# The proton–deuteron scattering length in pionless effective field theory

Sebastian König

in collaboration with H.-W. Hammer

**EMMI RRTF Workshop:**
**“The systematic treatment of the Coulomb interaction in few-body systems”**

TU Darmstadt

Jan 13/14, 2016

SK, H.-W. Hammer, PRC **90** 034005, 1312.2573 [nucl-th]

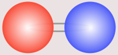
# Status of $p$–$d$ scattering lengths

## Proton

- spin $^1/_2$
- isospin $^1/_2$

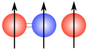
## Deuteron

- spin $1$
- isospin $0$

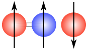


$\hookrightarrow$ **two S-wave channels:**

$$\mathbf{1} \otimes \frac{\mathbf{1}}{\mathbf{2}} = \frac{\mathbf{3}}{\mathbf{2}} \left(\sim \right.$$



$$\left.\right) \oplus \frac{\mathbf{1}}{\mathbf{2}} \left(\sim \right.$$



$$\left. + \cdots \right)$$

**Quartet channel**

| Ref. | $^4a_{p-d}$ (fm) |
|---|---|
| van Oers, Brockman (1967) | $11.4^{+1.8}_{-1.2}$ |
| Arvieux (1973) | $11.88 \pm 0.4$ |
| Huttel *et al.* (1983) | $\approx 11.1$ |
| Chen *et al.* (1989) | $13.8$ |
| Kievsky *et al.* (1994) | $13.76$ |
| Black *et al.* (1999) | $14.7 \pm 2.3$ |

**Doublet channel**

| Ref. | $^2a_{p-d}$ (fm) |
|---|---|
| van Oers, Brockman (1967) | $1.2 \pm 0.2$ |
| Arvieux (1973) | $2.73 \pm 0.10$ |
| Huttel *et al.* (1983) | $\approx 4.0$ |
| Chen *et al.* (1989) | $0.17$ |
| Kievsky *et al.* (1997) | $0.024$ |
| Black *et al.* (1999) | $-0.13 \pm 0.04$ |

# Motivation

**Goal**

**Precise and controlled extraction from EFT calculation!**

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**Precise and controlled extraction from EFT calculation!**

## Scope of method

- **Nuclear astrophysics**
  - Scattering parameters $\leftrightarrow$ shallow bound states

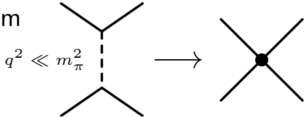
    SK, Lee, Hammer 2013

    Sparenberg, Capel, Baye 2010
  - Low-energy nuclear reactions in Halo-EFT
  - $\rightarrow$ one-neutron halo states in $^{11}\mathrm{Be}$
  - $\rightarrow$ one-proton halo state in $^{8}\mathrm{B}$ ?
- **Cold-atom systems**
  - EFT with van-der-Waals tails?

# Pionless EFT

# Pionless EFT

- at very low energies even pions can be integrated out
  $\hookrightarrow$ only nucleons left as effective degrees of freedom
- non-relativistic framework
- large scattering lengths in $N$-$N$ scattering
  $\hookrightarrow$ additional low-energy scale




$$\gamma_d = \frac{1}{a_d}\left(1 + \mathcal{O}(a_d/r_d)\right)$$

Kaplan, Savage, Wise 1998; van Kolck 1997/98

# Pionless EFT

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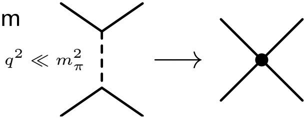


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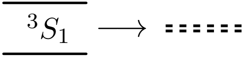



- convenient description of three-body sector with dibaryon fields

Bedaque, Hammer, van Kolck 1998

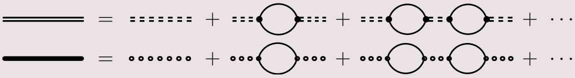
# Resummations

**Power counting** $\hookrightarrow$ resum certain classes of diagrams!

## Full dibaryon propagators

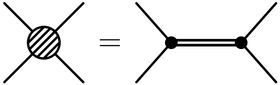
$^3S_1$ : $\Delta_d =$

$^1S_0$ : $\Delta_t =$



**Fix parameters from $N$-$N$ scattering!**

$$\Delta_d\left(p_0 = \frac{\mathbf{k}^2}{2M_N}, \mathbf{p} = 0\right) \sim \frac{\mathrm{i}}{\underbrace{k\cot\delta_d}_{=-\gamma_d+\frac{\rho_d}{2}(k^2+\gamma_d^2)+\cdots} - \mathrm{i}k}$$



$$\Delta_d(p) \sim \frac{1}{-\gamma_d + \sqrt{\frac{\mathbf{p}^2}{4} - M_N p_0 - \mathrm{i}\varepsilon} - \frac{\rho_d}{2}\left(\frac{\mathbf{p}^2}{4} - M_N p_0 - \gamma_d^2\right)}$$

$Q \sim \gamma_d \approx 45$ MeV, $\Lambda \sim m_\pi \approx 140$ MeV

$\mathcal{O}(Q/\Lambda) \sim \mathcal{O}(r/a) \sim 1/3$

expand in $\rho_d$, $r_t$ $\rightarrow$ NLO, N$^2$LO, ...

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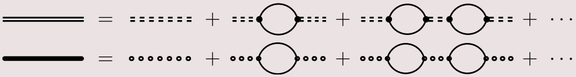
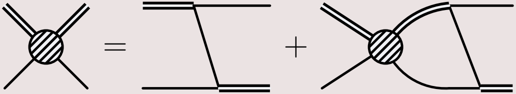


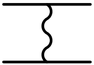
## Three-body amplitude

$\sim \cdots$ all of same order $\rightarrow$ **Integral equation!**



**Lippmann–Schwinger equation** $\rightsquigarrow$ **solve numerically!**

# Coulomb contributions and subtraction

**Coulomb photons:**  $\sim \; (\mathrm{ie}) \, \frac{\mathrm{i}}{\mathbf{q}^2} \, (\mathrm{ie}) \;\longrightarrow\; (\mathrm{ie}) \, \frac{\mathrm{i}}{\mathbf{q}^2 + \lambda^2} \, (\mathrm{ie})$

- long (infinite) range $\rightarrow$ very strong at small momentum transfer
- two-body Coulomb scattering can be solved analytically

$\hookrightarrow$ **subtract the pure Coulomb contribution!**

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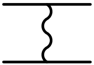
**Coulomb photons:**  $\sim \;(\mathrm{ie})\,\frac{\mathrm{i}}{\mathbf{q}^2}\,(\mathrm{ie}) \;\longrightarrow\; (\mathrm{ie})\,\frac{\mathrm{i}}{\mathbf{q}^2+\lambda^2}\,(\mathrm{ie})$

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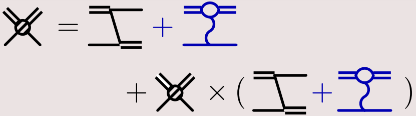
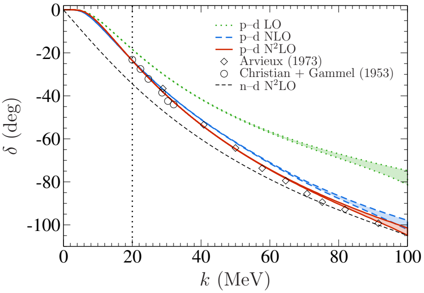
## Bottom line



$\rightarrow$ full amplitude $\mathcal{T}_{\mathrm{full}}$

$\rightarrow$ Coulomb amplitude $\mathcal{T}_{\mathrm{c}}$

$$\tilde{\delta}(k) \approx \delta_{\mathrm{diff}}(k) \equiv \delta_{\mathrm{full}}(k) - \delta_{\mathrm{c}}(k)$$




Rupak, Kong (2003); SK, Hammer (2011)

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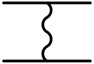
**Coulomb photons:**  $\sim\ (\mathrm{ie})\,\dfrac{\mathrm{i}}{\mathbf{q}^2}\,(\mathrm{ie}) \longrightarrow (\mathrm{ie})\,\dfrac{\mathrm{i}}{\mathbf{q}^2+\lambda^2}\,(\mathrm{ie})$

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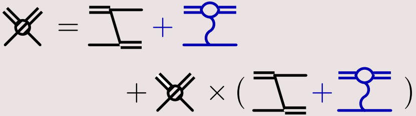

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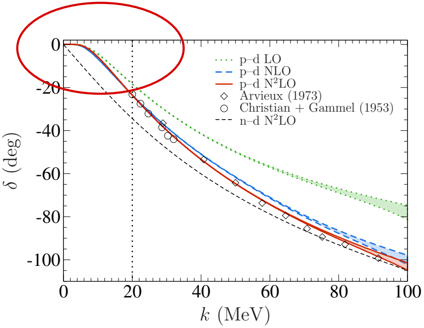


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# Modified effective range expansion

## Ordinary effective range expansion

$$k \cot \delta(k) = -\frac{1}{a_0} + \frac{r_0}{2} k^2 + \cdots$$

$a$ = scattering length

$r$ = effective range

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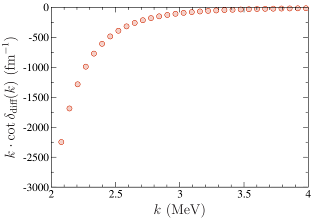
## Modified effective range expansion

$$C_{\eta,0}^2 \, k \cot \delta_{\text{diff}}(k) + \alpha\mu \, h_0(\eta) = -\frac{1}{a_0^C} + \cdots$$

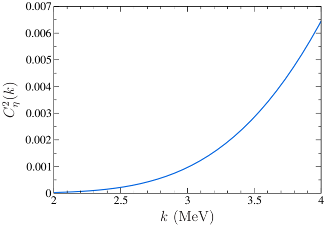
**Gamow factor**

$$C_{\eta,0}^2 = \frac{2\pi\eta}{e^{2\pi\eta} - 1}$$

$$\eta = \alpha\mu/k$$




$\times$




**= finite value**

# The Gamow factor

$$C_{\eta,0}^2 = \frac{2\pi\eta}{e^{2\pi\eta} - 1}$$

**But we have a screened Coulomb potential!**

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- **note:** $C_{\eta,0}^2 = (\text{Jost function})^{-2} = \left|\psi_{\mathbf{k}}^{(+)}(\mathbf{r} = \mathbf{0})\right|^2$
- cf. general modified ERE by van Haeringen and Kok — van Haeringen, Kok 1982
- **furthermore:** $\psi_{\mathbf{k},0}^{(+)}(p) = \frac{2\pi^2}{k^2}\delta(k-p) - \frac{2\mu\, Z_0 \mathcal{T}(E;p,k)}{k^2 - p^2 + \mathrm{i}\varepsilon}$ , $E = E(k)$

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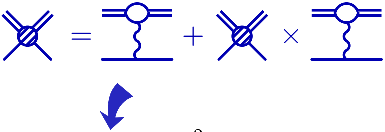


## Solution

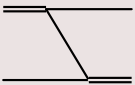
$$\rightsquigarrow C^2_{\eta,\lambda} = \left|1 + \frac{2M_N}{3\pi^2}\int_0^\Lambda \frac{\mathrm{d}p\, p^2}{p^2 - k^2 - \mathrm{i}\varepsilon} Z_0 \mathcal{T}_{\mathrm{c},\lambda}(E;p,k)\right|^2$$

$\hookrightarrow$ **consistent extraction from numerical calculation!**

# Perturbative expansion

## Strong sector

**Leading order**

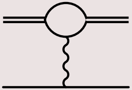


**Corrections**

$$\Delta_d^{\mathrm{LO}}(p) \times \frac{\rho_d}{2}\left(p_0 - \frac{\mathbf{p}^2}{4M_N}\right) \times \Delta_d^{\mathrm{LO}}$$

## Coulomb sector

**Leading order**



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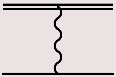
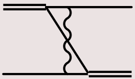
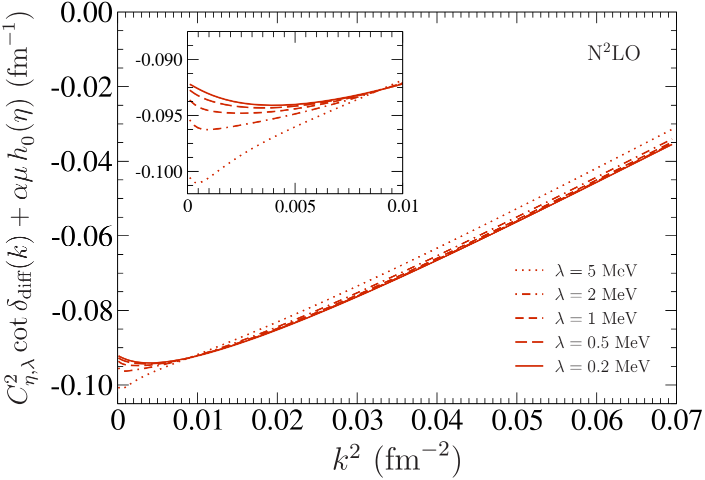




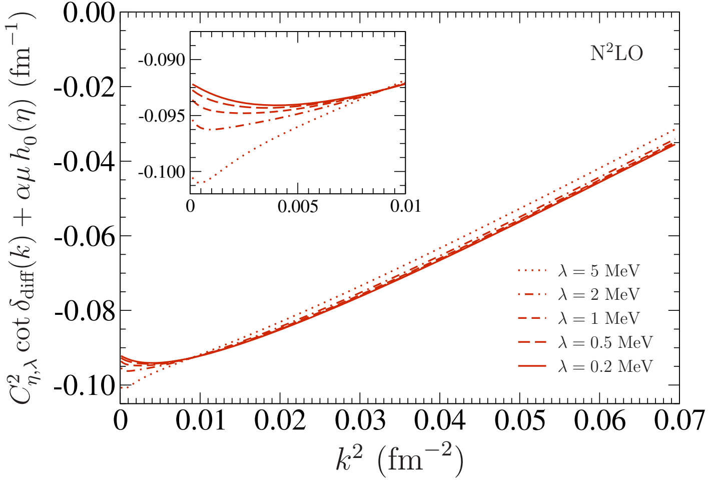
- simultaneous expansion in $Q/\Lambda$ and $p/(\alpha M_N)$, $p \ll Q \sim 45\ \mathrm{MeV}$ Rupak, Kong 2003
- alternative: simply count powers of $\alpha$ SK, Grießhammer, Hammer 2015
- efficient implementation of fully perturbative scheme Vanasse 2013

# Effective range function




- convergent in the limit $\lambda \to 0$ ✓

# Effective range function




- convergent in the limit $\lambda \to 0$ ✓ $\quad C^2_{\eta,\lambda}\left[k\cot\delta_{\mathrm{diff}}(k)\right] + \gamma\, h(\eta) = -\dfrac{1}{a_{p\text{-}d}} + \mathcal{O}(k^2)$
- curvarture $\leftrightarrow$ missing screening corrections in $h(\eta)$ ?

# More screening corrections

$$C^2_{\eta,\lambda}\,[k\cot\delta_{\text{diff}}(k)] + \gamma\,h_\lambda(\eta) = -\frac{1}{a_{p\text{–}d}} + \frac{r_{p\text{–}d}}{2}k^2 + \mathcal{O}(k^4)$$

**We want the $h_\lambda(\eta)$ !**

**General modified ERE**

$$|\mathcal{F}_\ell(k)|^{-2}k^{2\ell+1}\left[\cot\tilde{\delta}_\ell(k) - \mathrm{i}\right] + M_\ell(k) = -\frac{1}{\tilde{a}_\ell} + \frac{\tilde{r}_\ell}{2}k^2 + \cdots$$

- $h_\lambda(\eta) = \mathrm{Re}\,M_0(k) \sim \mathrm{Re}\left\{\lim_{r\to 0}\frac{\mathrm{d}}{\mathrm{d}r}f_0(k,r)\right\}$ (derivative of Jost solution)

  van Haeringen, Kok (1982)

- but this does not work here because of logarithmic terms!

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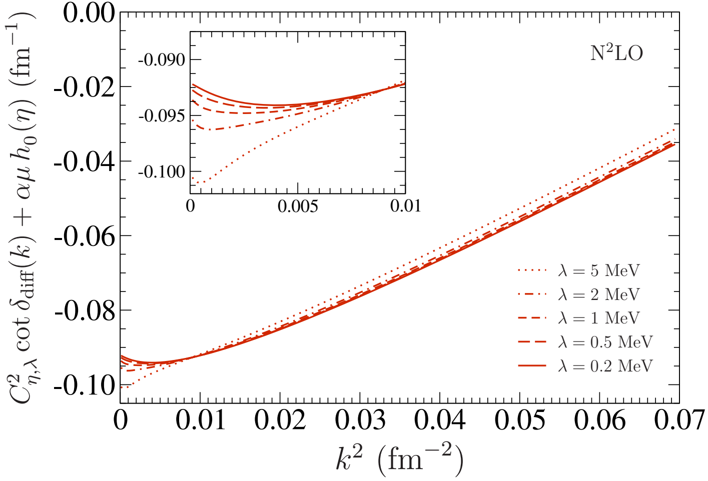
- but this does not work here because of logarithmic terms!
- **alternative: relate this to the Green's function**

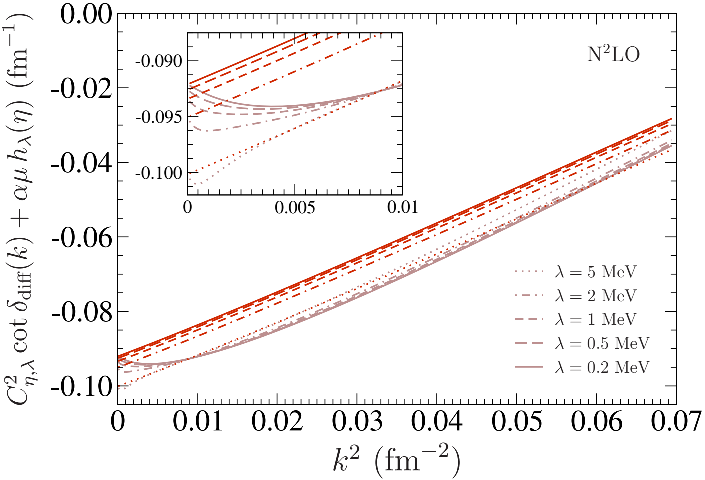
  *cf.* Kong+Ravndal (1999), Steele+Furnstahl (1999)

$$\hookrightarrow h_\lambda(\eta) = \frac{k^2}{\alpha\mu}\frac{1}{\pi}\,\mathrm{P}\int_0^\Lambda \mathrm{d}q\,\frac{C^2_{\eta,\lambda}(q)}{(q+k)(q-k)} \quad , \quad \mu = 2M_N/3$$

# Effective range function revisited

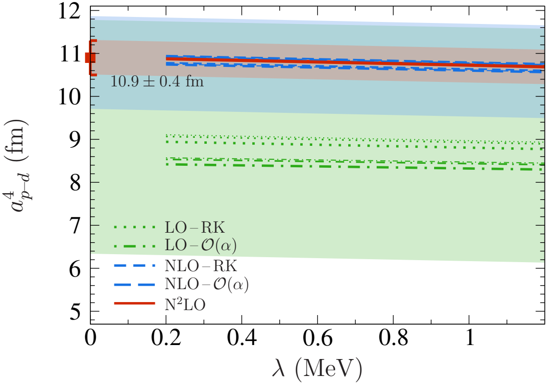


# Effective range function revisited

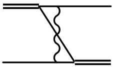



- consistent calculation of $h_\lambda(\eta)$ removes non-analyticity! ✓
- scattering length is unaffected! ✓

# Quartet-channel result



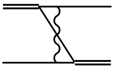

“RK”



as $Q/\Lambda$ correction

Rupak, Kong 2003

“$\mathcal{O}(\alpha)$”



already at LO

SK, Grießhammer, Hammer 2015

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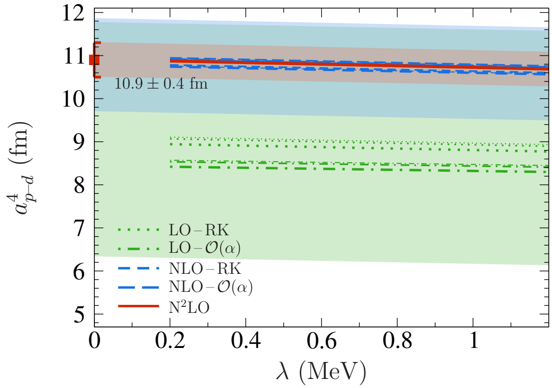
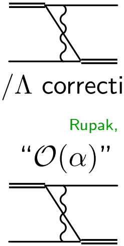



## Older experimental determinations

- $a^4_{p-d} = 11.88 \pm 0.4$ fm Arvieux (1973)
- $a^4_{p-d} = 11.1$ fm Huttel *et al.* (1983)

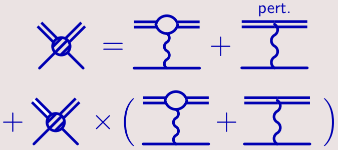
## More recent results

- $a^4_{p-d} \approx 13.8$ fm Chen *et al.*(1989) Kievsky *et al.*(1994)
- $a^4_{p-d} = 14.7 \pm 2.3$ fm Black *et al.* (1999)

**How do we explain/resolve this discrepancy?**

# Coulomb subtraction

## EFT calculation (momentum space)




- $$\cot\delta_{\mathrm{diff}} = \frac{2\pi}{\mu}\frac{\mathrm{e}^{2\mathrm{i}\delta_{\mathrm{c}}}}{T_{\mathrm{full}} - T_{\mathrm{c}}} + \mathrm{i}k$$
- $$C^2_{\eta,\lambda} = \left|1 + \frac{2M_N}{3\pi^2}\int_0^\Lambda \frac{\mathrm{d}p\,p^2}{p^2 - k^2 - \mathrm{i}\varepsilon} T_c(E;p,k)\right|^2$$

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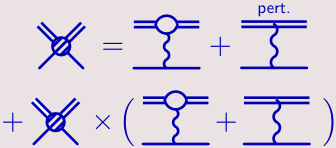


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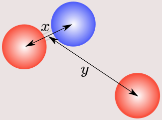
## Potential-model calculation (configuration space)

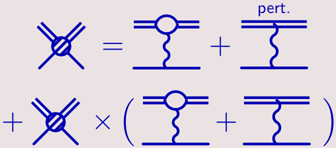
$$\psi(x,y) \overset{y\to\infty}{\longrightarrow} \Big[F(\eta,ky)\cot\tilde{\delta}(k) + G(\eta,ky)\Big]\,u(x)$$

*cf.* Chen, Payne, Friar, Gibson 1989



# Coulomb subtraction

## EFT calculation (momentum space)




- $\cot\delta_{\mathrm{diff}} = \dfrac{2\pi}{\mu}\dfrac{\mathrm{e}^{2\mathrm{i}\delta_\mathrm{c}}}{T_{\mathrm{full}} - T_\mathrm{c}} + \mathrm{i}k$
- $C^2_{\eta,\lambda} = \left| 1 + \dfrac{2M_N}{3\pi^2}\displaystyle\int_0^\Lambda \dfrac{\mathrm{d}p\,p^2}{p^2 - k^2 - \mathrm{i}\varepsilon} T_c(E;p,k)\right|^2$

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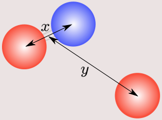


**These two subtractions are not quite equivalent!**

- bubble diagram $\leftrightarrow$ three-body effects in Coulomb sector
- in configuration space, the subtraction is effectively at the two-body level

# Subtraction with simple Yukawa potential

Just get $T_c$ from a pure two-body (Yukawa) calculation

$\hookrightarrow$ **no perturbative expansion in pure Coulomb sector!**

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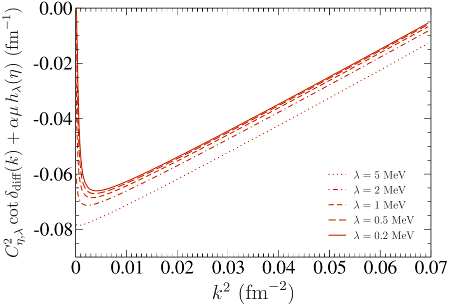

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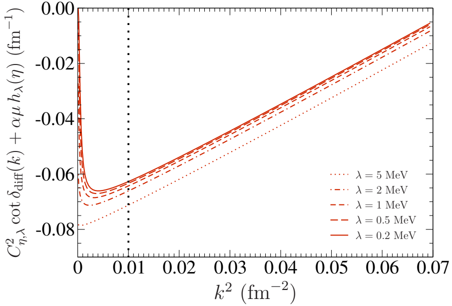


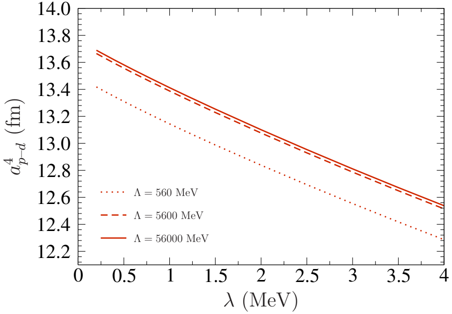

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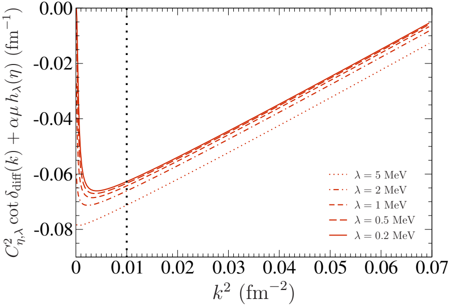


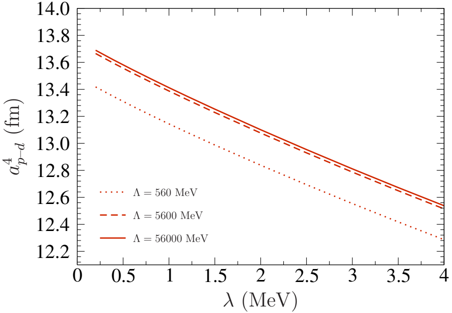


- ✓ gives a value consistent with potential-model calculations
- ✓ slope (effective range) also in good agreement
- ✗ artifacts at small $k$ and stronger cutoff dependence
- ✗ no longer a "clean" EFT calculation

# Summary and outlook

- non-perturbative Coulomb effects are hard to include consistently
- screened Gamow factor can be calculated numerically...
- ...as well as screening corrections to remaining ERE function
- discrepancy with potential-model calculations can be resolved by adjusting the Coulomb subtraction
- pionless EFT calculation only uses two input parameters ($\gamma_d$, $\rho_d$)
- scattering lengths are scheme-dependent quantities, in particular beyond the two-body sector!
- still interesting to study $p$–$d$ doublet channel in this framework!

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**Thanks for your attention!**