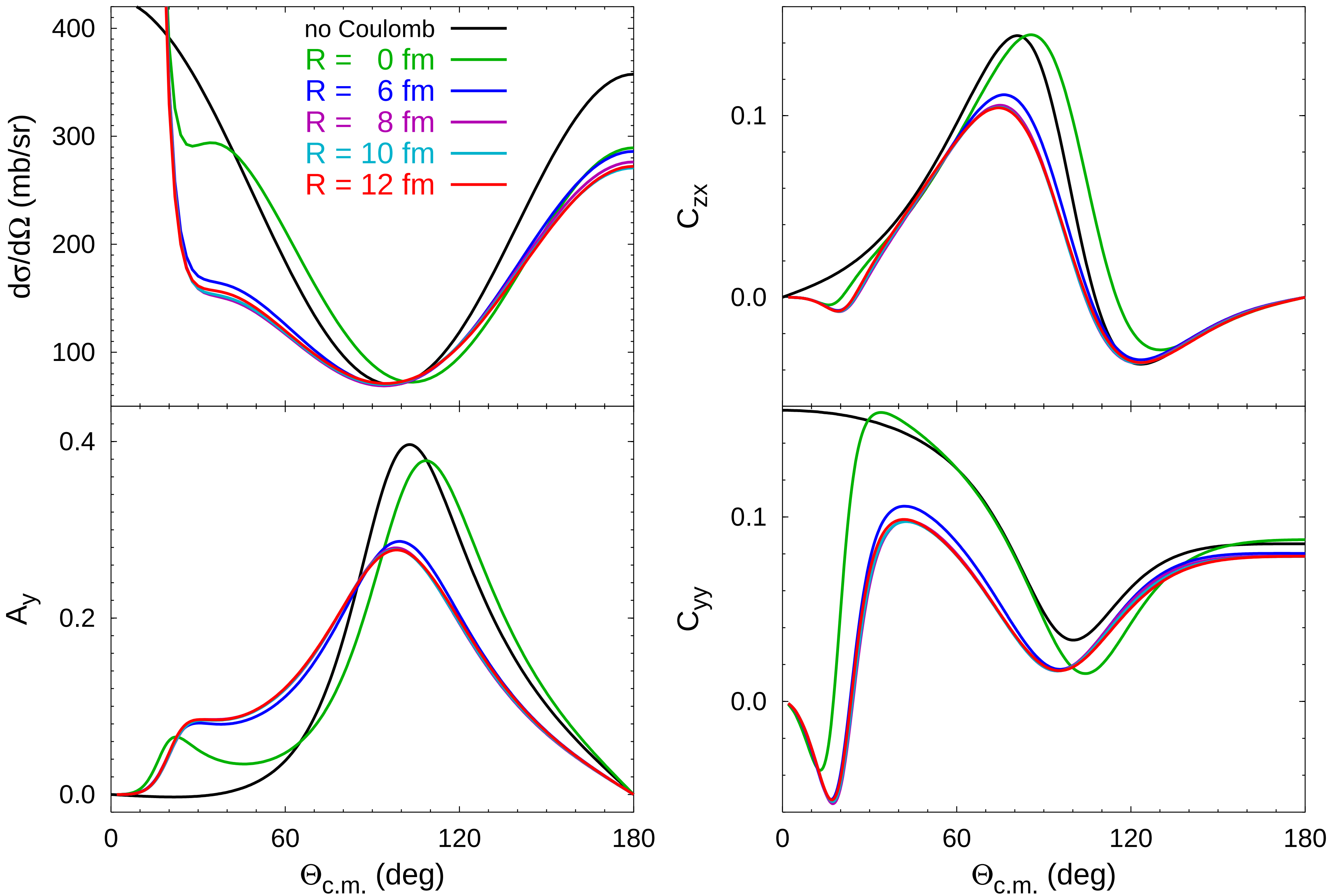
Renormalization:

$$\mathcal{T}_{fi} = \lim_{R\to\infty} Z_{Rf}^{-\frac{1}{2}} \mathcal{T}_{fi}^{(R)} Z_{Ri}^{-\frac{1}{2}}$$

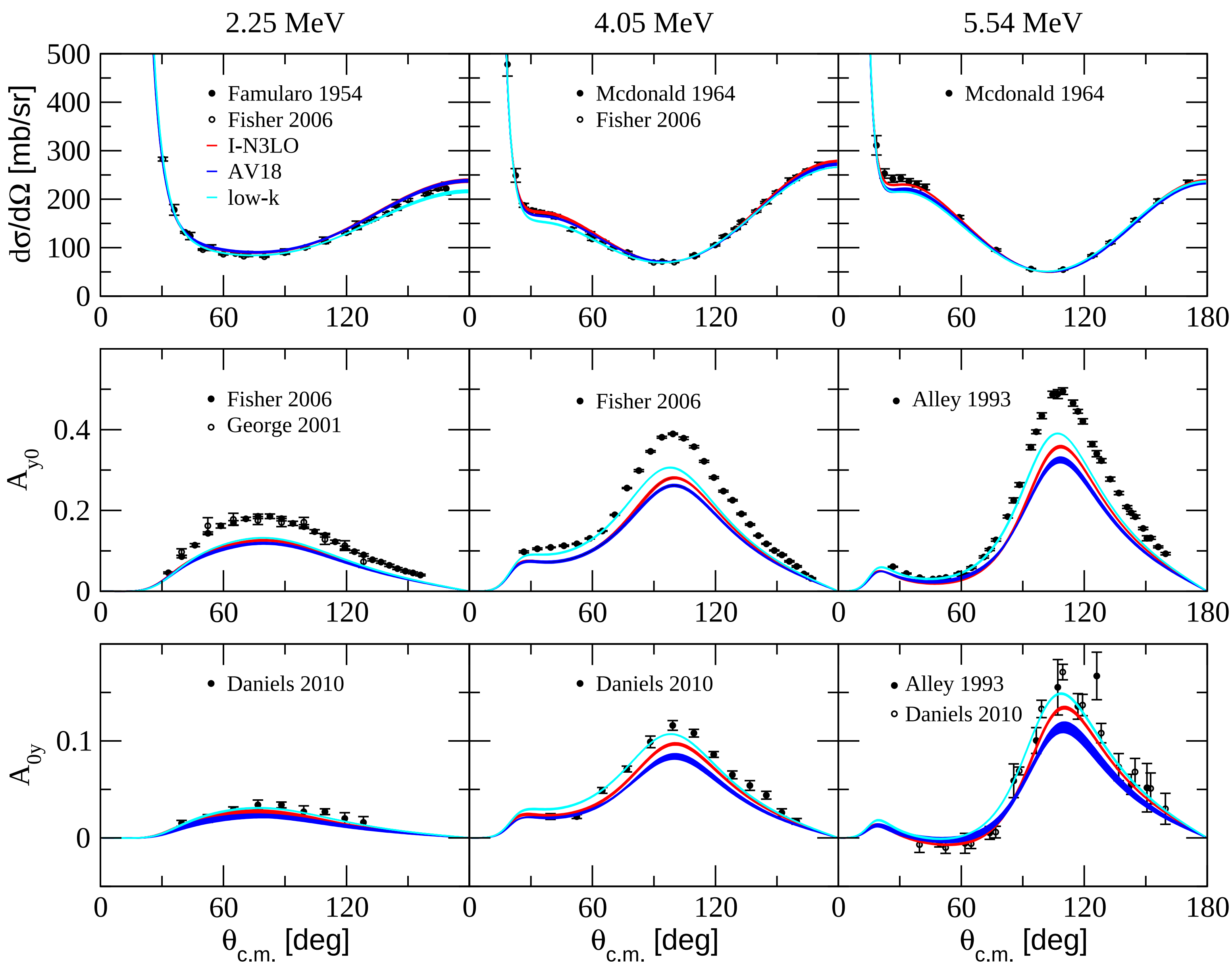
$$= \delta_{fi} T_{Ci}^{\text{c.m.}} + \lim_{R\to\infty} Z_{Rf}^{-\frac{1}{2}} [\mathcal{T}_{fi}^{(R)} - \delta_{fi} T_{Ri}^{\text{c.m.}}] Z_{Ri}^{-\frac{1}{2}}$$

Coulomb-distorted short-range part: fast convergence with R

Convergence with R: p-^{3}He scattering at $E_p = 4$ MeV

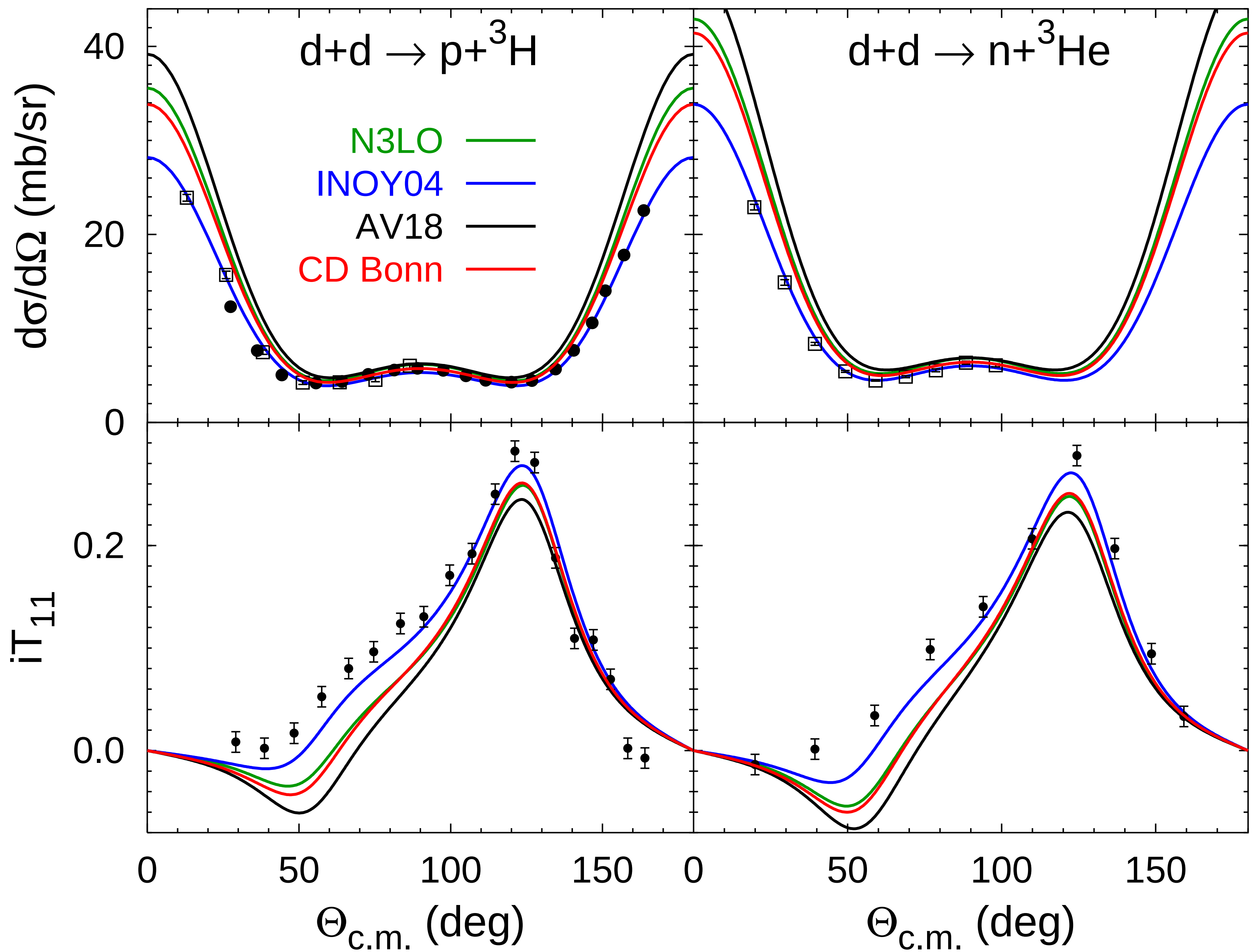


p-^{3}He scattering



AGS/HH/FY (Lisbon/Pisa/Strasbourg, PRC 84, 054010)

$d + d \to N + [3N]$ transfer at $E_d = 3$ MeV



Screening and renormalization (DFS version)

- **standard scattering equations**:
  transition operators, momentum space,
  partial waves, without separable approximation,
  straightforward extension to 4b scattering

- **limitations in practical applicability**:
  no more than 2 charged clusters for breakup,
  not too large screening radius and angular momentum,
  not too low energy and not too large charge

[Deltuva,Fonseca,Sauer:
PRC 71, 054005; PRC 72, 054004; PRC 73, 057001;
PRL 98, 162502; PRC 80, 064002; EPJ WoC 3, 01003]