Assessing Theory Errors from Residual Cutoff Dependence

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Providing reliable theoretical uncertainties,
testing non-perturbative EFTs.

hg: *Nucl. Phys.* **A744** (2004) 192;
hg: NNPSS 2008, Saclay workshop 04.03.2013, Benasque workshop 24.07.2014;
hg: Chiral Dynamics proceedings [arXiv:1511.00490 [nucl-th]]

1. The EFT Promise: Serious Theorists Have Error Bars

Scientific Method: Quantitative results with corridor of theoretical uncertainties for *falsifiable predictions*.

"Double-Blind" Theory Errors: Assess with pretense of no/very limited data.

PHYSICAL REVIEW A **83**, 040001 (2011)

Editorial: Uncertainty Estimates

The purpose of this Editorial is to discuss the importance of including uncertainty estimates in papers involving theoretical calculations of physical quantities.

It is not unusual for manuscripts on theoretical work to be submitted without uncertainty estimates for numerical results. In contrast, papers presenting the results of laboratory measurements would usually not be considered acceptable for publication in *Physical Review A* without a detailed discussion of the uncertainties involved in the measurements. For example, a graphical presentation of data is always accompanied by error bars for the data points. The determination of these error bars is often the most difficult part of the measurement. Without them, it is impossible to tell whether or not bumps and irregularities in the data are real physical effects, or artifacts of the measurement. Even papers reporting the observation of entirely new phenomena need to contain enough information to convince the reader that the effect being reported is real. The standards become much more rigorous for papers claiming high accuracy.

The question is to what extent can the same high standards be applied to papers reporting the results of theoretical calculations. It is all too often the case that the numerical results are presented without uncertainty estimates. Authors sometimes say that it is difficult to arrive at error estimates. Should this be considered an adequate reason for omitting them? In order to answer this question, we need to consider the goals and objectives of the theoretical (or computational) work being done. Theoretical papers can be broadly classified as follows:

Workshop *"Predictive Capabilities of Nuclear Theories"*, Krakow (Poland), 25 Aug 2012

Special Issue *J. Phys. G* (Feb 2015):

"Enhancing the Interaction between Nuclear Experiment and Theory through Information and Statistics"

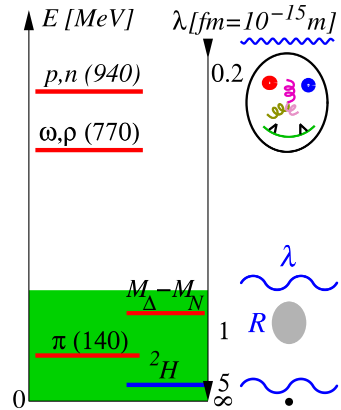
(a) Power-Counting Non-Perturbative EFTs

Correct long-range $+$ symmetries: Chiral SSB, gauge, iso-spin,. . .
Short-range: ignorance into minimal parameter-set at given order.

Systematic ordering in $Q = \dfrac{\text{typ. momentum } p_{\text{typ}}}{\text{breakdown scale } \overline{\Lambda}_{\text{EFT}}} \ll 1$

Controlled approximation: model-independent, error-estimate.

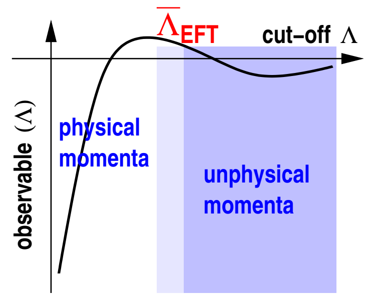
$\Longrightarrow$ **Chiral Effective Field Theory χEFT $\equiv$ low-energy QCD**
$\Longrightarrow$ **Pion-less Effective Field Theory EFT($\not\pi$) $\equiv$ low-energy χEFT**



Shallow real/virtual QCD bound states $\Longrightarrow$ Few-N non-perturbative!

$$T_{\text{LO}} = V_{\text{LO}} + V_{\text{LO}}\, G\, T_{\text{LO}}$$
$$T_{\text{NLO}} = (\mathbb{1} + T^\dagger_{\text{LO}})\, V_{\text{NLO}}\, (\mathbb{1} + T_{\text{LO}}) \quad \text{strict perturbation about LO}$$

$\Longrightarrow$ Analytic results rare; regularisation by **cut-off** $\Lambda \overset{!!}{\neq} \overline{\Lambda}_{\text{EFT}}$.



$\Longrightarrow$ saturated at $\overline{\Lambda}_{\text{EFT}} \lesssim \Lambda$.

(b) NN χEFT Power Counting Comparison

Derived with explicit & implicit assumptions; contentious issue.
Proposed order Q^n at which counter-term enters *differs*. $\Longrightarrow$ Predict *different* accuracy, # of parameters.

<table>
<tr><th>wave</th><th>order</th><th>Yang/Long
PRC86(2012) 024001 etc.</th><th>Pavon Valderrama
PRC74 (2006) 054001 etc.</th><th>Birse
PRC74 (2006) 014003</th></tr>
<tr><td rowspan="3">$^1S_0$</td><td>LO</td><td colspan="3">-1</td></tr>
<tr><td>NLO</td><td colspan="3">0</td></tr>
<tr><td>N^{2}LO</td><td>1</td><td colspan="2">2</td></tr>
<tr><td rowspan="2">$^3S_1$</td><td>LO</td><td colspan="3">-1</td></tr>
<tr><td>NLO</td><td>1</td><td>2</td><td>$\frac{1}{2}$</td></tr>
<tr><td rowspan="2">$^3SD_1$</td><td>LO</td><td>1</td><td>$-\frac{1}{2}$</td><td>-1</td></tr>
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<tr><td>NLO</td><td></td><td>2</td><td>$\frac{1}{2}$</td></tr>
<tr><td>$^3P_0$ (attr. triplet)</td><td>LO</td><td colspan="2">-1</td><td>$-\frac{1}{2}$</td></tr>
<tr><td>TPE</td><td>LO</td><td>1</td><td colspan="2">2</td></tr>
<tr><td colspan="2"># of param. at Q^{-1}</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td colspan="2"># of param. at Q^{0}</td><td>4</td><td>6</td><td>6</td></tr>
<tr><td colspan="2"># of param. at Q^{1}</td><td>8</td><td>6</td><td>9</td></tr>
</table>

Weinberg: LO: 2; NLO: $+0$; N^{2}LO: $+7=9$ – different channels; consistency questioned Beane/...2002; Nogga/...2005

With same χ^2, proposal with least parameters *wins*: minimum information bias.

(c) (Some) Ways to Estimate Theoretical Uncertainties at fixed k

Choose most conservative/worst-case error for final estimate! Clearly state your choice!

Expansion parameter $Q = \dfrac{\text{typ. low scale } p_{\text{typ}}}{\text{typ. high scale } \overline{\Lambda}_{\text{EFT}}} \Longrightarrow \mathcal{O} = \sum\limits_{i=0}^{n} c_i(\Lambda)\, Q^i$ complete up to order Q^n ($n=0$ is LO).

- A priori: Q^{n+1} of LO.
- Convergence pattern of series: smaller corrections LO $\to$ NLO $\to$ N^{2}LO $\to \dots$

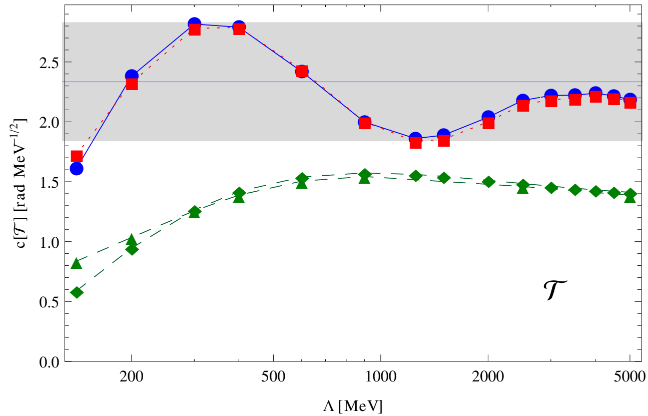
  $\Longrightarrow$ **Bayesian estimate:** error $Q^{n+1} \times \max\limits_i |c_i|$ captures corridor with $\dfrac{n+1}{n+2} \times 100\%$ degree of belief.

  Furnstahl/Klco/Phillips/Wesolowski (BUQEYE) 2015
- Less dependence on particular low-E data taken for LECs. (e.g. Z-param. vs. ERE; fit H_0 to a_3 vs. B_3,...)
- Include selected higher-order RG- & gauge-invariant effects: *does not increase accuracy.*
- Corridor mapped by cutoff Λ *in wide range*.

  Should decrease order-by-order.

  Example: PV coefficient in nd at $k=0$.

  hg/Schindler/Springer 2012



3. Error Plots Test Power Counting & Renormalisation

(a) Using Cut-Offs to Your Advantage

Observable $\mathcal{O}(k)$ at momentum k, order Q^n in EFT, cut-off Λ:

$$\mathcal{O}_n(k;\mu) = \underbrace{\sum_i^n \left(\frac{k, p_{\text{typ.}}}{\overline{\Lambda}_{\text{EFT}}}\right)^i \mathcal{O}_i}_{\text{renormalised, } \Lambda\text{-indep.}} \underbrace{+\, \mathcal{C}(\Lambda; k, p_{\text{typ}}, \overline{\Lambda}_{\text{EFT}}) \left(\frac{k, p_{\text{typ.}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1}}_{\text{residual } \Lambda\text{-dependence}}$$

parametrically small

$\mathcal{C}$ "of natural size"

$\Longrightarrow$ **Difference between *any two* cut-offs:** $\dfrac{\mathcal{O}_n(k;\Lambda_1) - \mathcal{O}_n(k;\Lambda_2)}{\mathcal{O}_n(k;\Lambda_1)} = \left(\dfrac{k, p_{\text{typ.}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1} \times \dfrac{\mathcal{C}(\Lambda_1) - \mathcal{C}(\Lambda_2)}{\mathcal{C}(\Lambda_1)}$

Isolate breakdown scale $\overline{\Lambda}_{\text{EFT}}$, order n by double-ln plot of "derivative of observable w. r. t. cut-off".

Test consistency: Does numerics match predicted convergence pattern?

Renormalisation Group Evolution: $\Lambda_1 \to \Lambda_2 \Longrightarrow \dfrac{\Lambda}{\mathcal{O}} \dfrac{d\mathcal{O}}{d\Lambda} = \left(\dfrac{k, p_{\text{typ.}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1} \dfrac{d\ln\mathcal{C}(\Lambda)}{d\ln\Lambda} \to 0$ if exact RGE.

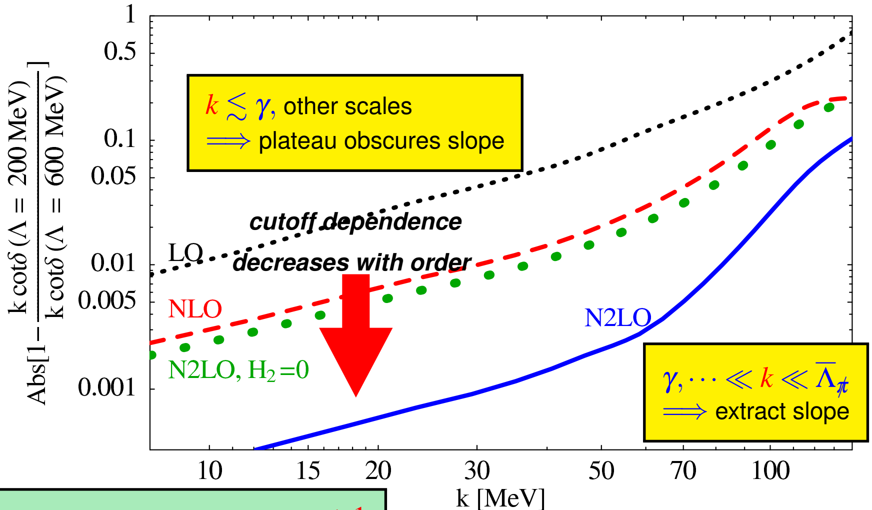
Residual Λ-dependence decreases parametrically order-by-order.

Complication: Several intrinsic low-energy scales in few-N EFT:

scattering momentum k, m_π, inverse NN scatt. lengths $\gamma(^3\mathrm{S}_1) \approx 45$ MeV, $\gamma(^1\mathrm{S}_0) \approx 8$ MeV,...

(b) Example: nd Doublet-S Wave in EFT($\not\pi$)

Does momentum-dependent 3NI H_2 enter at N^{2}LO hg/…2002-4 **– or higher** Platter/Phillips 2006**?**



$$\left| 1 - \frac{k \cot\delta(\mu = 200\ \text{MeV})}{k \cot\delta(\mu = \infty)} \right| \sim \underbrace{\left(\frac{k, p_{\text{typ.}}}{\overline{\Lambda}_{\not\pi}} \right)^{n+1}}_{Q^{n+1}}$$

	LO	NLO	N^{2}LO	N^{2}LO without H_2
$n+1$ fitted	~ 1.9	2.9	4.8	3.1
$n+1$ predicted	2	3	4	**not renormalised**

$\Longrightarrow$ Fit to $k \in [70; 100 \dots 130]$ MeV $\gg \gamma, \dots$ **:** H_2 **is N^{2}LO**; re-confirmed by Ji/Phillips 2013

Slope Confirms Power Counting; Estimates $\overline{\Lambda}_{\not\pi} \approx 140$ MeV; Determines Mom.-Dep. Uncertainties.

(c) Comments: It's Not The Golden Bullet, but Worth A Try

$$\boxed{\frac{\mathcal{O}_n(k;\Lambda_1)-\mathcal{O}_n(k;\Lambda_2)}{\mathcal{O}_n(k;\Lambda_1)} = \left(\frac{k, p_{\text{typ.}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1} \times \frac{\mathcal{C}(\Lambda_1)-\mathcal{C}(\Lambda_2)}{\mathcal{C}(\Lambda_1)}}$$

- $n, \overline{\Lambda}_{\text{EFT}}$ regulator independent. – But not $\mathcal{C}$: flexible regulator...
- Non-integer powers, non-analyticities: $n+1 \to n+\mathrm{Re}[\alpha]$ with $n \notin \mathbb{Z}$.
- Fit quality also tests assumed functional dependencies.
- Estimate k-dependence of expansion parameter $Q(k) = \left(\frac{k, p_{\text{typ.}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1}$ $\Longrightarrow$ Res. theoret. error.
- Needs "Window of Opportunity" $p_{\text{typ}} \ll k \ll \overline{\Lambda}_{\text{EFT}}$.
- Any two cutoffs Λ_1, Λ_2 – Numerical leverage?!

(c) Comments: It's Not The Golden Bullet, but Worth A Try

$$\frac{\mathcal{O}_n(k;\Lambda_1)-\mathcal{O}_n(k;\Lambda_2)}{\mathcal{O}_n(k;\Lambda_1)} = \left(\frac{k, p_{\text{typ.}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1} \times \frac{\mathcal{C}(\Lambda_1)-\mathcal{C}(\Lambda_2)}{\mathcal{C}(\Lambda_1)}$$

What observable to choose?: Avoid Accidental Zeroes $\mathcal{O}(\Lambda_1)-\mathcal{O}(\Lambda_2)=0$ & Infinities $\mathcal{O}(\Lambda)=0$.

Best if unconstrained: Isolate dynamics!

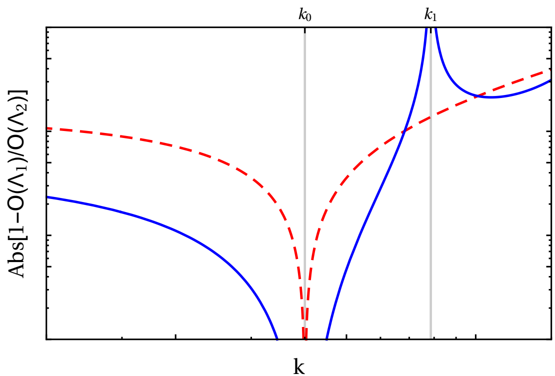
e.g. $k^{2l+1}\cot\delta_l(k)$ for lth scattering wave.

Not $\delta_l(k)$: $\delta_l(k\to 0)\propto k^{2l+1}$: constrained.

Best if same sign for all $k\lesssim\overline{\Lambda}_{\text{EFT}}$ $\Longrightarrow$ Peruse Λ_1, Λ_2.

If LECs need fitting, do for small $k\to 0,\ k\sim p_{\text{typ}}$.

Slope may still emerge for $k\nearrow\overline{\Lambda}_{\text{EFT}}$; larger LEC fit error.



Some Limitations:

– Cannot see LECs which do *not absorb cutoff-dependence*.

– Can be numerically indecisive (e.g. small coefficients).

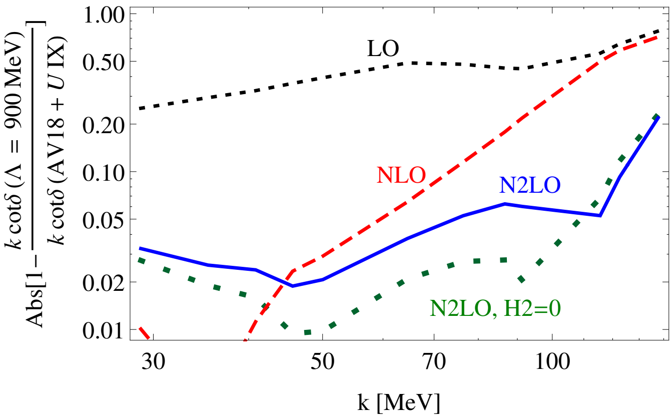
(c) Comments: It's Not The Golden Bullet, but Worth A Try

$$\frac{\mathcal{O}_n(k;\Lambda_1)-\mathcal{O}_n(k;\Lambda_2)}{\mathcal{O}_n(k;\Lambda_1)} = \left(\frac{k,p_{\text{typ.}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1} \times \frac{\mathcal{C}(\Lambda_1)-\mathcal{C}(\Lambda_2)}{\mathcal{C}(\Lambda_1)}$$

These Are *Not* "Lepage-Plots" $\dfrac{\mathcal{O}_n(k;\Lambda)-\mathcal{O}(\textbf{data})}{\mathcal{O}(\textbf{data})}$.

Lepage: nucl-th/9706029; Steele/Furnstahl: nucl-th/9802069; ...

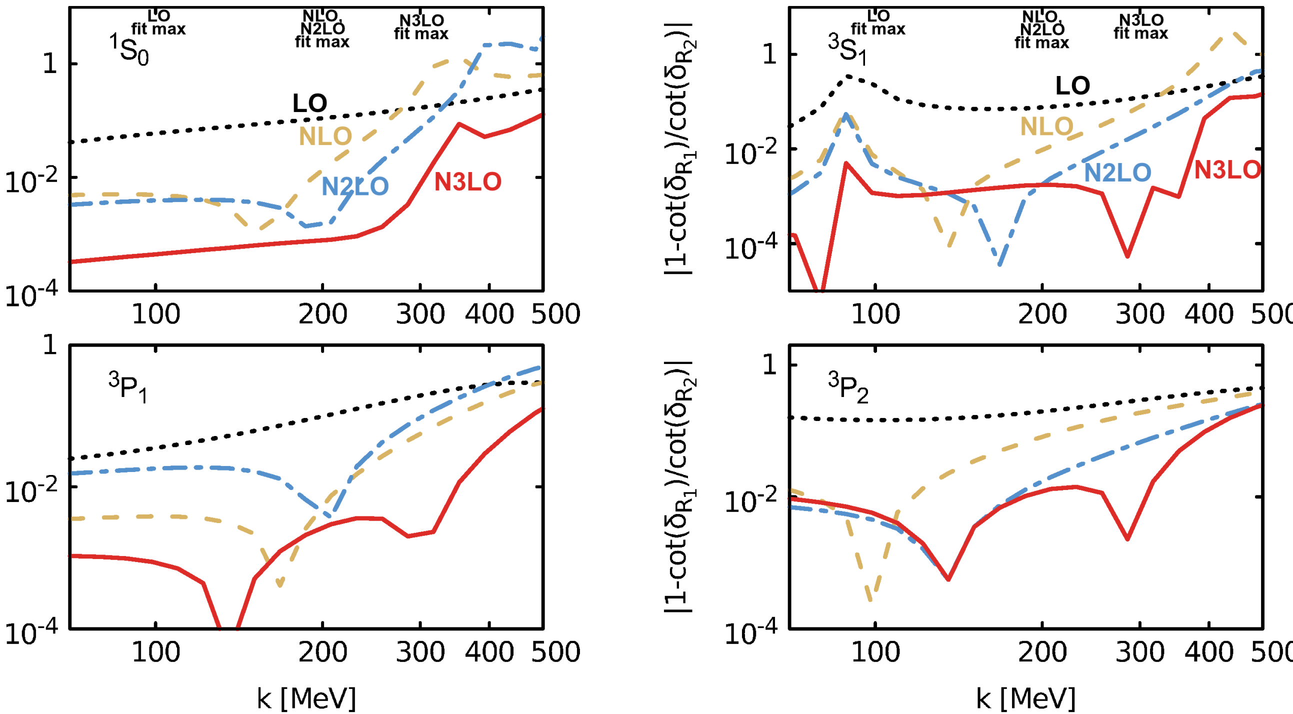
"Lepage" needs data/pseudo-data. $\Longrightarrow$ No consistency test; not double-blind; compromise predictive power.



EFT may converge by itself, but not to data. – Example χEFT without dynamical $\Delta(1232)$ at $k \sim 300$ MeV.

(d) What About NN?: Unsolicited Comments

Plot stolen from **Epelbaum/Krebs/Meißner EPJA51 (2015) 5, 53**.



Inconclusive: Breakdown scale $400 - 500$ MeV $\Longleftrightarrow \Delta(1232)$? NLO, N^{2}LO parallel? Slopes?

Coupled channels; attractive tensor? Fit- & slope-regions not clearly separated.

4. Concluding Questions

– EFTs make **quantitative, falsifiable predictions** how its uncertainties evolve with momentum.

– **Non-perturbative EFTs**: Power-counting not established analytically. Test by residual cut-off dependence:

"Momentum-dependent Renormalisation Group flow of observable with cut-off":

$$\frac{\mathcal{O}_n(k;\Lambda_1)-\mathcal{O}_n(k;\Lambda_2)}{\mathcal{O}_n(k;\Lambda_1)} \propto \left(\frac{k,p_{\text{typ.}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1}$$ **for any two cut-offs** $\Lambda_1,\Lambda_2 \gtrsim \overline{\Lambda}_{\text{EFT}}$.

- For ***order*** $\mathcal{O}(Q^n)$ ***to which result is complete***: slope at $k \gg$low scales;
- For ***breakdown scale*** $\overline{\Lambda}_{\textit{EFT}}$: k at which different orders show same-size variations;
- For lower bound on ***expansion parameter*** Q: vary Λ_1,Λ_2 over wide range.

– Using widely separate cutoffs Λ_1,Λ_2 increases leverage; decreases numerical noise.

– Straightforward extension to include non-analytic running $\sim \ln[k,p_{\text{typ.}}],\dots$.

– **Not all observables equally suited (avoid constraints!).**

– **Self-consistency test of EFT: No resort to data.** — "Not A Lepage Plot".

– **Results may be inconclusive.** – One of hopefully many arrows in the quiver.

What will this look like for few-nucleon observables in χEFT?