Hadron Structure from Lattice QCD

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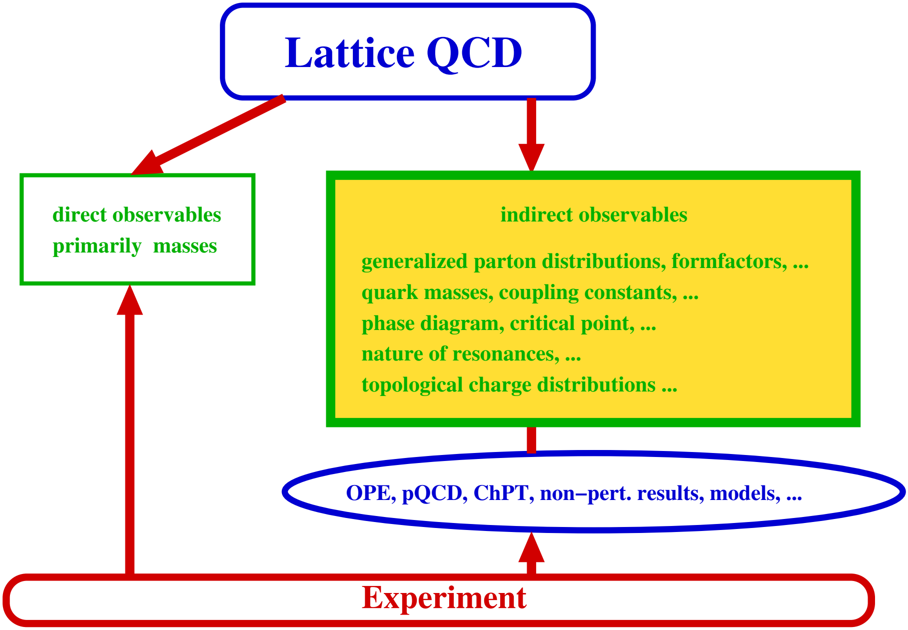
Outline

- Some introductory remarks on lattice QCD
- A collection of results
- Nucleon form factors
    - The status: dipole or not dipole ?
    - Twisted boundary conditions
    - When does asymptotic dynamics start ?
- Conclusions


Lattice QCD
direct observables
primarily masses
indirect observables
generalized parton distributions, formfactors, ...
quark masses, coupling constants, ...
phase diagram, critical point, ...
nature of resonances, ...
topological charge distributions ...
OPE, pQCD, ChPT, non−pert. results, models, ...
Experiment

OPE and pQCD allows to link experimental observations to correlators in a well-defined manner:

$$\langle \textit{Hadron} \mid \text{quark and gluon field operators} \mid \textit{Hadron}' \rangle$$

$\langle P(p)\vert\bar{q}(x)\gamma_\mu D_{\mu_1}...D_{\mu_n}q(x)\vert P(p)\rangle$	momentum distribution of quarks
$\langle P(p')\vert\bar{q}(x)\gamma_\mu q(x)\vert P(p)\rangle$	form factors of a proton
$\langle 0\vert\overline{Cq(x)}\Gamma_1 q(x)\Gamma_2 q(x)\vert P(p)\rangle$	proton distribution amplitude
$\langle P(p)\vert\bar{q}(x)\Gamma_\mu q(x)\bar{q}'(x)\Gamma'_\nu q'(x)\vert P(p)\rangle$	diquark correlations in a proton
$\langle P(p,s)\vert\bar{q}(x)\gamma_\mu\tilde{G}_{\nu\lambda}(x)q(x)\vert P(p,s)\rangle$	color magnetic field in a proton

These can be calculated on the lattice.

Lattice QCD

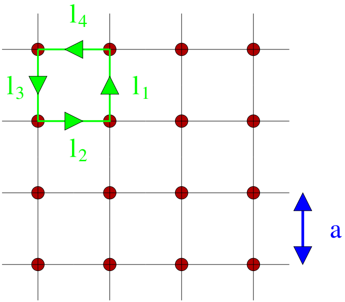
QCD is contained in the generating functional:

$$\begin{aligned} Z[J_\mu^a, \bar{\eta}^i, \eta^i] &= \int \mathcal{D}[A^{a\mu}, \bar{\psi}^i, \psi^i] \\ & \exp\left(\mathrm{i} \int d^4x \left[\mathcal{L}_{\mathrm{QCD}} - J_\mu^a A_\mu^a - \bar{\psi}^i \eta^i - \bar{\eta}^i \psi^i\right]\right) \end{aligned}$$

A numerical integration is made possible by analytic continuation to imaginary time:

$$\begin{aligned} t &\leftrightarrow -\mathrm{i}\tau \\ S = \int d^4x(T - V) &\leftrightarrow \mathrm{i} \int d^4x_E(T + V) = \mathrm{i}S_E \\ e^{\mathrm{i}S} &\leftrightarrow e^{-S_E} \end{aligned}$$

Discretized space time ⇒ e.g. the Wilson action



$$U(l_1) = \exp\left(-igA^b(l_1)\,\frac{\lambda^b}{2}\,a\right)$$

$$W_\square = \mathrm{Tr}\{U(l_1)U(l_2)U(l_3)U(l_4)\}$$

$$\sum_\square \frac{2}{g^2}(3 - \mathcal{R}e\,W_\square) = \frac{1}{4}\int d^4x\left(F^a_{\mu\nu}F^a_{\mu\nu} + O(a^2)\right)$$

Hadronic 2- and 3- Point functions

One needs combinations of field operators which have the wanted quantum numbers, e.g. for the nucleon ($C = i\gamma^2\gamma^4 = C^{-1}$):

$$\hat{B}_\alpha(t,\vec{p}) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \epsilon_{ijk} \hat{u}^i_\alpha(x)\ \hat{u}^j_\beta(x)(C^{-1}\gamma_5)_{\beta\gamma}\hat{d}^k_\gamma(x)$$

$$\begin{aligned}\langle 0|T\left\{\hat{B}(y_4)\hat{A}(x_4)\right\}|0\rangle &= e^{-(T-y_4+x_4)E_B}\langle B|\hat{B}(0)|0\rangle\langle 0|\hat{A}(0)|B\rangle \\ &+ e^{-(y_4-x_4)E_A}\langle 0|\hat{B}(0)|A\rangle\langle A|\hat{A}(0)|0\rangle\end{aligned}$$

$\hat{B}$ generates the antiparticle of $\hat{A}$. One has (anti)periodic boundary conditions.

To get the hadron masses one simply has to determine the slopes.

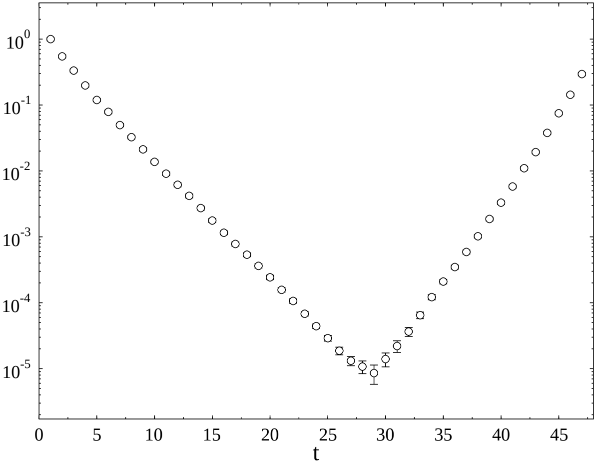
$$e^{-(y_4-x_4)M_N}\langle 0|\hat{N}^\dagger(0)|N\rangle\langle N|\hat{N}(0)|0\rangle$$

$$\begin{aligned}|B\rangle \quad &\sim \quad c_0|N\rangle \;+\; c_1|N'\rangle \;+\; c_2|N\pi\rangle \;+\; \ldots \\ &\Rightarrow \quad c_0 e^{-E_N t}|N\rangle \;+\; c_1 e^{-E_{N'} t}|N'\rangle \;+\; c_2 e^{-E_{N\pi} t}|N\pi\rangle \;+\; \ldots\end{aligned}$$

Note: A quark propagator is the inverse of the Dirac operator on the lattice, which is just a large matrix.

$$\begin{aligned}&\langle B_\alpha(t,\vec{p})\bar{B}_\beta(0,\vec{p})\rangle \\ &= \sum_{\substack{x \\ x_4=t}} \sum_{\substack{y \\ y_4=0}} \mathrm{e}^{\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})}\epsilon_{ijk}\epsilon_{i'j'k'}(C^{-1}\gamma_5)_{\alpha'\alpha''}(\gamma_5 C)_{\beta'\beta''} \\ &\left\langle G^{ki'}_{\alpha''\beta'}(x,y)\left(G^{jj'}_{\alpha'\beta''}(x,y)G^{ik'}_{\alpha\beta}(x,y) - G^{ij'}_{\alpha\beta''}(x,y)G^{jk'}_{\alpha'\beta}(x,y)\right)\right\rangle_g\end{aligned}$$

A nucleon 2-point function

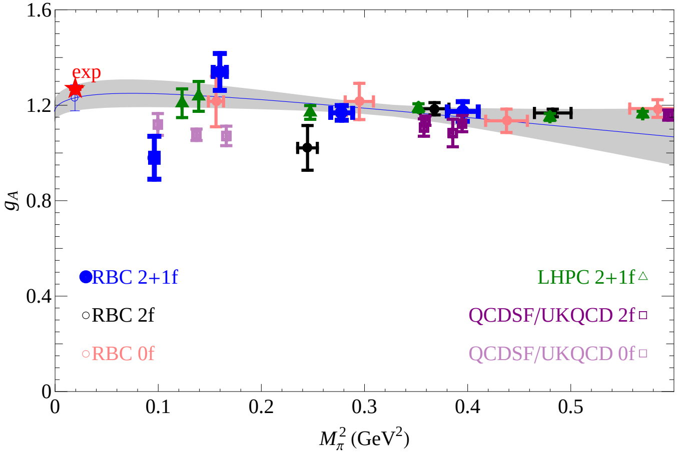


Once the propagation in imaginary time has projected the original source onto the physical wave function on can calculate physical correlators from

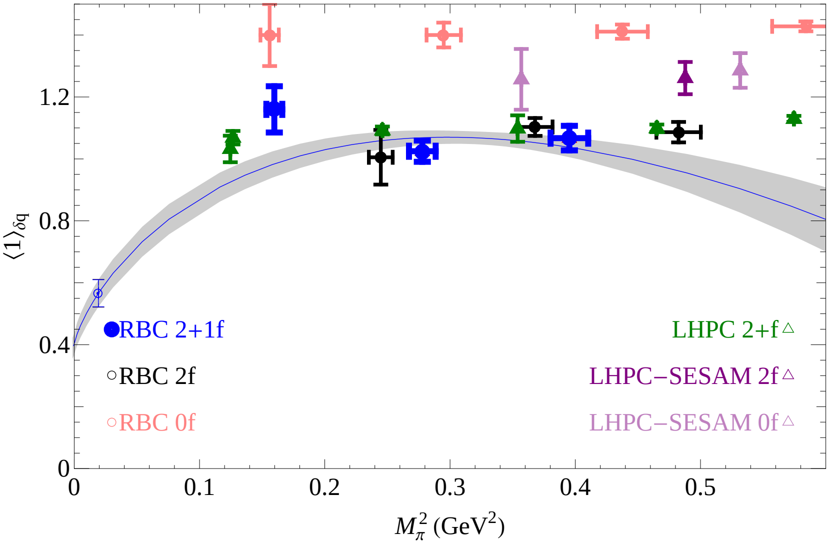
$$\frac{\tilde{\Gamma}_{\alpha\beta}\langle B_\beta(t,\vec{p})\mathcal{O}\bar{B}_\alpha(0,\vec{p})\rangle}{\Gamma_{\alpha\beta}\langle B_\beta(t,\vec{p})\bar{B}_\alpha(0,\vec{p})\rangle}$$

H.-W. Lin arXiv:0903.4080 (RBC and LHPC)

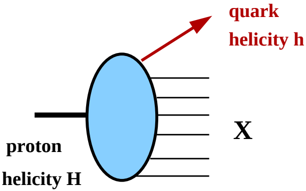
g_A, a case for which ChPT predicts little m_π dependence, but strong V dependence



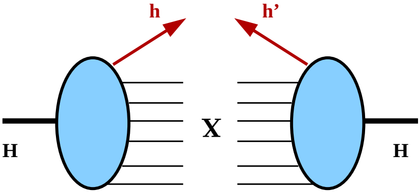
$\delta u - \delta d$, a case for which ChPT predicts a strong m_π dependence



Note: Longitudinal and transverse spin structure are substantially different



$$\Psi_k \sim \langle Xq|p\rangle$$



$$\sim \sum_X \; \Psi^*_{k'} \; \Psi_k$$

$k = (H, h)$ hadron and quark helicities

$$\Psi_{total} = \sum_k c_k \Psi_k \;\Rightarrow\; 0 \leq \sum_X \left| \sum_k c_k \Psi_k \right|^2 = \sum_X \sum_{k,k'} c^*_{k'} \left(\Psi^*_{k'} \Psi_k \right) c_k$$

This implies that the matrix $M_{k'k} = \sum_X \Psi^*_{k'} \Psi_k$ is positive semi-definite. With the notation:

$$M = \frac{1}{2} \begin{pmatrix} q(x) + \Delta q(x) & 0 & 0 & 2\delta q(x) \\ 0 & q(x) - \Delta q(x) & 0 & 0 \\ 0 & 0 & q(x) - \Delta q(x) & 0 \\ 2\delta q(x) & 0 & 0 & q(x) + \Delta q(x) \end{pmatrix} \begin{matrix} k = (Hh) \\ (++) \\ (-+) \\ (+-) \\ (--) \end{matrix}$$

$$k' = \quad (++) \quad (-+) \quad (+-) \quad (--)$$

The helicity of the state X is fixed in each term of the sum over X, thus $H - h = H' - h' = h_X$.

The positive definiteness of M implies

$$q(x) - \Delta q(x) \geq 0 \quad , \quad \det \begin{pmatrix} q(x) + \Delta q(x) & 2\delta q(x) \\ 2\delta q(x) & q(x) + \Delta q(x) \end{pmatrix} \geq 0$$

The last constraint gives Soffer's bound:

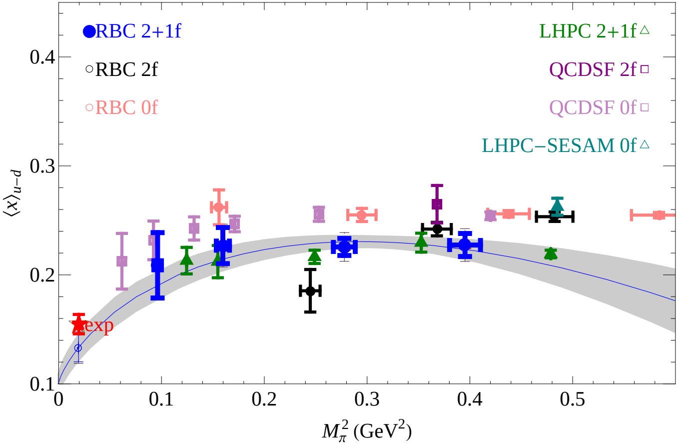
$$2|\delta q(x)| \leq q(x) + \Delta q(x)$$

The relationship between longitudinal spin-flip and transverse spin asymmetry:

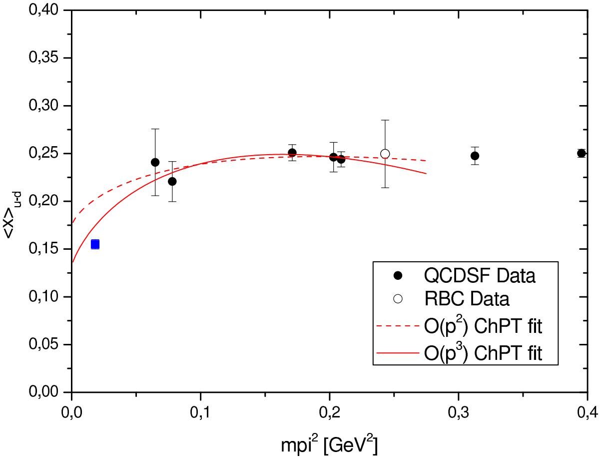
$$\frac{1}{\sqrt{2}}\left(\langle\uparrow| + \langle\downarrow|\right)\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) - \frac{1}{\sqrt{2}}\left(\langle\uparrow| - \langle\downarrow|\right)\frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)$$
$$= \langle\uparrow|\downarrow\rangle + \langle\downarrow|\uparrow\rangle$$

$\Rightarrow$ probability for transverse quark spin in a transversely polarized nucleon $=$ longitudinal spin flip

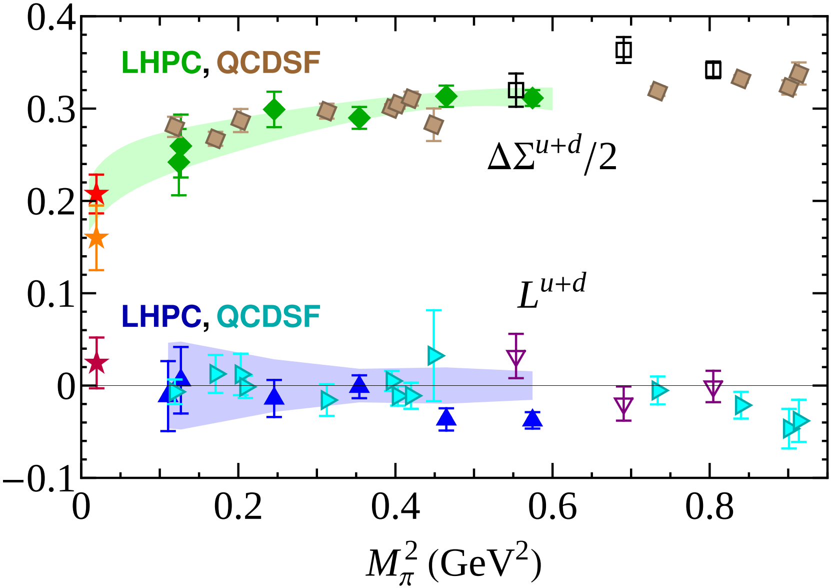
$\langle x \rangle_{u-d}$, another case with strong m_π dependence



The problem of $\int_0^1 x[u(x) - d(x)]dx$ seems indeed to be solved. A recent ChPT calculation of Th. Hemmert



The spin structure of the nucleon, $'L' := J - S$



OPE for Ji's sumrule

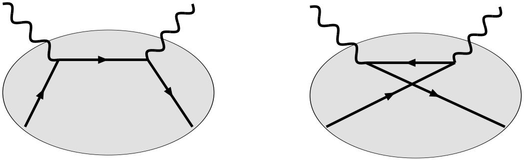
part 1: The energy momentum tensor of QCD

$$\mathcal{L}_{QCD} \rightarrow \mathbf{T}_{\mu\nu} = \frac{1}{2}[\bar{q}\gamma^{(\mu} i \overleftrightarrow{D}^{\nu)} q + \bar{q}\gamma^{(\mu} i \overleftrightarrow{D}^{\nu)} q]$$
$$+ \quad \frac{1}{4} g^{\mu\nu} F^2 - F^{\mu\alpha} F^{\nu}_{\alpha}$$

This gives the total angular momentum as matrix element of a specific operator

$$J^i_{q,g} = \frac{1}{2} \epsilon^{ijk} \int d^3x (\mathbf{T}^{0k}_{q,g} x^j - \mathbf{T}^{0j}_{q,g} x^k)$$

part 2: The DVCS amplitude



$$T^{\mu\nu}(P,\Delta,t) \;=\; \mathrm{i}\int d^4y\; e^{\mathrm{i}(q+q')\cdot\frac{y}{2}} \left\langle P_2, S_2\right| \mathcal{T}\left\{\hat{J}^{\mu\dagger}_{\mathrm{em}}\left(\frac{y}{2}\right)\hat{J}^{\nu}_{\mathrm{em}}\left(\frac{y}{2}\right)\right\}\left|P_1, S_1\right\rangle$$

$$\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle P_2|\,\bar{q}(-\frac{1}{2}z)\,\gamma^+ q(\frac{1}{2}z)\,|P_1\rangle\Big|_{z^+=0,\,z_\perp=0}$$
$$= \frac{1}{P^+}\Big[H_q(x,\xi,t)\,\bar{N}(P_2)\gamma^+ N(P_1) + E_q(x,\xi,t)\,\bar{N}(P_2)\frac{i\sigma^{+\alpha}\Delta_\alpha}{2M}N(P_1)\Big]$$

In both cases one gets the same matrix elements, but for $J_{q,g}$ the initial and final momentum is equal, i.e. $\Delta^\mu = 0$.

$$\mathcal{O}_q^{\mu\mu_1\ldots\mu_n} := \textbf{Sym}\ \bar{q}(x)\gamma^\mu \mathrm{i} \overset{\leftrightarrow}{D}{}^{\mu_1} \ldots \mathrm{i} \overset{\leftrightarrow}{D}{}^{\mu_n} q(x)$$

$$\begin{aligned}
\left\langle P_2 \middle| \mathcal{O}_q^{\mu\mu_1\ldots\mu_n} \middle| P_1 \right\rangle = \ & \textbf{Sym}\ \bar{N}(P_2)\gamma^\mu N(P_1) \sum_{i=0,even}^{n} A^q_{n+1,i}(t)\Delta^{\mu_1}\ldots\Delta^{\mu_i} P^{\mu_{i+1}}\ldots P^{\mu_n} \\
+ \ & \textbf{Sym}\ \bar{N}(P_2)\frac{\mathrm{i}\sigma^{\mu\alpha}\Delta_\alpha}{2M} N(P_1) \sum_{i=0,even}^{n} B^q_{n+1,i}(t)\Delta^{\mu_1}\ldots\Delta^{\mu_i} P^{\mu_{i+1}}\ldots P^{\mu_n} \\
+ \ & \textbf{Sym}\ \bar{N}(P_2)\frac{\Delta_\mu}{M} N(P_1)\ C^q_{n+1}(t)\mathrm{mod}(n,2)\Delta^{\mu_1}\ldots\Delta^{\mu_n}
\end{aligned}$$

identifying equal terms gives

$$\left\langle J_q^3 \right\rangle = \frac{1}{2}[A_{2,0}^q(0) + B_{2,0}^q(0)] \qquad \text{Ji's sumrule}$$

in addition moments of GPDs are related to the GFFs.

$$\int_{-1}^{1} dx\, x^{n-1}\, H(x,\xi,t) = \sum_{\substack{k=0 \\ \text{even}}}^{n-1} (2\xi)^k A_{n,k}(t) + \mathrm{mod}(n+1,2)\,(2\xi)^n C_n(t)$$

$$\int_{-1}^{1} dx\, x^{n-1}\, E(x,\xi,t) = \sum_{\substack{k=0 \\ \text{even}}}^{n-1} (2\xi)^k B_{n,k}(t) - \mathrm{mod}(n+1,2)\,(2\xi)^n C_n(t)$$

Some more GPD results from QCDSF

spin-correlated transverse densities

$$\frac{1}{(2\pi)^2}\int d^2\Delta_\perp e^{ib_\perp\cdot\Delta_\perp}\int\frac{dz^-}{2\pi}e^{ix\bar{P}^+z^-}\langle P_2|\,\bar{q}(-\frac{1}{2}z)\,\gamma^+[1+\vec{s}\cdot\vec{\gamma})\gamma_5]q(\frac{1}{2}z)\,|P_1\rangle\Big|_{z^+=0}^{z_\perp=0}$$

$$= \frac{1}{2}\left[F+s^iF_T^i\right]$$

$$= \frac{1}{2}\Big[H - S^i\epsilon^{ij}b^j\frac{1}{m}E' - s^i\epsilon^{ij}b^j\frac{1}{m}\left(E'_T+2\tilde{H}'_T\right)$$

$$+ s^iS^i\left(H_T-\frac{1}{4m^2}\Delta_b\tilde{H}_T\right)+s^i(2b^ib^j-b^2\delta^{ij})S^j\frac{1}{m^2}\tilde{H}''_T\Big]$$

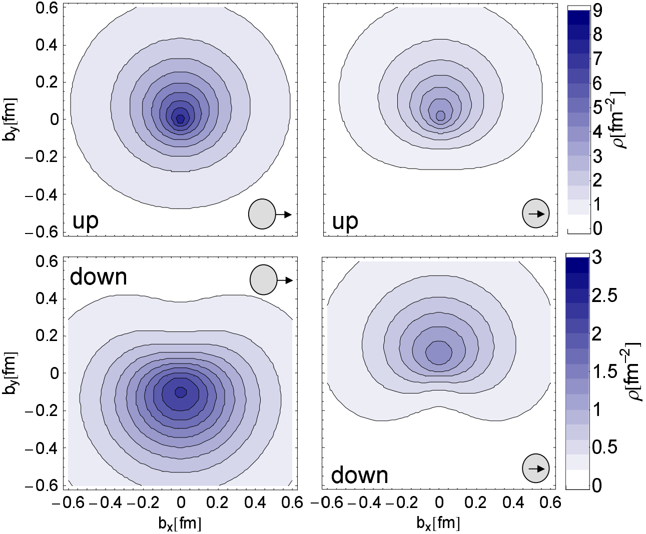
has a simple interpretation:

$S^i\epsilon^{ij}b^j$ coupling of proton spin to quark angular momentum

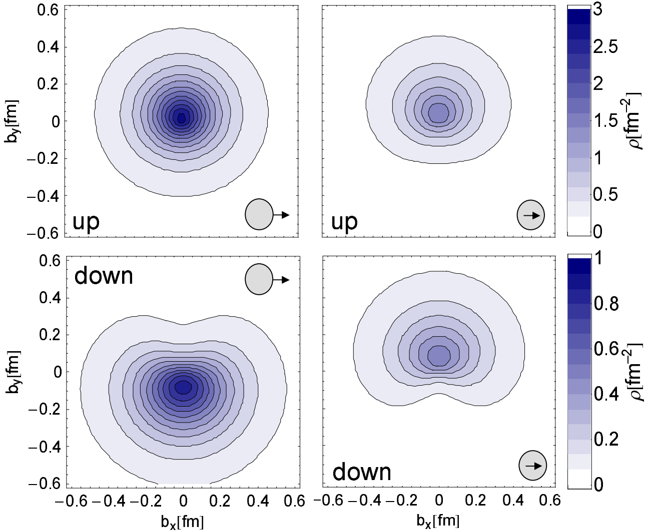
$s^i\epsilon^{ij}b^j$ coupling of quark spin to quark angular momentum

s^iS^i coupling of quark spin and proton spin

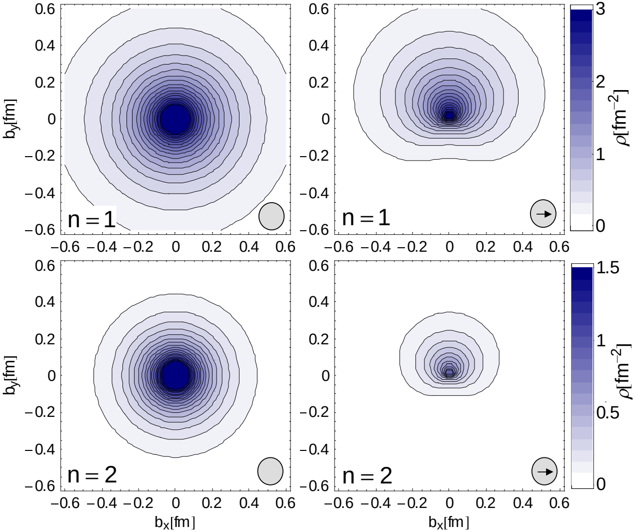
The nucleon, first moment ($n = 1$)



The nucleon, second moment ($n = 2$)



The pion first and second moment



Distribution amplitudes

Nucleon Wave Function

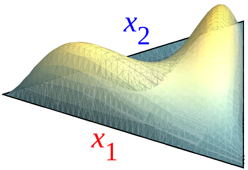
$$\Psi_{\mathrm{BS}}(x) = \langle 0| T \left[q(x_1, k_{1,\perp}) q(x_2, k_{2,\perp}) q(x_3, k_{3,\perp}) \right] |p\rangle$$

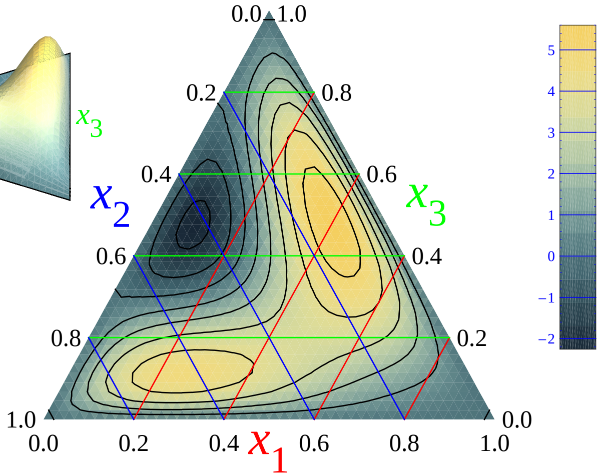
x_i: logitudinal momentum fractions carried by quarks
$k_{i,\perp}$: quarks transverse momenta
$|p\rangle$: proton state with momentum p.

Nucleon Distribution Amplitude

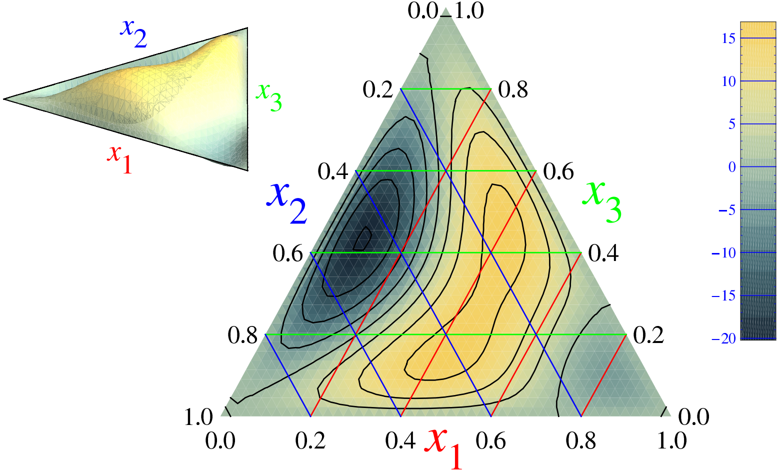
$$\Phi(x_i, \mu) = Z(\mu) \int^{|k_\perp| \leq \mu} \mathrm{d}^3 k_{i,\perp} \Psi_{\mathrm{BS}}(x, k_\perp)$$

With next-to-next-to-leading conformal spin (ϕ^{101}, ϕ^{200}, ϕ^{002})



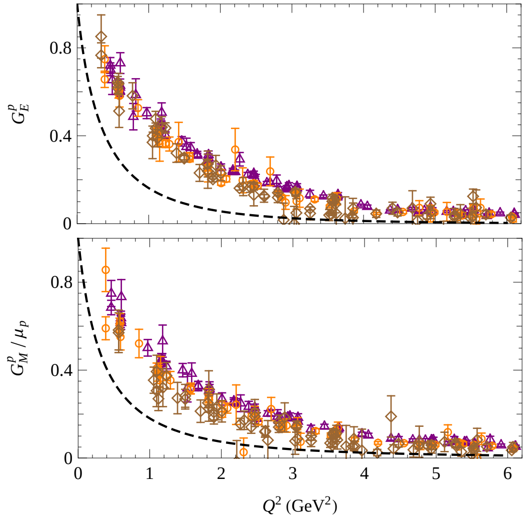


$N^*(1535)$



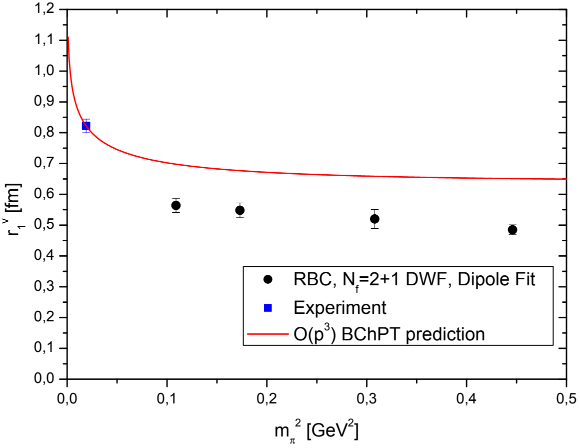
x2
x3
x1
0.0
1.0
0.2
0.8
0.4
0.6
0.6
0.4
0.8
0.2
1.0
0.0
0.0
0.2
0.4
0.6
0.8
1.0
x2
x3
x1
15
10
5
0
−5
−10
−15
−20

However, I want to concentrate one something which does not work: The nucleon form factors (H.-W. Lin (RBC))
(dashed line: Fit to experimental data)



and at order p^3 ChPT does not solve the problem

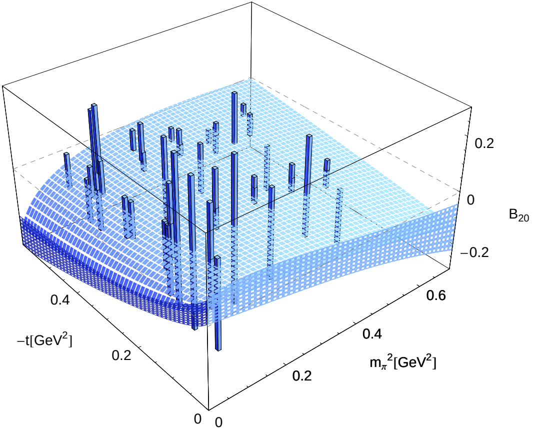
Th. Gail and Th. Hemmert: Isovector Dirac form factor



complicated m_π dependence, not easily described by dipole ansatz Do Generalized Form Factors have similar problems ?

There are two main problems:

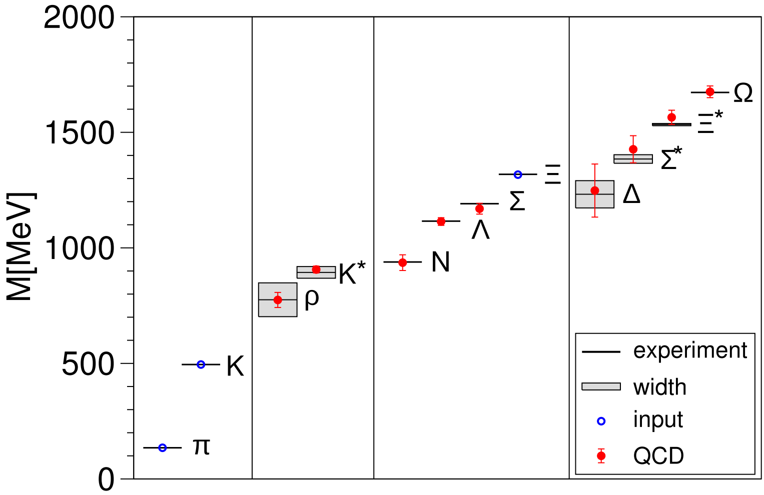
- extrapolation in m_π
- extrapolation in t



Combined fit for B_{20}^{u+d} (LHPC)

Simulations at physical pion mass become feasible

Dürr et al., BMW-Collaboration (Z. Fodor)

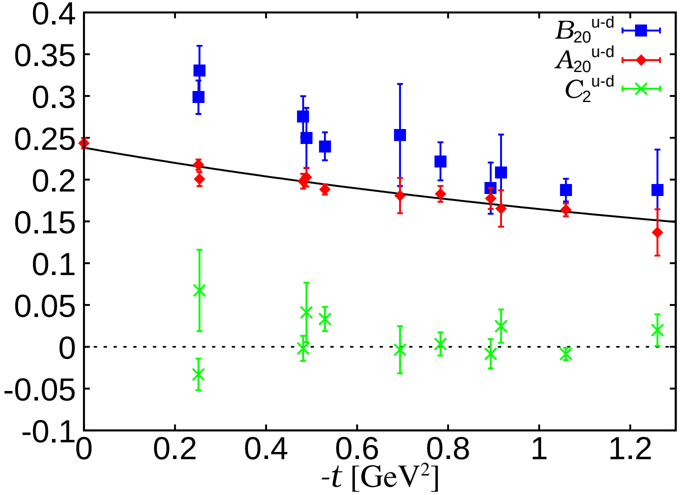


Multiplying a source with a site dependent factor $e^{\mathrm{i}\vec{p}\cdot\vec{x}_j}$ one can give it a 3-momentum $\vec{p}$.
The periodic spatial boundary conditions imply $p_n = \frac{2n\pi}{aL}$ and $q = p_n(\mathrm{source}) - p_m(\mathrm{sink})$

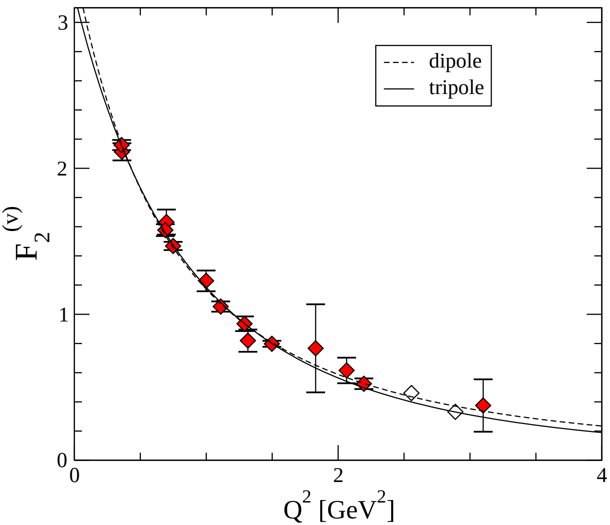
$$P_{n,min} = \frac{2\pi}{aL} \sim 400 - 600\mathrm{MeV}$$

This introduced large systematic errors for any extrapolation

The extrapolation of $B_{2,0}$ to $\Delta^2 = 0$ (QCDSF)



The same problem is encountered for the form factor F_2 (QCDSF)



The solution: twisted boundary conditions

$$
\begin{aligned}
q(x_i + L, ...) &= e^{i2\pi\theta}\, q(x_i, ...) \\
p_i &= \frac{2\pi}{L}(n_i - \theta)
\end{aligned}
$$

a practical problem: You do not want to generate new configurations for each θ

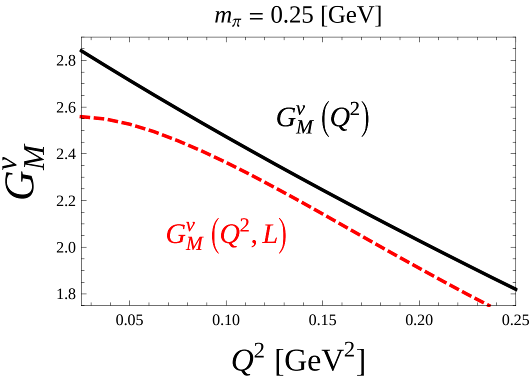
⇒ partially twisted boundary conditions (only for valence quarks

⇒ potentially large finite size artefacts

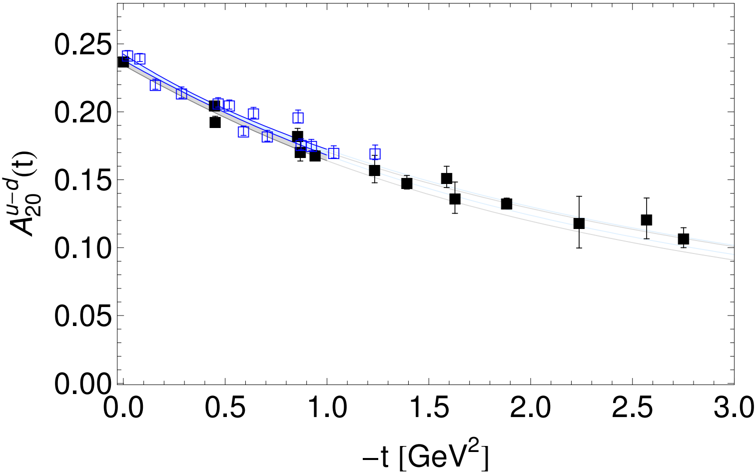
This is one of the many cases where one need input from effective field theory

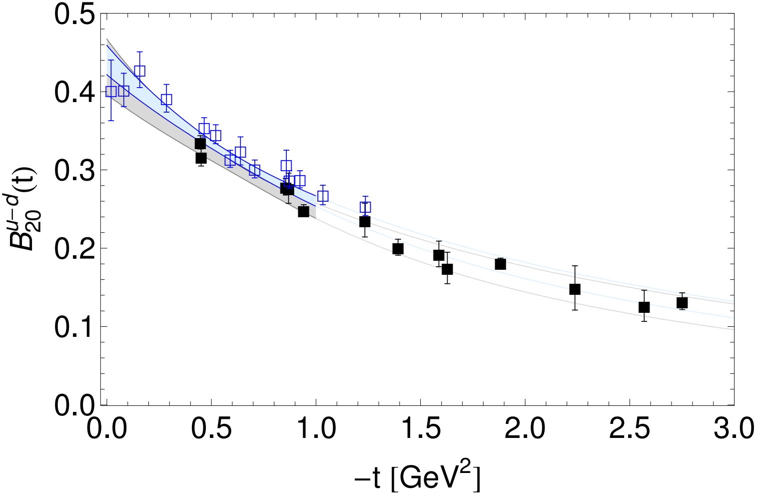
F.-J. Jiang and B.C. Tiburzi, arXiv:0810.1495

The isovector magnetic form factor for $L = 2.75$ fm

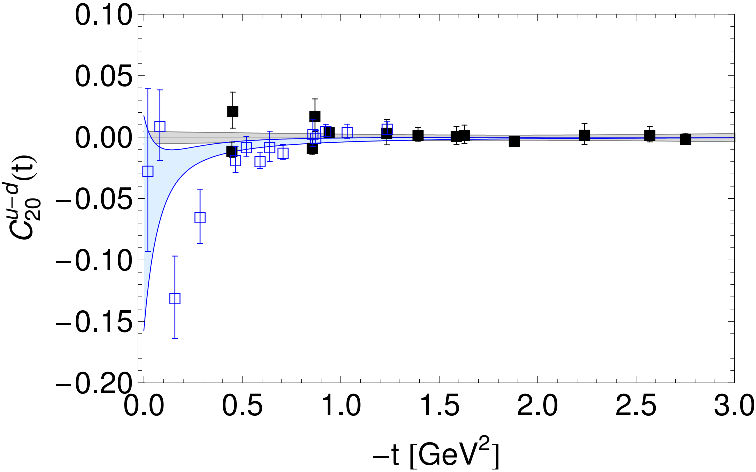


We (QCDSF) get relatively small effects for usual production lattices



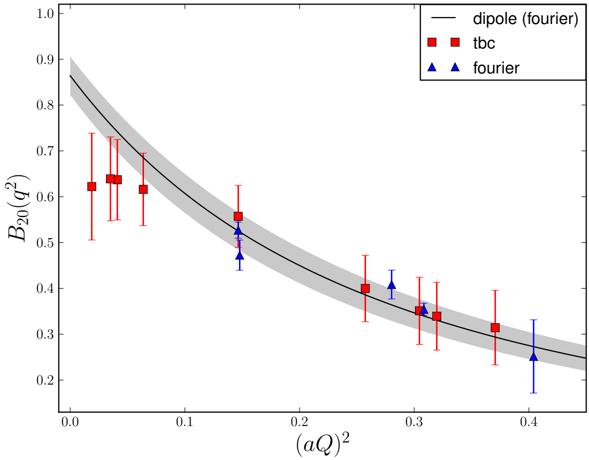


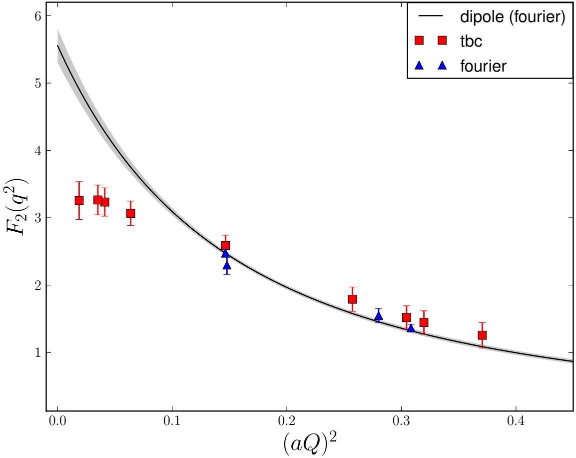
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$B_{20}^{u-d}(t)$
$-t$ [GeV^2]



0.10
0.05
0.00
−0.05
−0.10
−0.15
−0.20
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$C_{20}^{u-d}(t)$
$-t$ [GeV^2]

but for very small lattices ($L = 1.5$ fm) the effects get indeed large.





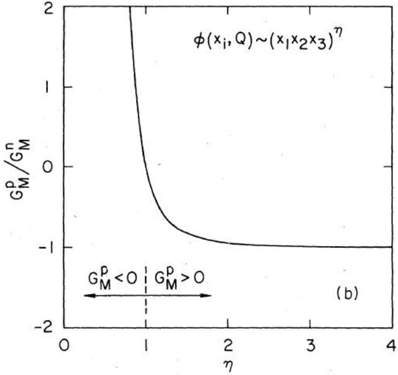
⇒ more precise investigations (Lattice QCD and effective field theory) are ongoing

This example was meant to illustrate that

- Lattice QCD has produced so many results that it has no longer to prove its usefulness
  $\Rightarrow$ The emphasis is now on control of systematic uncertainties.
- This requires typically close collaboration of Effective Field Theory, pQCD and Lattice QCD (and much work).
- But in most cases there exists a clear road-map for what has to be done

A side remark: It is known since 30 years that $F_2(t)$ (respectively $G_M(t)$) is difficult.
Brodsky and Lepage Phys. Rev **D22** (1980) 2157

$$G_E(t) = F_1(t) + \frac{t}{4m_N^2} F_2(t) \quad , \quad G_M(t) = F_1(t) + F_2(t)$$



Conclusions

- Lattice QCD provides already a very detailed picture of hadron structure
- In the next decades will bring further substantial progress
- To control systematic uncertainties one needs Lattice results with different lattice actions
- One also needs ever more precise experimental input
- Most details of hadron structure still have to be understood

The people who did the work QCDSF)

A. Ali Khan, V. Braun, D. Brömmel, M. Göckeler, Ph. Hägler, T. Hemmert, R. Horsley, T. Kaltenbrunner, D. Pleiter, M. Ohtani, P. Rakow, G. Schierholz, N. Warkentin, J. Zanotti, et al.