

Effective theory of ${}^3\text{H}$ and ${}^3\text{He}$

Sebastian König

in collaboration with H.W. Griedhammer, H.-W. Hammer, and U. van Kolck

EMMI RRTF Workshop:

“The systematic treatment of the Coulomb interaction in few-body systems”

TU Darmstadt

January 11, 2016

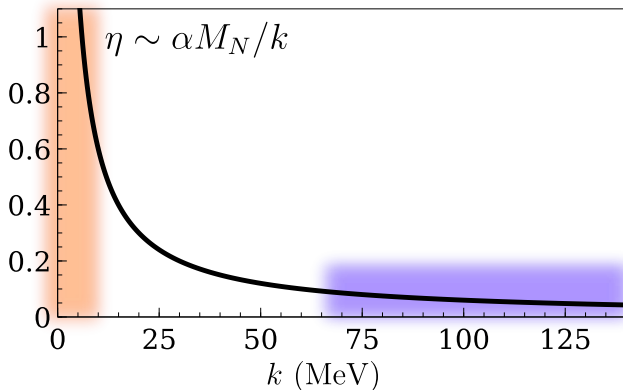


THE OHIO STATE UNIVERSITY

NUCLEI
Nuclear Computational Low-Energy Initiative

Motivation

- **scaling of Coulomb contributions**



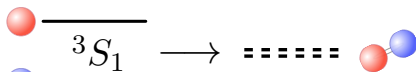
- **three-body forces**

$$H(\Lambda) = H_{0,0}(\Lambda) + H_{0,1}(\Lambda) + H_{0,1}^{(\alpha)}(\Lambda)$$

Vanasse, Egolf, Kerin, SK, Springer, PRC **89** 064003 (2014)

Two-body sector

Introduce dibaryon fields. . .

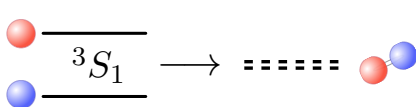


Bedaque *et al.* (1998)



Two-body sector

Introduce dibaryon fields...



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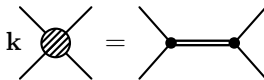


...and resum bubble-insertions to all orders!

Full dibaryon propagators

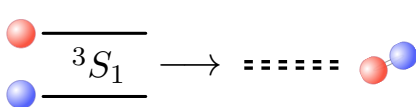
$$\begin{aligned}
 ^3S_1: \quad \Delta_d &= \text{double line} = \text{dashed line} + \text{dashed line} \circ \text{dashed line} + \text{dashed line} \circ \text{dashed line} \circ \text{dashed line} + \dots \\
 ^1S_0: \quad \Delta_t &= \text{solid line} = \text{dotted line} + \text{dotted line} \circ \text{dotted line} + \text{dotted line} \circ \text{dotted line} \circ \text{dotted line} + \dots
 \end{aligned}$$

$$\begin{aligned}
 \Delta_d(k) &\sim \frac{i}{\underbrace{k \cot \delta_d}_{-\gamma_d + \frac{\rho_d}{2}(k^2 + \gamma_d^2)} - ik} \\
 &= -\gamma_d + \frac{\rho_d}{2}(k^2 + \gamma_d^2) + \dots
 \end{aligned}$$



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 {}^3S_1: \quad \Delta_d &= \text{double line} = \text{double dashed line} + \text{double dashed line} \text{ with bubble} + \text{double dashed line} \text{ with two bubbles} + \dots \\
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 &= -\gamma_d + \frac{\rho_d}{2}(k^2 + \gamma_d^2) + \dots
 \end{aligned}$$

$$\gamma_d \rho_d \sim Q/\Lambda_{\pi} = \mathcal{O}(1/3)$$

Propagator renormalization

Full dibaryon propagators

$$\begin{aligned}
 {}^3S_1: \quad \Delta_d &= \text{double line} = \text{dashed line} + \text{dashed line} \circ \text{dashed line} + \text{dashed line} \circ \text{dashed line} \circ \text{dashed line} + \dots \\
 {}^1S_0: \quad \Delta_t &= \text{thick solid line} = \text{dotted line} + \text{dotted line} \circ \text{dotted line} + \text{dotted line} \circ \text{dotted line} \circ \text{dotted line} + \dots
 \end{aligned}$$

$$i\Delta_d^{(0)}(p_0, \mathbf{p}) = \frac{-i}{\sigma_d^{(0)} + y_d^2 I_0(p_0, \mathbf{p})}$$

$$\begin{aligned}
 I_0(p_0, \mathbf{p}) &= M_N \int^{\Lambda} d^3q \frac{1}{M_N p_0 - \mathbf{p}^2/4 - \mathbf{q}^2 + i\varepsilon} \\
 &= -\frac{M_N}{4\pi} \left(\frac{2\Lambda}{\pi} + \underbrace{\sqrt{\frac{\mathbf{p}^2}{4} - M_N p_0 - i\varepsilon}}_{\rightarrow -ik} \right) + \mathcal{O}(1/\Lambda)
 \end{aligned}$$

- absorb linear divergence: $\sigma_d^{(0)} = \frac{2\Lambda}{\pi} - \gamma_d$
- $1/\Lambda$ effects are neglected (equivalent: PDS, $2\Lambda/\pi \rightarrow \mu_R$)
- without dibaryons: resummation of  $\sim C_0 = -4\pi/(M_N \sigma)$

Range corrections

Dibaryon kinetic-energy terms

$$\text{Diagram: two horizontal lines connected by a cross} \sim i\Delta_d^{\text{LO}}(p) \times (-ic_d) \left(p_0 - \frac{\mathbf{p}^2}{4M_N} \right) \times i\Delta_d^{\text{LO}}(p)$$

↪ effective-range corrections

① resummed. . .

$$\Delta_d(p) \sim \frac{1}{-\gamma_d + \sqrt{\frac{\mathbf{p}^2}{4} - M_N p_0} - i\epsilon - \frac{\rho_d}{2} \left(\frac{\mathbf{p}^2}{4} - M_N p_0 - \gamma_d^2 \right)}$$

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② ... and re-expanded

$$\begin{aligned} \Delta_d(E - q^2/(2M_N); q) &\equiv D_d^{(0)}(E; q) + D_d^{(1)}(E; q) + \dots \\ &= \frac{-1}{-\gamma_d + \sqrt{3q^2/4 - M_N E} - i\epsilon} \times \left[1 + \frac{\rho_d}{2} \frac{(3q^2/4 - M_N E - \gamma_d^2)}{-\gamma_d + \sqrt{3q^2/4 - M_N E} - i\epsilon} + \dots \right] \end{aligned}$$

Three-body sector

Nucleon

- spin $1/2$
- isospin $1/2$



Deuteron

- spin 1
- isospin 0



↪ **two S-wave channels:**

$$1 \otimes \frac{1}{2} = \frac{3}{2} \left(\sim \begin{array}{c} \uparrow \uparrow \uparrow \\ \text{red blue red} \end{array} \right) \oplus \overbrace{\frac{1}{2} \left(\sim \begin{array}{c} \uparrow \uparrow \downarrow \\ \text{red blue red} \end{array} + \dots \right)}^{\text{spin doublet} \rightarrow {}^3\text{H}, {}^3\text{He}}$$

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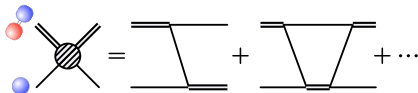
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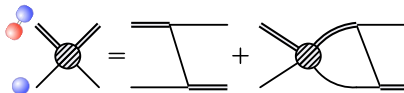
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↪ **solve integral equations to get phase shifts and binding energies**

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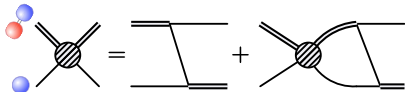
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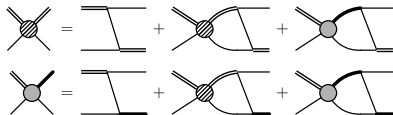
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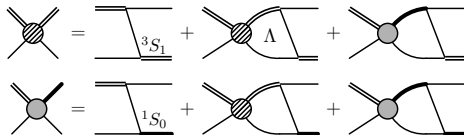


—doublet channel—

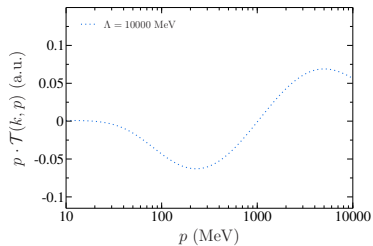
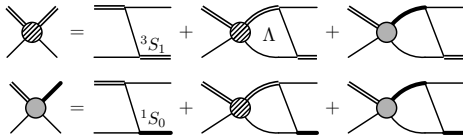


↪ **solve integral equations to get phase shifts and binding energies**

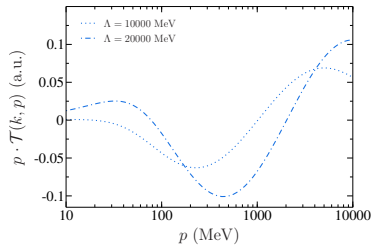
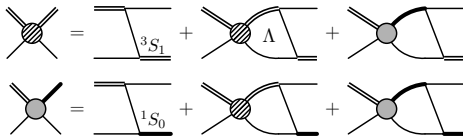
The triton



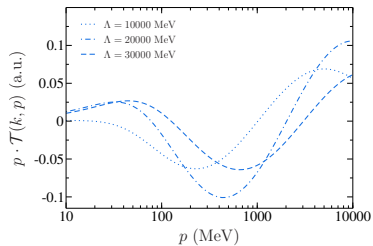
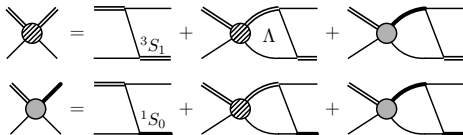
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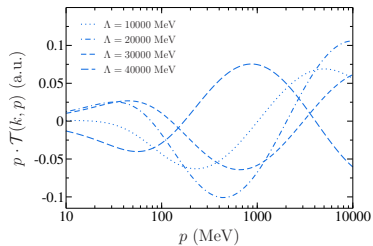
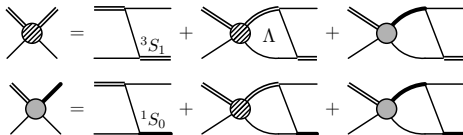
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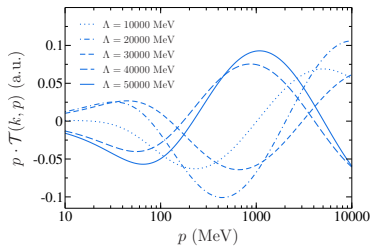
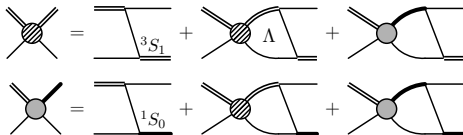
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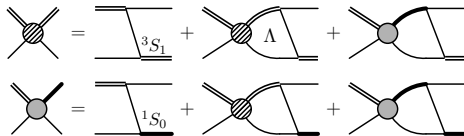
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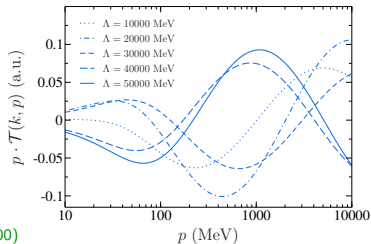


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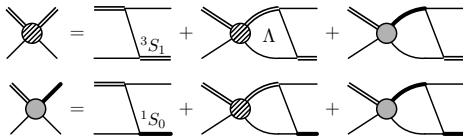


amplitude has no well-defined limit!

Bedaque *et al.*, NPA **676** 357 (2000)

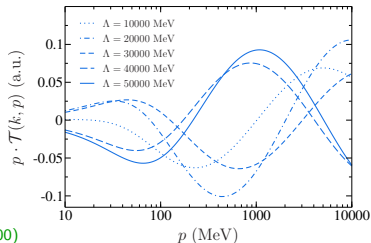


The triton



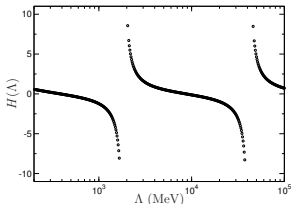
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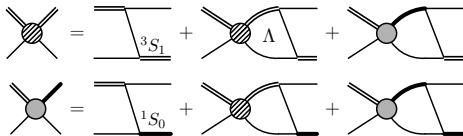
Three-body force promotion

already at LO: ,



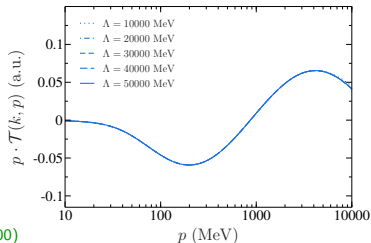
- independent of spin and isospin
→ $SU(4)$ -symmetry
- RG limit cycle $\leftrightarrow$ Efimov effect
- **makes amplitude cutoff-independent**

The triton



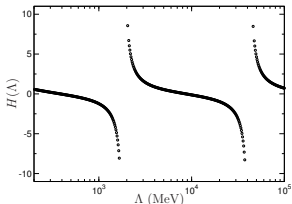
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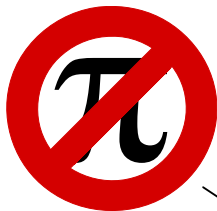


Three-body force promotion

already at LO: $\rightarrow$, $\sim$



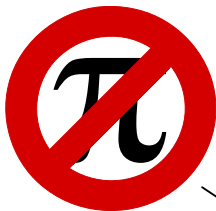
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**large
scattering
length**

Braaten, Hammer
Phys. Rept. 428 259 (2006)

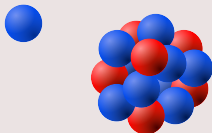
Universality



halo EFT

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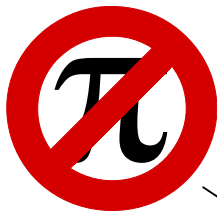
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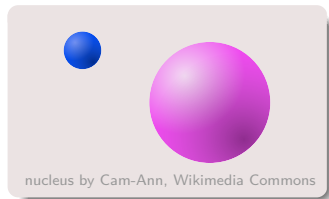
nucleus by Cam-Ann, Wikimedia Commons

van Kolck, NPA **645**, 273 (1999)
Bertulani *et al.*, NPA **712**, 37 (2002)
...
Zhang *et al.*, 1507.07239 [nucl-th]

Universality



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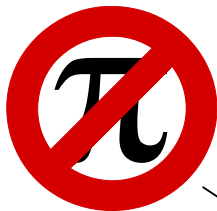
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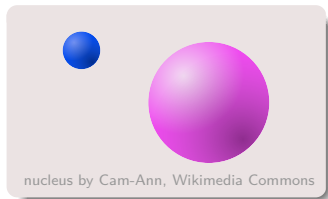
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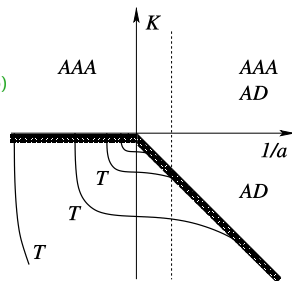
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large
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cold atoms

- tunable scattering length!
- Efimov effect!



Coulomb contributions

Coulomb photons:  $\sim (ie) \frac{i}{q^2} (ie) \longrightarrow (ie) \frac{i}{q^2 + \lambda^2} (ie)$

$\mathcal{O}(\alpha)$ diagrams



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$\mathcal{O}(\alpha)$ diagrams



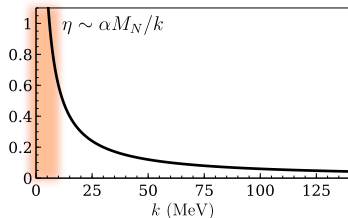
only generated by dibaryon kinetic term!

$\hookrightarrow$ **higher-order correction** $\sim \rho_d$

Coulomb subtraction

Coulomb photons: $\text{---} \text{---} \text{---}$ $\sim (\text{ie}) \frac{i}{\mathbf{q}^2} (\text{ie}) \longrightarrow (\text{ie}) \frac{i}{\mathbf{q}^2 + \lambda^2} (\text{ie})$

- long (infinite) range $\rightarrow$ very strong at small momentum transfer



Coulomb nonperturbative for $\eta \sim 1$
important for p - d scattering length!

$\hookrightarrow$ SK, Hammer, PRC **90** 034005 (2014)

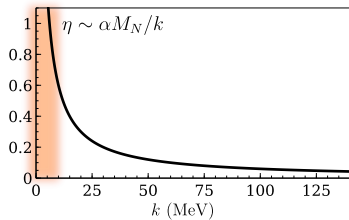
$$C_\eta^2 k \cot \delta_{\text{diff}}(k) + \alpha \mu h(\eta) = -\frac{1}{a_{p-d}} + \dots$$

(modified ERE)

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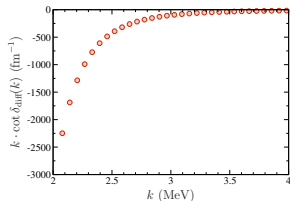


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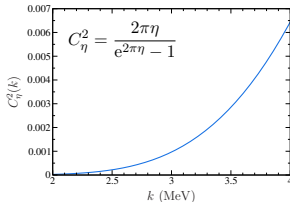
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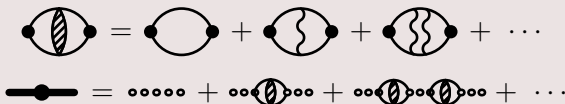
×



= finite value

Coulomb effects in the proton–proton channel

Coulomb-dressed propagator



$$\Delta_{t,pp}(p) \sim \frac{1}{-1/a_{p-p} - \alpha M_N H(\eta)} \quad , \quad \eta = \alpha M_N / (2i\sqrt{\mathbf{p}^2/4 - M_N p_0} - i\varepsilon)$$

↪ **Coulomb-modified effective range expansion**

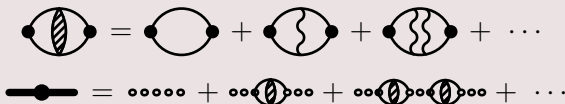
Kong, Ravndal (1999)

Bethe (1949)

cf. Ando, Birse (2010)

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Bethe (1949)

cf. Ando, Birse (2010)

$$\Delta_{t,pp}(p_0, \mathbf{p}) = \frac{-i}{\underbrace{\sigma_{t,pp} - \frac{2\Lambda}{\pi} + \alpha M_N \left(\log \frac{2\Lambda}{\alpha M_N} - C_E \right)}_{=1/a_C} - \alpha M_N H(\eta)}$$

note: two divergences absorbed into a single parameter!

He-3 binding energy

bound-state $\leftrightarrow$ pole!

$$\text{pole diagram} \sim \frac{\text{residue diagram}}{E + E_B} + \text{regular terms}$$

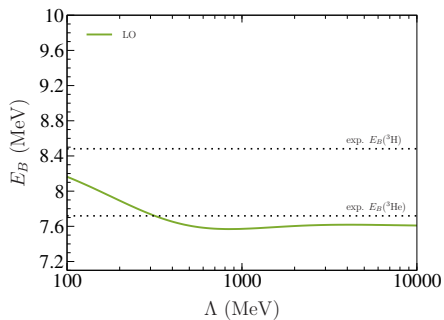
$\hookrightarrow$ calculate ${}^3\text{He}$ binding energy!

$$\begin{aligned} \text{pole diagram} &= \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{pole diagram} \times (\text{diagram 1} + \text{diagram 2} + \text{diagram 3}) \\ &\quad + \text{pole diagram} \times (\text{diagram 4} + \text{diagram 5}) + \text{pole diagram} \times (\text{diagram 6} + \text{diagram 7}) \end{aligned}$$

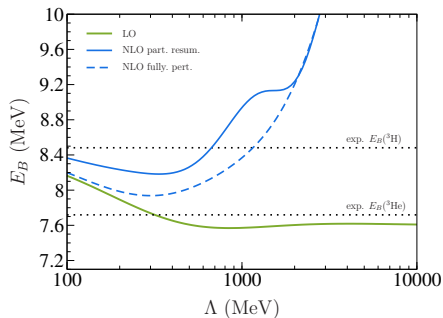
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He-3 beyond leading order



He-3 beyond leading order



NLO corrections

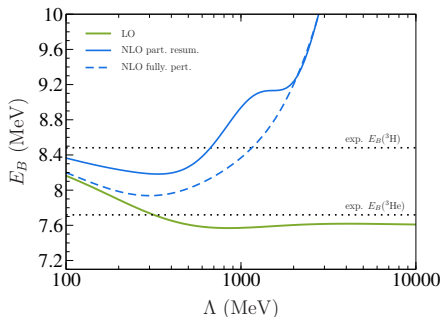
- effective ranges

$$\frac{\times}{\sim \rho_d}, \quad \frac{\times}{\sim r_{0t}}$$

- dibaryon-photon coupling

$$\frac{\text{---}}{\text{---}} \sim \alpha \rho_d$$

He-3 beyond leading order



NLO corrections

- effective ranges

$$\frac{\times}{\sim \rho_d}, \quad \frac{\times}{\sim r_{0t}}$$

- dibaryon-photon coupling

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \sim \alpha \rho_d$$

- NLO result is not cutoff stable $\hookrightarrow$ incomplete renormalization!
- refitting the three-body force to $E_B(^3\text{He})$ gives stable p - d phase shifts!

SK, Ph.D. thesis (2013)

SK et al., JPG 42 045101 (2015)

- form of new p - d specific counterterm can be derived analytically!

$$\rightsquigarrow \text{three body-force } H(\Lambda) = H_{0,0}(\Lambda) + H_{0,1}(\Lambda) + H_{0,1}^{(\alpha)}(\Lambda)$$

Vanasse, Egolf, Kerin, SK, Springer, PRC 89 064003 (2014)

Nonperturbative vs. perturbative and helium

$$\text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 1} \times (\text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4}) \\ + \text{Diagram 5} \times (\text{Diagram 2} + \text{Diagram 4}) + \text{Diagram 6} \times (\text{Diagram 7} + \text{Diagram 8})$$



$$\text{Diagram 5} = \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 1} \times (\text{Diagram 2} + \text{Diagram 4}) \\ + \text{Diagram 5} \times (\text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4}) \\ + \text{Diagram 6} \times (\text{Diagram 7} + \text{Diagram 8})$$

$$\text{Diagram 6} = \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 1} \times (\text{Diagram 2} + \text{Diagram 4}) \\ + \text{Diagram 5} \times (\text{Diagram 2} + \text{Diagram 4})$$

- iterate $\mathcal{O}(\alpha)$ diagrams. . .
- get ${}^3\text{He}$ pole directly

Nonperturbative vs. perturbative and helium

$$\text{Diagram 1} = \text{Diagram 1.1} + \text{Diagram 1.2} + \text{Diagram 1.3} + \text{Diagram 1.4} \times (\text{Diagram 1.1.1} + \text{Diagram 1.1.2} + \text{Diagram 1.1.3})$$

$$+ \text{Diagram 1.5} \times (\text{Diagram 1.1.1} + \text{Diagram 1.1.2}) + \text{Diagram 1.6} \times (\text{Diagram 1.1.1} + \text{Diagram 1.1.2})$$

$$\text{Diagram 2} = \text{Diagram 2.1} + \text{Diagram 2.2} + \text{Diagram 2.3} \times (\text{Diagram 2.1.1} + \text{Diagram 2.1.2})$$

$$+ \text{Diagram 2.4} \times (\text{Diagram 2.1.1} + \text{Diagram 2.1.2})$$

$$+ \text{Diagram 2.5} \times (\text{Diagram 2.1.1} + \text{Diagram 2.1.2})$$

$$\text{Diagram 3} = \text{Diagram 3.1} + \text{Diagram 3.2} + \text{Diagram 3.3} \times (\text{Diagram 3.1.1} + \text{Diagram 3.1.2})$$

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- iterate $\mathcal{O}(\alpha)$ diagrams. . .
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Nonperturbative vs. perturbative and helium

$$\text{X} = \text{---} + \text{---} + \text{---} + \text{X} \times (\text{---} + \text{---} + \text{---}) + \text{X} \times (\text{---} + \text{---}) + \text{X} \times (\text{---} + \text{---})$$



$$\text{Y} = \text{---} + \text{---} + \text{X} \times (\text{---} + \text{---}) + \text{Y} \times (\text{---} + \text{---}) + \text{Y} \times (\text{---} + \text{---})$$

$$\text{Z} = \text{---} + \text{---} + \text{X} \times (\text{---} + \text{---}) + \text{Z} \times (\text{---} + \text{---})$$

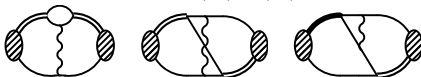
- use trinucleon wavefunctions
- fully perturbative in α !

$$\Delta E : \text{---} - \text{---}$$

- iterate $\mathcal{O}(\alpha)$ diagrams...
- get ${}^3\text{He}$ pole directly

$$\begin{aligned} \text{X} &= \text{---} + \text{---} + \text{---} \\ \text{Y} &= \text{---} + \text{---} + \text{---} \end{aligned}$$

$$\Delta E = \langle \psi | V_C | \psi \rangle$$



SK, Grießhammer, Hammer, J. Phys. G 42 045101 (2015)

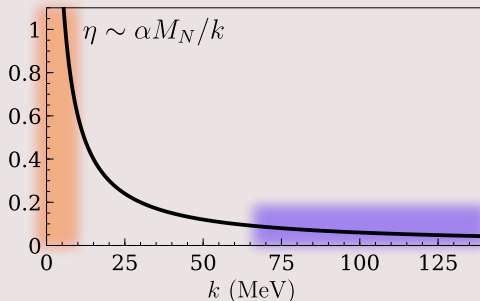
Coulomb corrections for nuclei



How much Coulomb should we really iterate?

Coulomb regimes

trinucleon binding momentum ~ 80 MeV ...

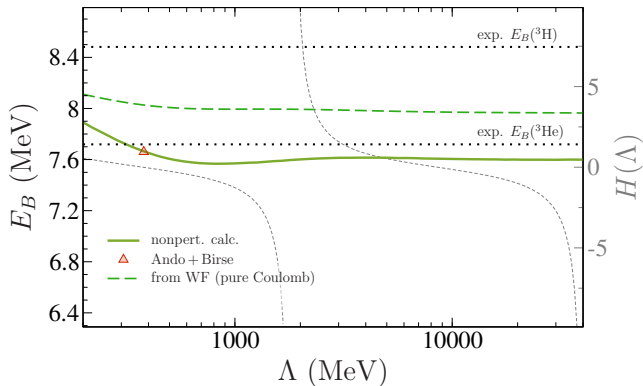


↪ should Coulomb not be a small perturbative correction?

Perturbative vs. nonperturbative helium

$$\Delta E : \text{[Diagram of } ^3\text{He} \text{]} - \text{[Diagram of } ^3\text{H} \text{]} \hookrightarrow \text{compare to non-perturbative result!}$$

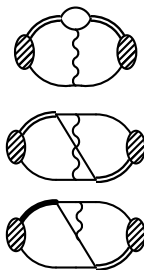
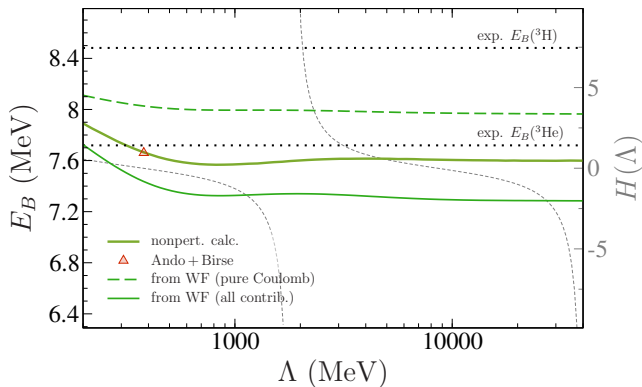
SK, Griebhammer, Hammer, J. Phys. G 42 045101 (2015)



Perturbative vs. nonperturbative helium

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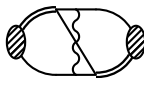
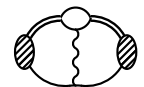
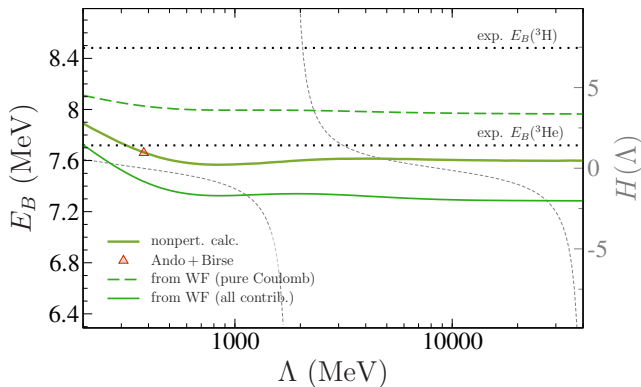
SK, Griebhammer, Hammer, J. Phys. G 42 045101 (2015)



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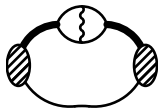
SK, Griebhammer, Hammer, J. Phys. G 42 045101 (2015)



Still missing here!

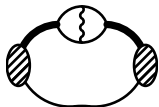


Coulomb bubble divergence



- the additional diagram is logarithmically divergent. . .
- . . . but this divergence comes from the photon-bubble subdiagram!

Coulomb bubble divergence



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Strategy

SK, Grißhammer, Hammer, van Kolck, 1508.05085 [nucl-th]

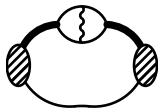
- 1 isolate divergence: 
- 2 take the leading-order 1S_0 in the **unitarity limit!**

$$a_{1S_0} = -23.7 \approx \infty \rightsquigarrow 1/a_{1S_0} \approx 0$$

- 3 include divergent diagram together with finite a_{1S_0}

$$\text{bubble diagram} + \text{finite diagram} = \text{finite}$$

Coulomb bubble divergence



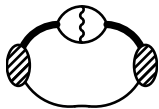
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↪ **determine counterterm from p - p scattering!**

$$\Delta_{t,pp}(p_0, \mathbf{p}) = \frac{-i}{\underbrace{\sigma_{t,pp} - \frac{2\Lambda}{\pi} + \alpha M_N \left(\log \frac{2\Lambda}{\alpha M_N} - C_E \right)}_{=1/a_C} - \alpha M_N H(\eta)}$$

cf. Kong, Ravndal (1999)

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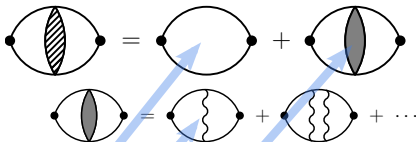
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Important to isolate divergence for consistent renormalization!

Dressed bubble integral

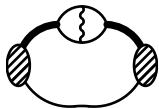
$$J_0(k) = G_C(k^2/M_N; \mathbf{0}, \mathbf{0})$$



$$\hat{G}_C = \hat{G}_0^{(+)} + \hat{G}_0^{(+)} \hat{V}_C \hat{G}_C = \hat{G}_0^{(+)} + \hat{G}_0^{(+)} \hat{T}_C \hat{G}_0^{(+)}$$

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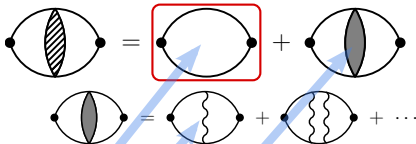
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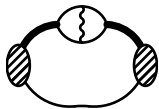
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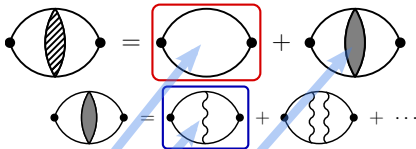
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New perturbative approach

Take the leading-order singlet channel in the unitarity limit!

$$a_t = -23.4 \approx \infty !$$

$$\sigma_t = \sigma_t^{(0)} + \sigma_t^{(1)}$$

$$i\Delta_t^{(0)}(p_0, \mathbf{p}) \frac{-i}{\sigma_t + y_t^2 I_0(p_0, \mathbf{p})} \rightarrow \frac{-i}{\sqrt{\frac{\mathbf{p}^2}{4} - M_N p_0 - i\varepsilon}}$$

$$\boxed{\sigma_t^{(0)} - 2\Lambda/\pi = 0}$$

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NLO corrections

- scattering length: $\sigma_t^{(1)} \sim -1/a_t$
- effective range: $c_t^{(1)} = \frac{M_N r_{0t}}{2}$


$$\sim -i\sigma_{t(pp)}^{(1)}$$


$$\sim -ic_t^{(1)}$$



$$i\Delta_t^{(1)}(p_0, \mathbf{p}) = i\Delta_t^{(0)}(p_0, \mathbf{p}) \times \left[-i\sigma_t^{(1)} - ic_t^{(1)} \left(p_0 - \frac{\mathbf{p}^2}{4M_N} \right) \right] \times i\Delta_t^{(0)}(p_0, \mathbf{p})$$

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- new 1S_0 LO is **isospin-symmetric** and parameter-free
- **allows matching between perturbative and non-perturbative Coulomb regimes**

New perturbative scheme

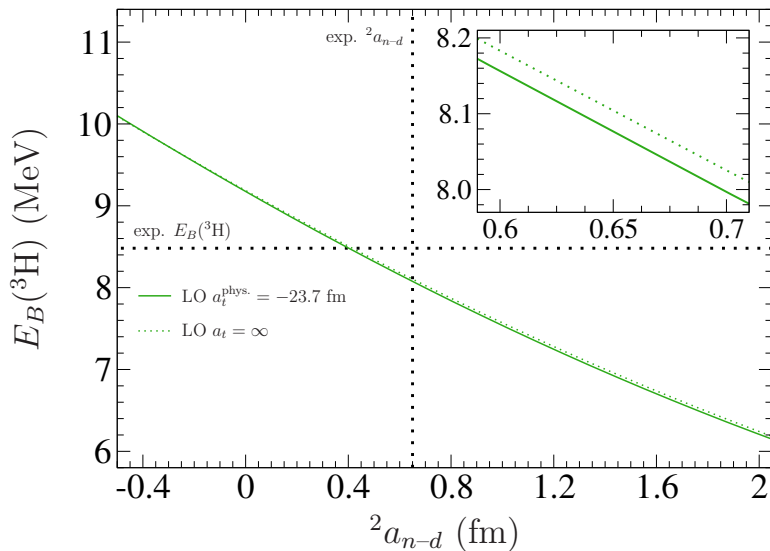
Leading order

- standard NN spin-triplet (pionless) amplitude (parameter γ_d)
- unitary NN spin-singlet amplitude (parameter-free)
- contact three-body force (parameter Λ_*)

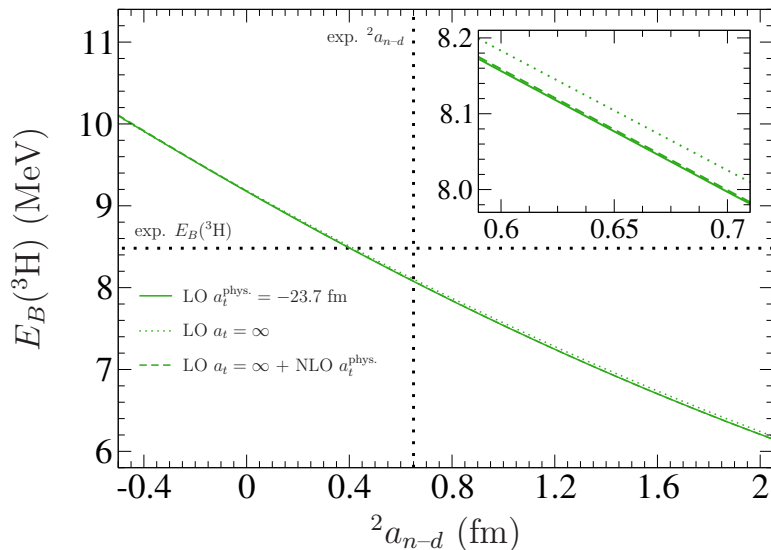
Next-to-leading order

- effective range in the NN spin-triplet channel (parameter ρ_d)
- isospin-symmetric range in the NN spin-singlet channel (parameter r_t)
- scattering-length correction to unitarity in the NN spin-singlet np and nn channels (parameter a_t),
- scattering-length correction to unitarity in the NN pp channel (parameter a_{p-p})
- one-photon exchange (parameter $\alpha = 1/137$)

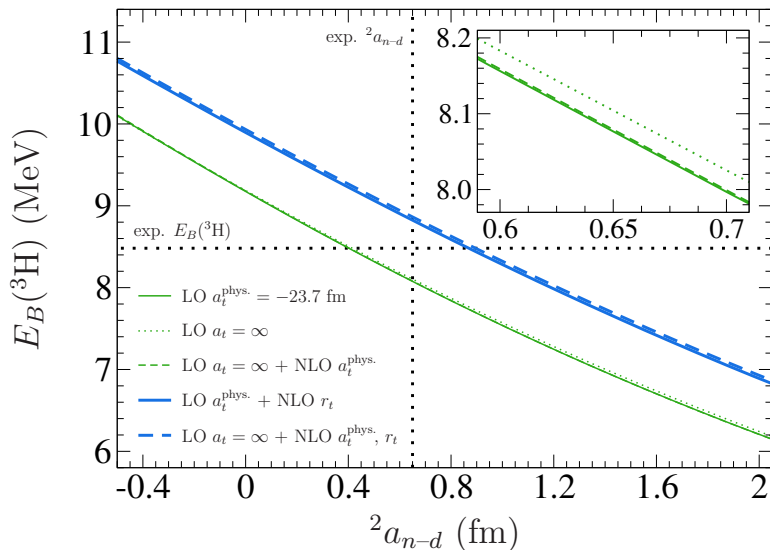
Phillips line



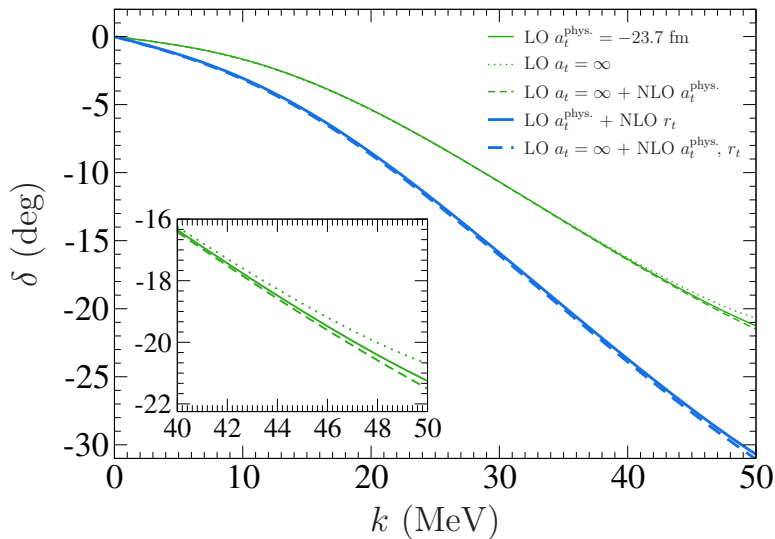
Phillips line



Phillips line



Doublet-channel phase shift



Two-body sector

Consider two regimes in the p - p sector:

① $Q \sim \alpha M_N \hookrightarrow$ Coulomb non-perturbative!  =  +  + ...

② $\alpha M_N \lesssim 1/a_{p-p} \ll Q \ll \Lambda_{\pi} \hookrightarrow$ Coulomb perturbative! 

- determine counterterm in non-pert. regime (match to modified ERE)
- important to use consistent regularization scheme!

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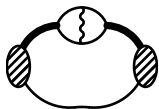
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Coulomb regularization

$$\overbrace{\delta I_0(k) \sim \left[\log \frac{2\Lambda}{\alpha M_N} - C_\zeta + \log i\eta \right]}^{\text{single photon}}, \quad \underbrace{\delta J_0(k) \sim - \left[\psi(i\eta) + \frac{1}{2i\eta} + C_\Delta \right]}_{H(\eta)} \overbrace{+ \frac{M_N}{4\pi} i k}^{\text{two and more}}$$

Coulomb bubble divergence



- the additional diagram is logarithmically divergent...
- ...but this divergence comes from the photon-bubble subdiagram!

↪ determine counterterm from p - p scattering!

$$\Delta_{t,pp}(p_0, \mathbf{p}) = \frac{-i}{\underbrace{\sigma_{t,pp} - \frac{2\Lambda}{\pi} + \alpha M_N \left(\log \frac{2\Lambda}{\alpha M_N} - C_E \right)}_{=1/a_C}} - \alpha M_N H(\eta)$$

cf. Kong, Ravndal (1999)

Important to isolate divergence for consistent renormalization!

Dressed bubble integral

$$J_0(k) = G_C(k^2/M_N; \mathbf{0}, \mathbf{0})$$

The diagram shows two equations involving surfaces with two boundary components (represented by black dots). The first equation shows a surface with a vertical line (shaded with diagonal lines) equal to the sum of an empty surface and a surface with a vertical line (shaded with horizontal lines). The second equation shows a surface with a vertical line (shaded with horizontal lines) equal to the sum of a surface with a vertical wavy line and a surface with a vertical wavy line (shaded with horizontal lines), followed by an ellipsis indicating further terms.

$$\hat{G}_C = \hat{G}_0^{(+)} + \hat{G}_0^{(+)} \hat{V}_C \hat{G}_C = \hat{G}_0^{(+)} + \hat{G}_0^{(+)} \hat{T}_C \hat{G}_0^{(+)} \\ \hat{T}_C = \hat{V}_C + \hat{V}_C \hat{G}_0^{(+)} \hat{T}_C$$

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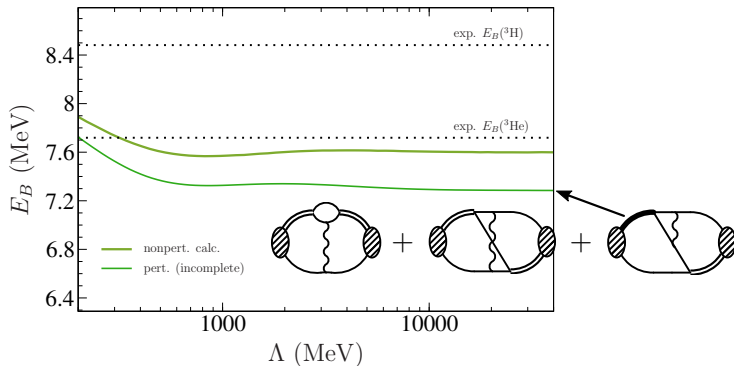
two and more

$\leadsto$ perturbative prediction for p - p phase shift:

$$k \cot \delta_{t,pp}(k) = -\frac{1}{a_{p-p}} + \alpha M_N C_\Delta + \frac{r_t}{2} k^2 + \alpha M_N \log \left(\frac{\alpha M_N}{2k} \right) + \dots$$

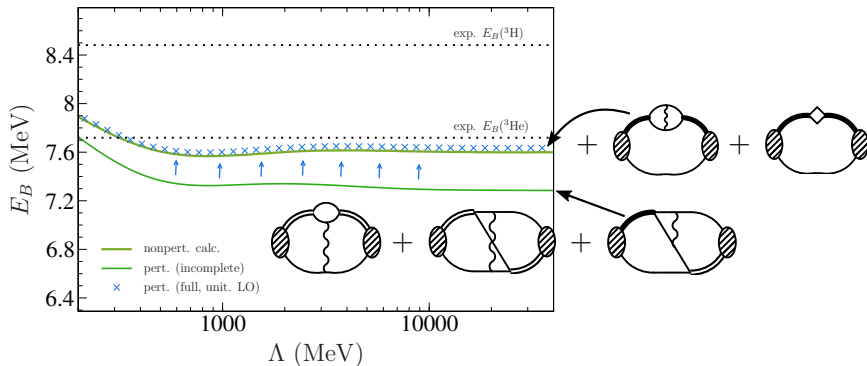
Perturbative ^3He result

New expansion ensures renormalization in the three-body sector!



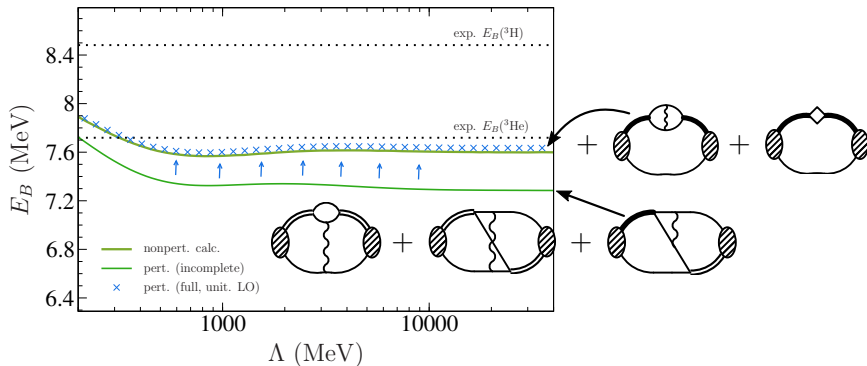
Perturbative ^3He result

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Perturbative ^3He result

New expansion ensures renormalization in the three-body sector!



- Coulomb indeed perturbative in ^3He ✓
- consistent renormalization crucial to achieve this!

New perturbative scheme

Leading order

- standard NN spin-triplet (pionless) amplitude (parameter γ_d)
- unitary NN spin-singlet amplitude (parameter-free)
- contact three-body force (parameter Λ_*)

Next-to-leading order

- effective range in the NN spin-triplet channel (parameter ρ_d)
- isospin-symmetric range in the NN spin-singlet channel (parameter r_t)
- scattering-length correction to unitarity in the NN spin-singlet np and nn channels (parameter a_t),
- scattering-length correction to unitarity in the NN pp channel (parameter a_{p-p})
- one-photon exchange (parameter $\alpha = 1/137$)

New perturbative scheme

Leading order

- standard NN spin-triplet (pionless) amplitude (parameter γ_d)
- unitary NN spin-singlet amplitude (parameter-free)
- contact three-body force (parameter Λ_*)

Next-to-leading order

- effective range in the NN spin-triplet channel (parameter ρ_d)
- isospin-symmetric range in the NN spin-singlet channel (parameter r_t)
- scattering-length correction to unitarity in the NN spin-singlet np and nn channels (parameter a_t),
- scattering-length correction to unitarity in the NN pp channel (parameter a_{p-p})
- one-photon exchange (parameter $\alpha = 1/137$)

Divergence dissection

Vanasse, Egolf, Kerin, SK, Springer, PRC **89** 064003 (2014)

Look at structure of p - d three-body force:

$$H_{0,1}^{(\alpha)}(\Lambda) = h_I^{(\alpha)}(\Lambda) + h_{\kappa}^{(\alpha)}(\Lambda)$$

$$h_I^{(\alpha)}(\Lambda) = -\frac{3\pi(1+s_0^2)}{16} \left\{ \boxed{\frac{1}{12}(r_{p-p} - r_t)\Lambda[1 - \dots]} \right.$$

+ various terms $\sim \log \Lambda$, all $\propto (r_{p-p} - r_t)$ or $\propto (\gamma_{p-p} - \gamma_t)$ $\left. \right\} / \sin^2(\dots)$

$$h_{\kappa}^{(\alpha)} = -\frac{\sqrt{3}\kappa\pi(1+s_0^2)}{48} \left\{ \text{various terms} \sim \log \Lambda \right.$$

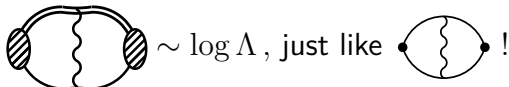
that do not vanish in the isospin limit + $\Psi(\Lambda)$ $\left. \right\} / \sin^2(\dots)$



- pieces associated with $r_{p-p} \neq r_t$ and $\alpha\rho_d, \alpha r_t$
- these have been relegated to a higher order in the new scheme!
- **but an otherwise remaining log-divergence is absent!**

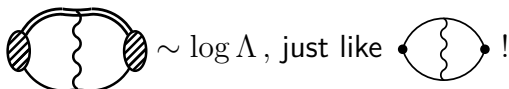
Lessons learned and open questions

- ① unnecessary iteration can cause spurious divergences
- ② isospin-breaking counterterms seem to be unavoidable
- ③ one-photon exchange generates logarithmic divergences:



Lessons learned and open questions

- 1 unnecessary iteration can cause spurious divergences
- 2 isospin-breaking counterterms seem to be unavoidable
- 3 one-photon exchange generates logarithmic divergences:



Questions

- is any of this related to the dibaryon formalism?
- does it matter that we took the two-body cutoff to infinity first? (more generally: regulator artifacts?)
- how much Coulomb has to be iterated in low-energy p - d scattering? (yet another counting above d -breakup threshold?)