QCD Equation of state and susceptibilities from the lattice

I Introduction

II Lattice results on the EoS

III Lattice results on susceptibilities

IV Conclusion

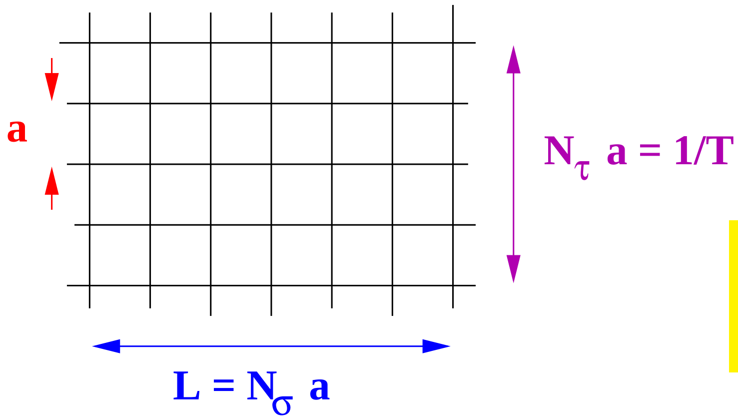
for the Bielefeld-Swansea and
the RBC-Bielefeld collaboration

I. Introduction

numerical treatment of QCD $\Rightarrow$ discretize (Euclidean) space-time

$\Rightarrow$ **lattice**

$N_\sigma^3 \times N_\tau$



$$Z(T,V) = \int \prod_{i=1}^{N_\tau N_\sigma^3} d\phi(x_i) \exp\{-S[\phi(x_i)]\}$$

finite yet high-dimensional path integral

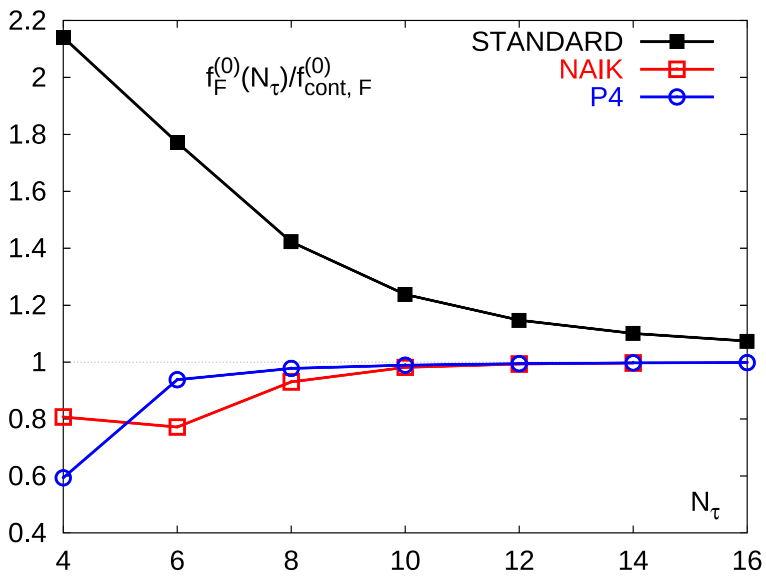
$\rightarrow$ **Monte Carlo**

- thermodynamic limit, IR - cut-off effects $LT = \frac{N_\sigma}{N_\tau} \rightarrow \infty$
- continuum limit, UV - cut-off effects $aT = \frac{1}{N_\tau} \rightarrow 0$
- chiral limit $m \rightarrow m_{\mathrm{phys}} \simeq 0$

  numerical effort $\sim (1/m)^p$

- reducing UV cut-off effects: Naik action

  p4 action



- improving flavor symmetry: various fat link prescriptions

  fat7, fat3, stout

- exact algorithm (RHMC (RHMC))

pressure $(\mu = (\mu_u, \mu_d, \ldots))$

$$\frac{p}{T^4} = \Omega(T,\mu) = \frac{1}{VT^3}\ln Z(T,\mu) = \sum_{n=0}^{\infty} c_n(T,m_q)\left(\frac{\mu}{T}\right)^n$$

with

$$c_n(T,m_q) = \frac{1}{n!}\frac{1}{VT^3}\frac{\partial^n \ln Z(T,\mu)}{\partial(\mu/T)^n}\Big|_{\mu=0}$$

number density

$$\frac{n_q}{T^3} = \frac{1}{VT^3}\frac{\partial \ln Z(T,\mu)}{\partial(\mu/T)} = \sum_{n=2}^{\infty} n c_n(T,m_q)\left(\frac{\mu}{T}\right)^{n-1}$$

interaction measure

$$\frac{\epsilon - 3p}{T^4} = \sum_{n=0}^{\infty} c_n'(T,m_q)\left(\frac{\mu}{T}\right)^n$$

with

$$c_n'(T,m_q) = T\frac{dc_n(T,m_q)}{dT}$$

from those, energy density

$$\frac{\epsilon}{T^4} = \sum_{n=0}^{\infty}(3c_n(T,m_q) + c_n'(T,m_q))\left(\frac{\mu}{T}\right)^n$$

and entropy density

$$\frac{s}{T^3} = \frac{\epsilon + p - \mu n_q}{T^4} = \sum_{n=0}^{\infty}((4-n)c_n(T,m_q) + c_n'(T,m_q))\left(\frac{\mu}{T}\right)^n$$

diagonal and off-diagonal susceptibilities

$$\frac{\chi_{ff}(T,\mu)}{T^2} = \frac{\partial^2\Omega(T,\mu)}{\partial(\mu_f/T)^2}$$

$$\frac{\chi_{fk}(T,\mu)}{T^2} = \frac{\partial^2\Omega(T,\mu)}{\partial(\mu_f/T)\partial(\mu_k/T)}$$

with $\mu_q = \frac{1}{2}(\mu_u + \mu_d)$ and $\mu_I = \frac{1}{2}(\mu_u - \mu_d)$

quark number susceptibility

$$\frac{\chi_q(T,\mu)}{T^2} = \frac{\partial^2\Omega(T,\mu)}{\partial(\mu_q/T)^2} = 2\,(\chi_{uu} + \chi_{ud})$$

isovector susceptibility

$$\frac{\chi_I(T,\mu)}{T^2} = \frac{\partial^2\Omega(T,\mu)}{\partial(\mu_I/T)^2} = 2\,(\chi_{uu} - \chi_{ud})$$

charge susceptibility

$$\frac{\chi_Q(T,\mu)}{T^2} = \frac{\partial^2\Omega(T,\mu)}{\partial(\mu_Q/T)^2} = \frac{1}{9}\,(5\chi_{uu} - 4\chi_{ud})$$

and higher moments/derivatives

(A) **high temperature** : **perturbation theory** [Vuorinen]

$$\Omega(T,\mu) = \Omega^{(0)}(T,\mu) + g^2\,\Omega^{(2)}(T,\mu) + g^3\,\Omega^{(3)}(T,\mu) + \mathcal{O}(g^4)$$

with Stefan-Boltzmann (free gas) limit

$$\frac{p_{SB}}{T^4} = \Omega^{(0)}(T,\mu) = \frac{8\pi^2}{45} + \sum_{f=u,d,..}\left[\frac{7\pi^2}{60} + \frac{1}{2}\left(\frac{\mu_f}{T}\right)^2 + \frac{1}{4\pi^2}\left(\frac{\mu_f}{T}\right)^4\right]$$

diagonal suscept.

$$\frac{\chi_{ff}(T,\mu)}{T^2} = 1 + \frac{3}{\pi^2}\left(\frac{\mu_f}{T}\right)^2 + \mathcal{O}(g^2)$$

off-diagonal suscept.

$$\frac{\chi_{fk}(T,\mu)}{T^2} = g^3\,\kappa\,\frac{\mu_f}{T}\frac{\mu_k}{T} + \mathcal{O}(g^4)$$

$$\frac{\chi_{fk}(T,0)}{T^2} = -\frac{5}{144\pi^6}\,g^6 \ln 1/g$$

(B) **low temperature** : **hadron resonance gas model**

$$\Omega_{HRG}(T,\mu_q,\mu_I) = \sum_{i\in\,mesons} \Omega^M_{m_i}(T,\mu_q,\mu_I) + \sum_{i\in\,baryons} \Omega^B_{m_i}(T,\mu_q,\mu_I)$$

where

$$\Omega^{M/B}_{m_i} = \frac{1}{2\pi^2}\left(\frac{m_i}{T}\right)^2 \sum_{\ell=1}^{\infty} \left\{ \frac{1}{(-1)^{\ell+1}} \right\} \ell^{-2} K_2\left(\frac{\ell m_i}{T}\right) z_i^\ell \qquad \text{with} \quad z_i = \exp((3B_i\mu_q + 2I_{3i}\mu_I)/T)$$

fugacities

- for baryons, $\ell \geq 2$ terms can safely be neglected[1] $\Rightarrow$ at $\mu_I = 0$:

$$\frac{p(T,\mu_q,\mu_I=0)}{T^4} \simeq G(T) + F(T)\cosh\left(\frac{3\mu_q}{T}\right) \qquad \Rightarrow \qquad \frac{\chi_q}{T^2} = 9F(T)\cosh\left(\frac{3\mu_q}{T}\right)$$

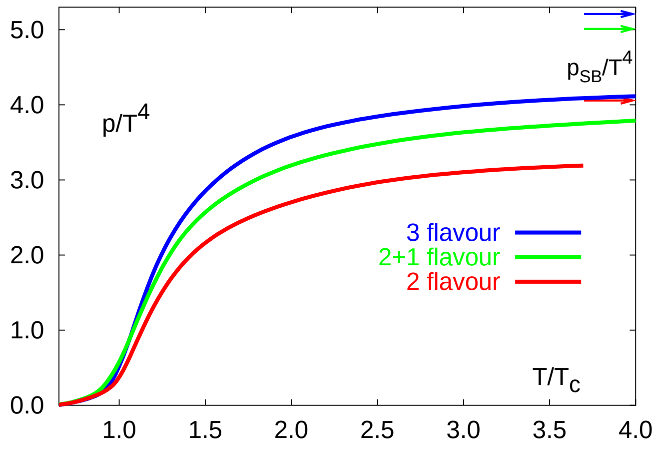
likewise, $\dfrac{\chi_I(T,\mu_q,\mu_I=0)}{T^2} \simeq G^I(T) + F^I(T)\cosh\left(\dfrac{3\mu_q}{T}\right)$

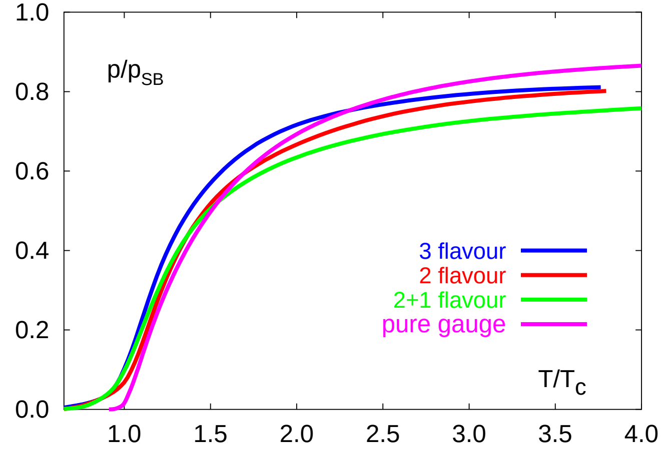
- For all quantities X of the form $X = G^X(T) + F^X(T)\cosh(3\mu_q/T)$:

$$X = \sum_{n=0}^{\infty} c_n^X(T)\,(\mu_q/T)^n \qquad \text{with} \qquad \frac{c^X_{2n+2}}{c^X_{2n}} = \frac{9}{(2n+2)(2n+1)} \quad \text{for } n \geq 1$$

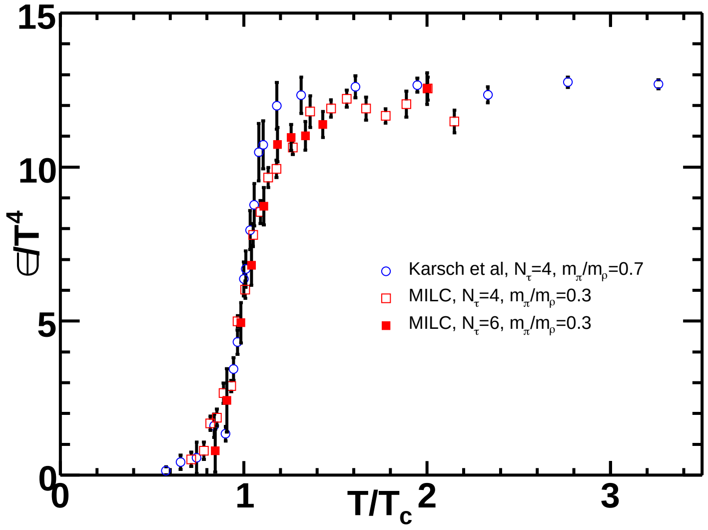
[1] $K_2(x) \sim e^{-x}(1+P(1/x))/\sqrt{x}$

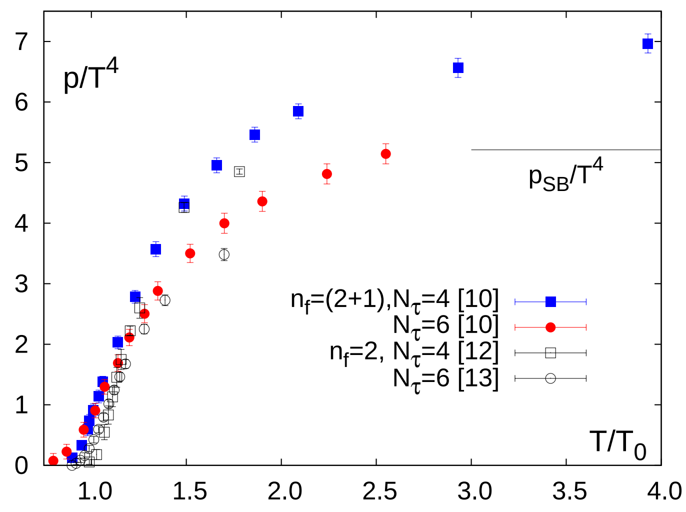
- old results: $16^3 \times 4, \quad m_\pi/m_\rho \simeq 0.7$ [Karsch, EL, Peikert]





- new results: MILC, LAT2005 Y.Aoki et al., JHEP 2006





quark masses seem to not matter too much – controlling/reducing UV effects important

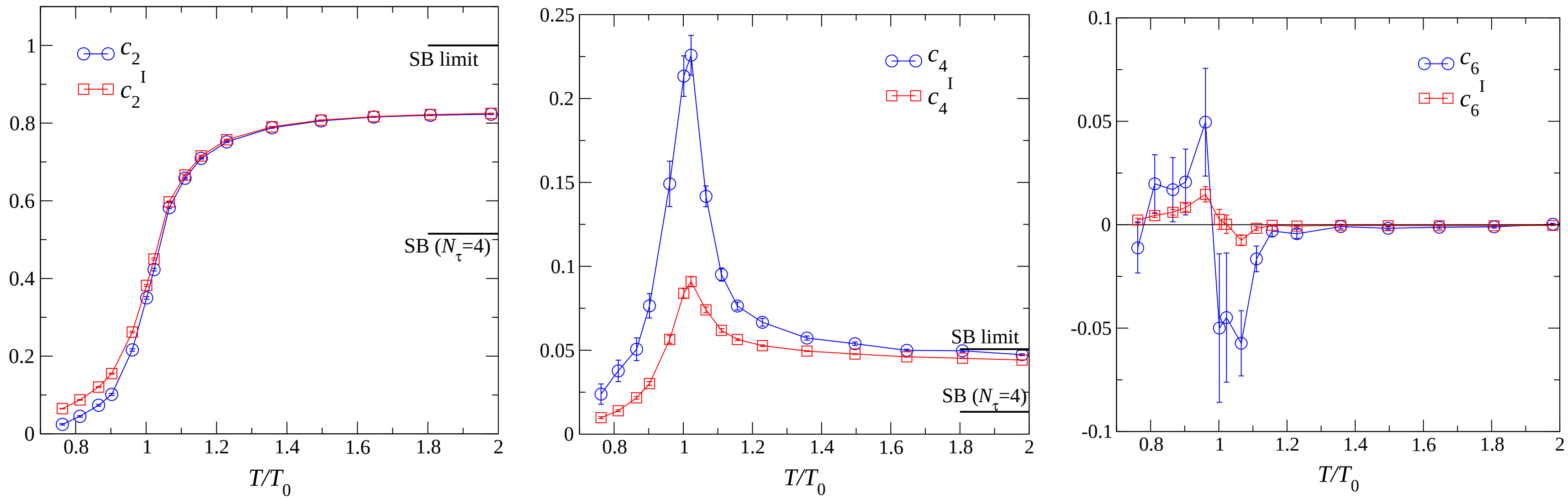
Taylor expansion coefficients

pressure
$$\frac{\Delta p}{T^4} = \frac{p(\mu_q)}{T^4} - \frac{p(\mu_q=0)}{T^4} = c_2\left(\frac{\mu_q}{T}\right)^2 + c_4\left(\frac{\mu_q}{T}\right)^4 + c_6\left(\frac{\mu_q}{T}\right)^6 + \cdots$$

quark number
$$\frac{n_q(T,\mu_q)}{T^3} = 2c_2\left(\frac{\mu_q}{T}\right) + 4c_4\left(\frac{\mu_q}{T}\right)^3 + 6c_6\left(\frac{\mu_q}{T}\right)^5 + \cdots$$

q number suscept.
$$\frac{\chi_q(T,\mu_q)}{T^2} = 2c_2 + 12c_4\left(\frac{\mu_q}{T}\right)^2 + 30c_6\left(\frac{\mu_q}{T}\right)^4 + \cdots$$

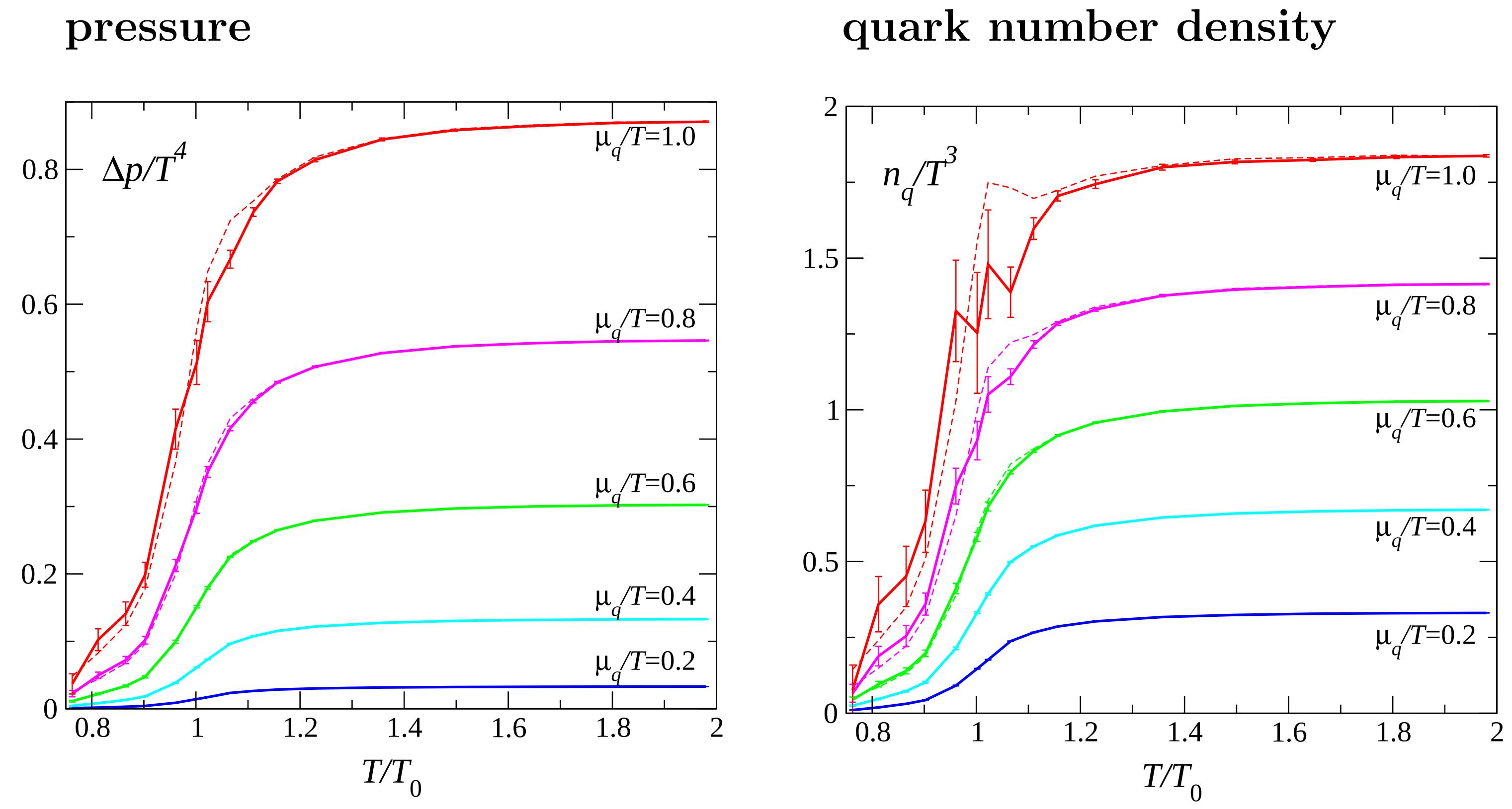
Isovector suscept.
$$\frac{\chi_I(T,\mu_q)}{T^2} = 2c_2^I + 12c_4^I\left(\frac{\mu_q}{T}\right)^2 + 30c_6^I\left(\frac{\mu_q}{T}\right)^4 + \cdots$$



- approaching SB
- discretisation effects !

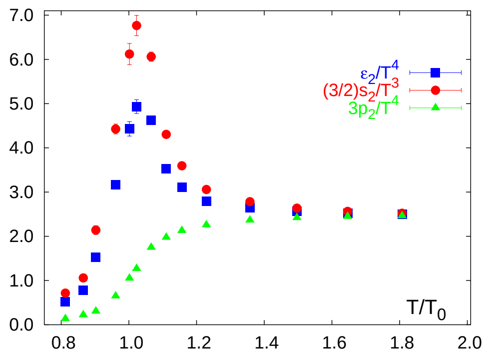
- peak around T_c

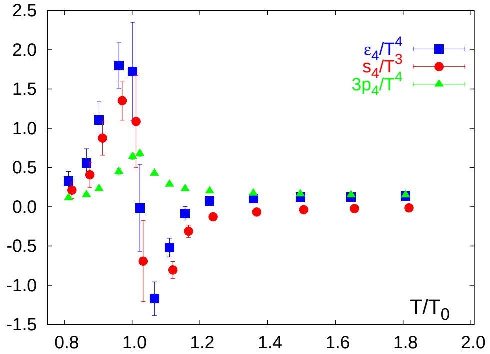
- sign change around $T \lesssim T_c$
- small

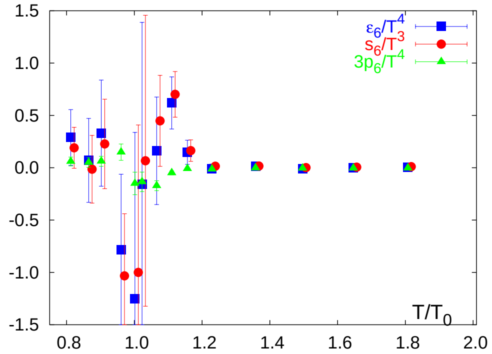


- comparison: up to $\mathcal{O}(\mu^4)$ (dashed) with up to $\mathcal{O}(\mu^6)$ (full) suggests rapid convergence
- contribution to total p is small: $p(\mu = 0)/T^4 \simeq \mathcal{O}(4)$

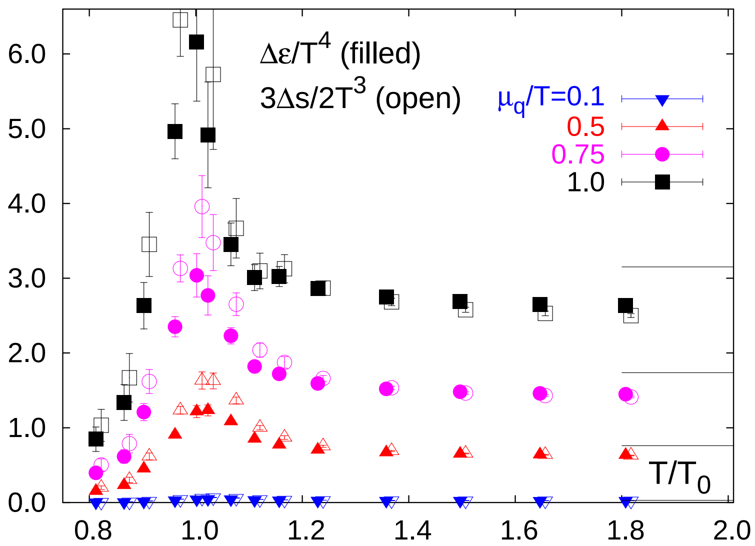
together with the c'_n coefficients ...



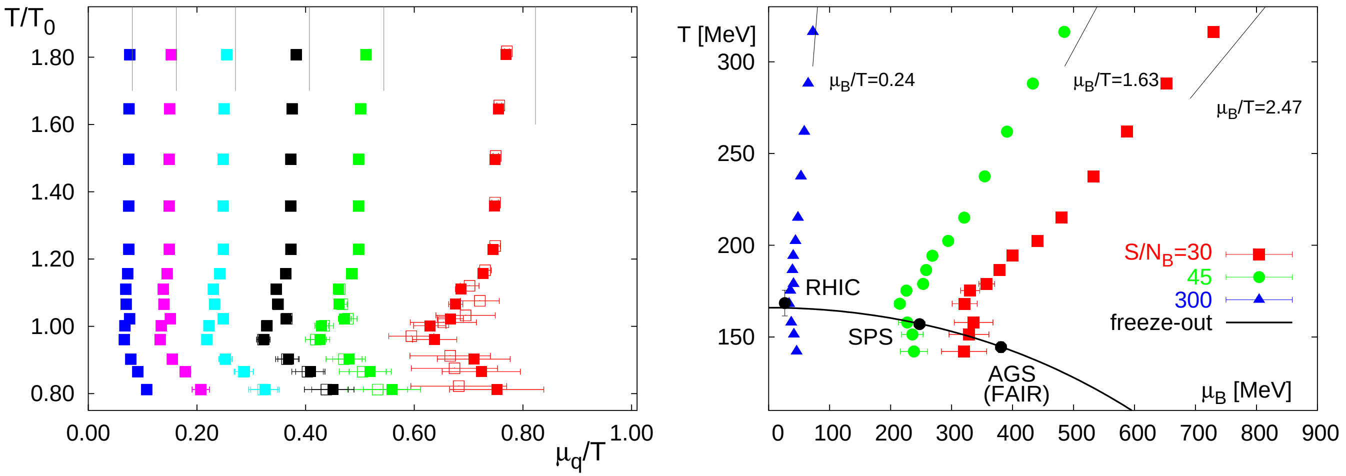




obtain **energy** and **entropy**



it is generally believed that the fireball expansion follows a line of fixed S/N_B



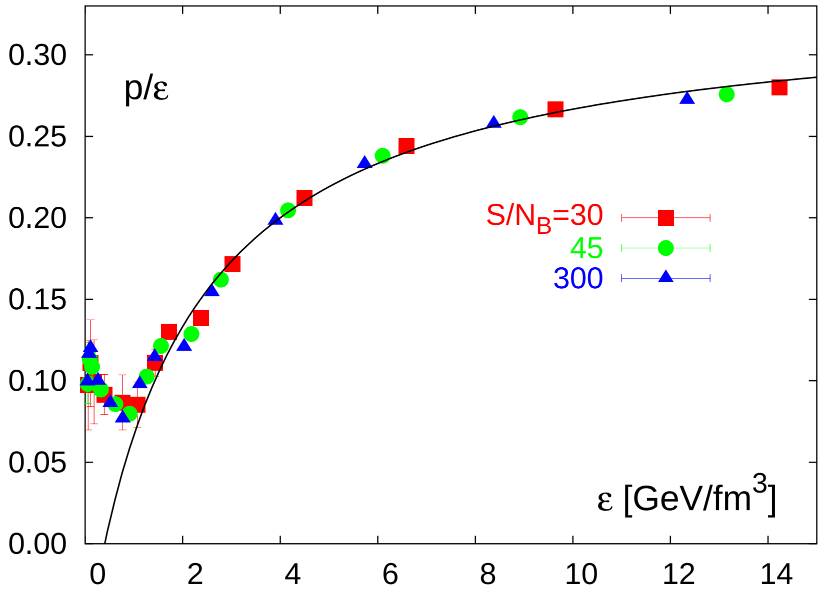
in the ideal gas limit
$$\frac{S}{N_B} = 3\frac{\frac{37\pi^2}{45} + (\frac{\mu_q}{T})^2}{\frac{\mu_q}{T} + \frac{1}{\pi^2}(\frac{\mu_q}{T})^3} \qquad \Rightarrow \frac{\mu_q}{T} = \text{const (vertical lines)}$$

isentropic expansion lines for SPS: $S/N_B \simeq 45$

RHIC: $S/N_B \simeq 300$

FAIR: $S/N_B \simeq 30$

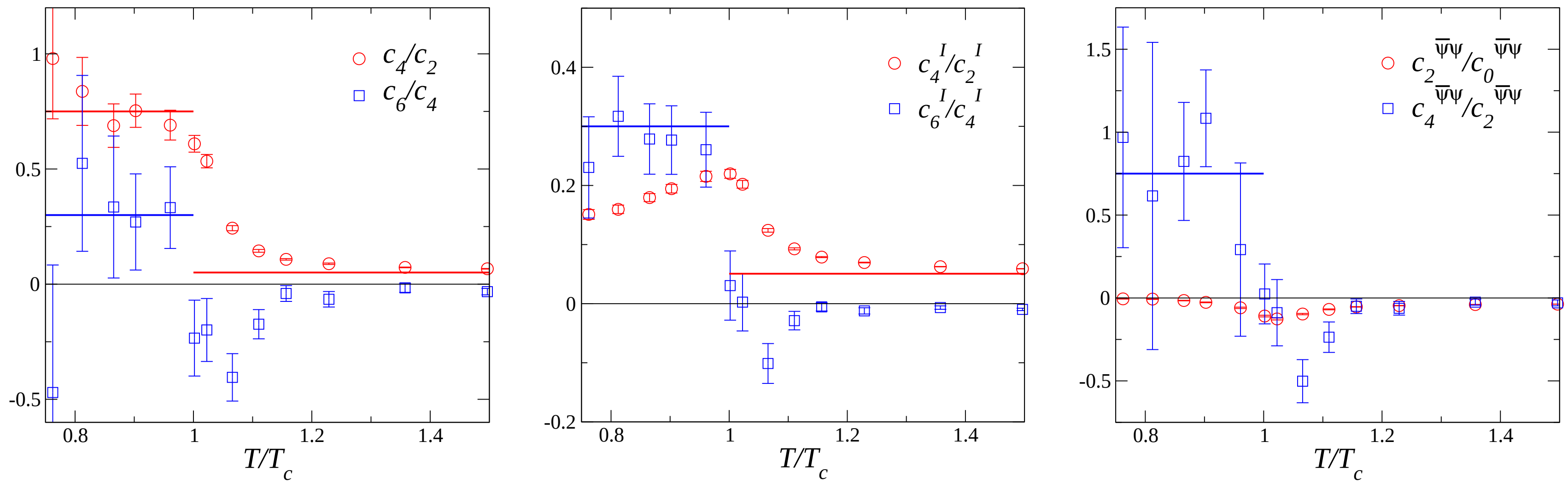
keep in mind: feasibility study of what one can do with lattice data



- $p(\epsilon)$ to a good approximation independent of S/N_B
- $p(\epsilon)$ well parametrized by

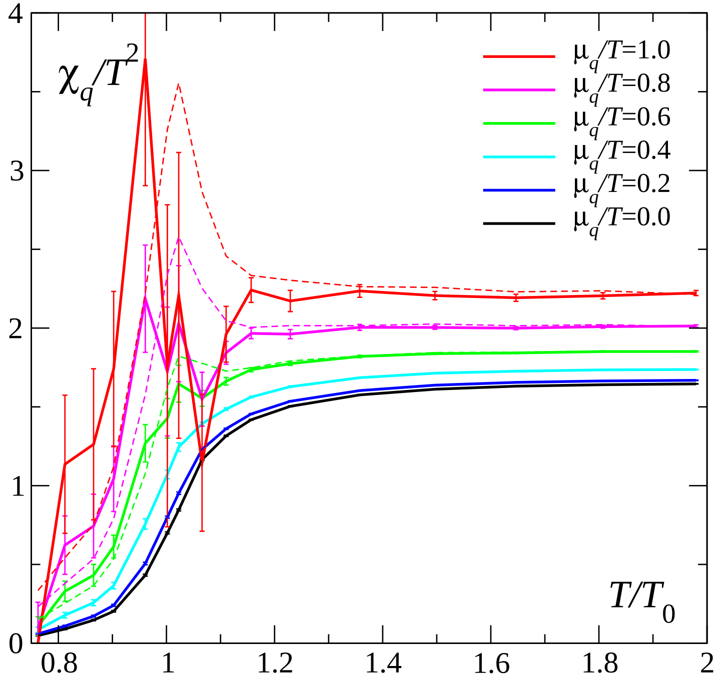
$$\frac{p}{\epsilon} \simeq \frac{1}{3}\left(1 - \frac{1.2}{1 + 0.5\,\epsilon\ \mathrm{fm}^3/\mathrm{GeV}}\right)$$

recall: ratios $\frac{c_{2n}}{c_{2n+2}}$ allow comparison with the hadron resonance gas model fairly detail independent

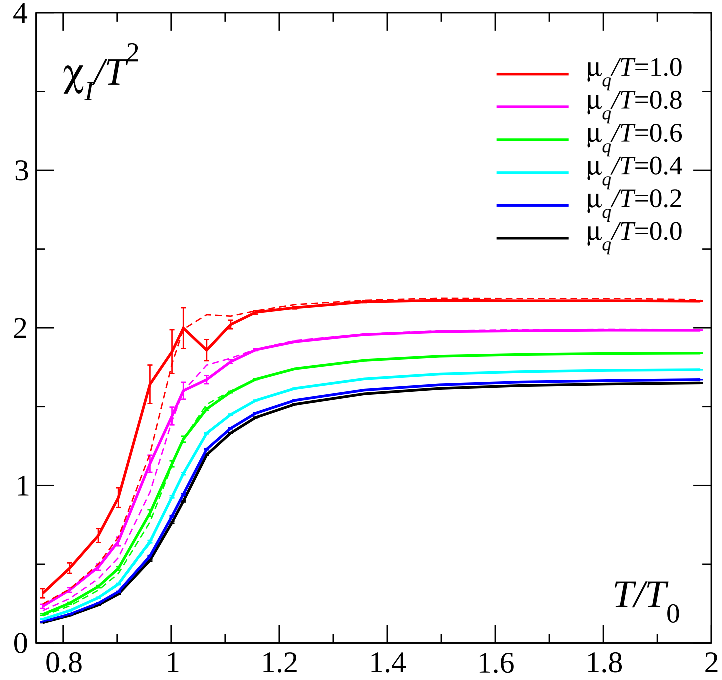


- above T_0: ratios approaching SB values
- below T_0: ratios except those involving $c_0, c_2^I, c_0^{\bar{\psi}\psi}$ (depend on $G^X(T)$) are
  - temperature independent
  - taking hadron resonance gas values $\rightarrow$ do not indicate critical behavior

quark number susceptibility

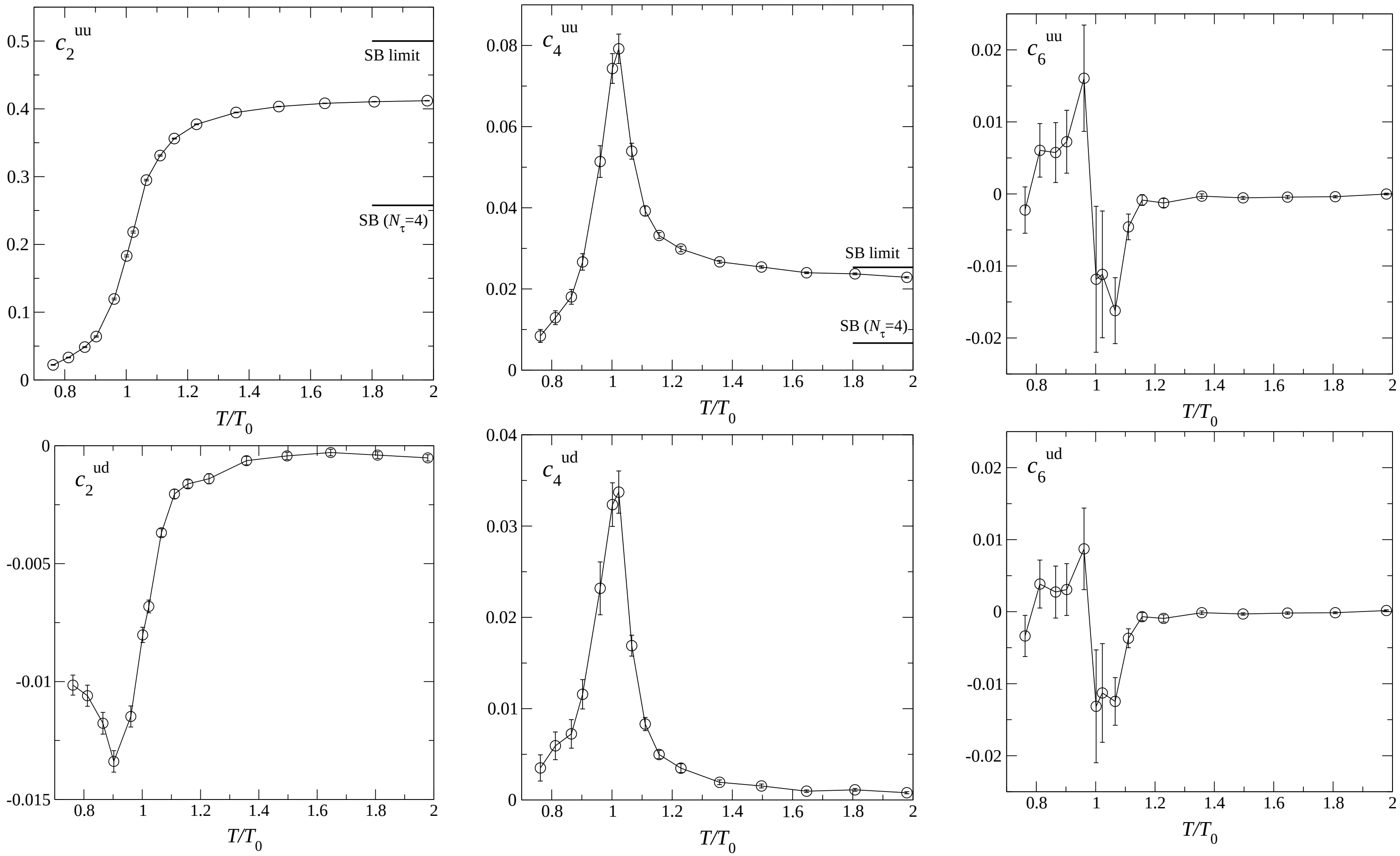


isovector susceptibility



- again comparison: up to $\mathcal{O}(\mu^4)$ (dashed) with up to $\mathcal{O}(\mu^6)$
- peak in χ_q developing with increasing μ, coming from c_4
- c_6 shifts peak in χ_q to smaller T
- peak less convincing because of error bars and dip $\longrightarrow$ more statistics needed here
- no peak in χ_I $\longrightarrow$ strong correlations between χ_{uu} and χ_{ud}

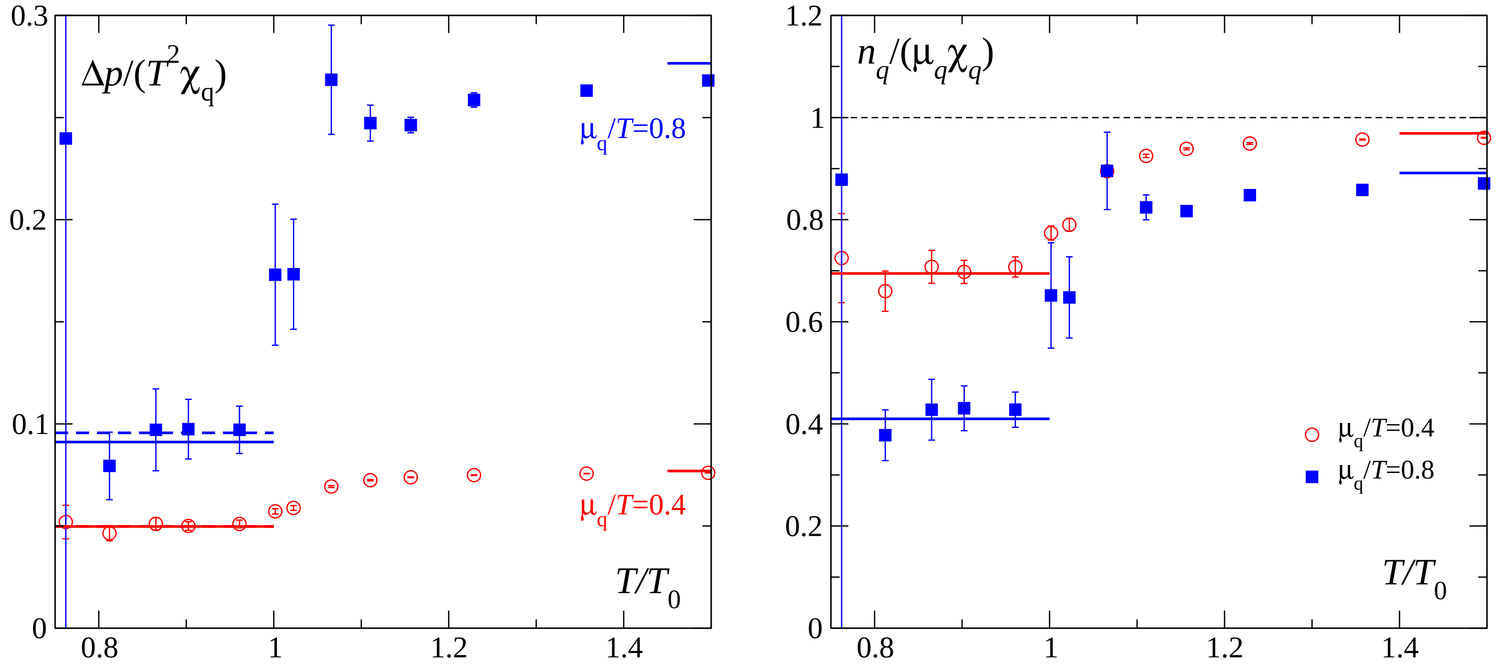
diagonal and off-diagonal susceptibilities



- at $T > T_0$: χ_{uu} and χ_{ud} approach SB limit, i.e. $\chi_{ud} \to 0$
- at $T > T_0$ signs in agreement with perturbation theory [Blaizot, Iancu, Rebhan]
- at $T \lesssim T_0$: $\chi_{ud} \neq 0$
- around T_0: $c_n^{ud} \simeq c_n^{uu}$ for $n > 2 \to$ at $\mu_q = \mu_c$, peaks in both, χ_{uu} and χ_{ud}

  $\to$ at $\mu_q > 0$, fluctuations in different flavor channels are correlated

- χ_q rises rapidly with increasing μ_q, but rise to a large extent due to rise in pressure



- $$\frac{\partial p}{\partial n_q} = \frac{\partial p/\partial \mu_q}{\partial n_q/\partial \mu_q} = \frac{n_q}{\chi_q} = \frac{1}{\kappa_T\, n_q} \to 0$$ at 2nd order phase transition (isothermal compressibility $\kappa_T \to \infty$)

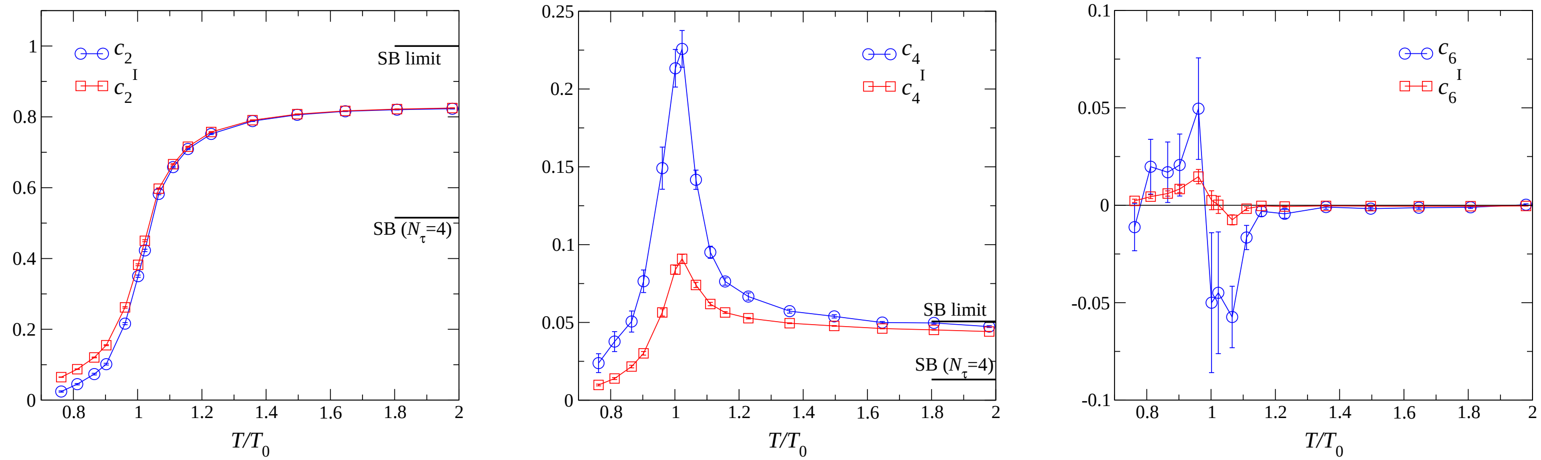
- no indication of criticality

- but, for $T \leq 0.96 T_c$, consistency with hadron resonance gas model (at $\mu_I = 0$)

$$\frac{n_q^{HRG}}{\mu_q \chi_q^{HRG}} = \frac{T}{3\mu_q} \tanh\left(\frac{3\mu_q}{T}\right)$$

(generalized) susceptibilities related to fluctuations, with $d_n = n!\, c_n$

$$d_2^x \sim \langle(\delta n_x)^2\rangle \qquad x = q, I, Q \qquad d_4^x \sim \langle(\delta n_x)^4\rangle - 3\langle(\delta n_x)^2\rangle^2$$

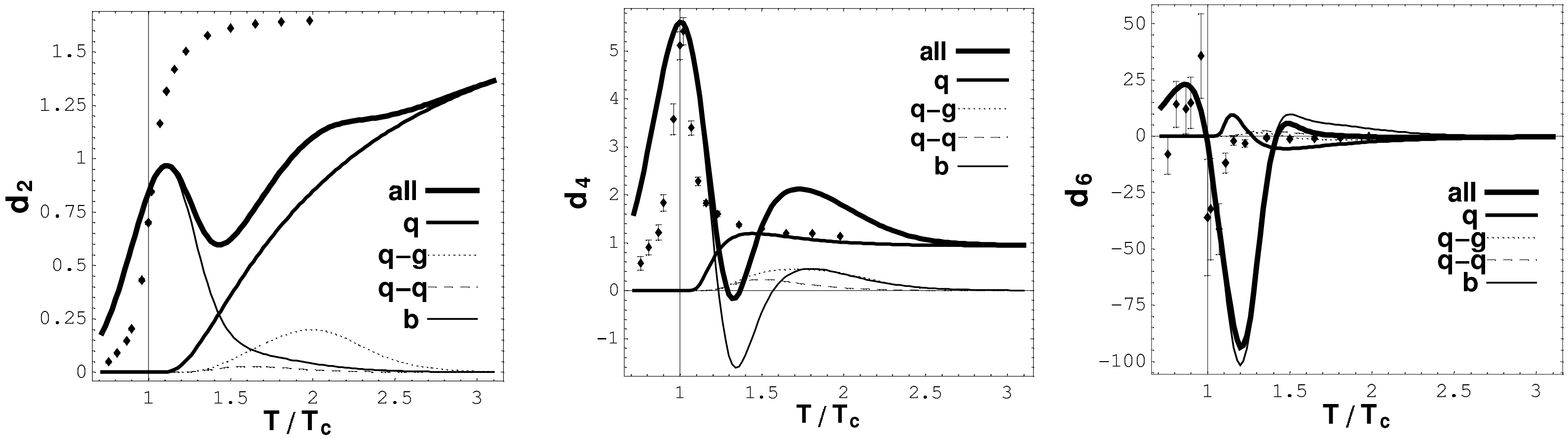


above T_c: drop and reasonable approach to the high temperature ideal gas values suggest

that for $T \gtrsim 1.5 T_c$ n_q and Q are carried by quarks [Ejiri, Karsch, Redlich] (generalized) susceptibilities related to fluctuations, with $d_n = n!\, c_n$

$$d_2^x \sim \langle (\delta n_x)^2 \rangle \qquad x = q, I, Q \qquad d_4^x \sim \langle (\delta n_x)^4 \rangle - 3 \langle (\delta n_x)^2 \rangle^2$$

compare with sQCD [Liao, Shuryak]



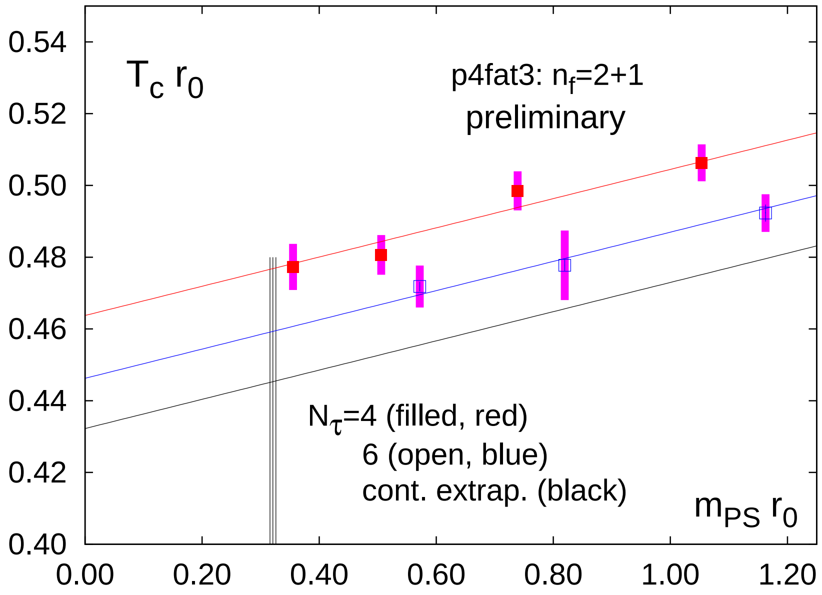
main features (peak in d_4 and wiggle in d_6) caused by T dependent baryon mass

IV. Conclusion

- ▷ caveat: most of the results presented were obtained on coarse lattices and large quark masses

- ▷ below T_c, EoS and suscepts. reasonably well described by hadron resonances
- ▷ the transition region also seems to be dominated by resonance rather than by chiral dynamics:
  - weak dependence on quark mass, $\epsilon_c/T_c^4 = 6 \pm 2$ still holds
- ▷ above $T \geq 1.5T_c$, net quark number and charge predominantly carried by d.o.fs with quark quantum numbers
- ▷ isentropic EoS fairly independent of S/N_B

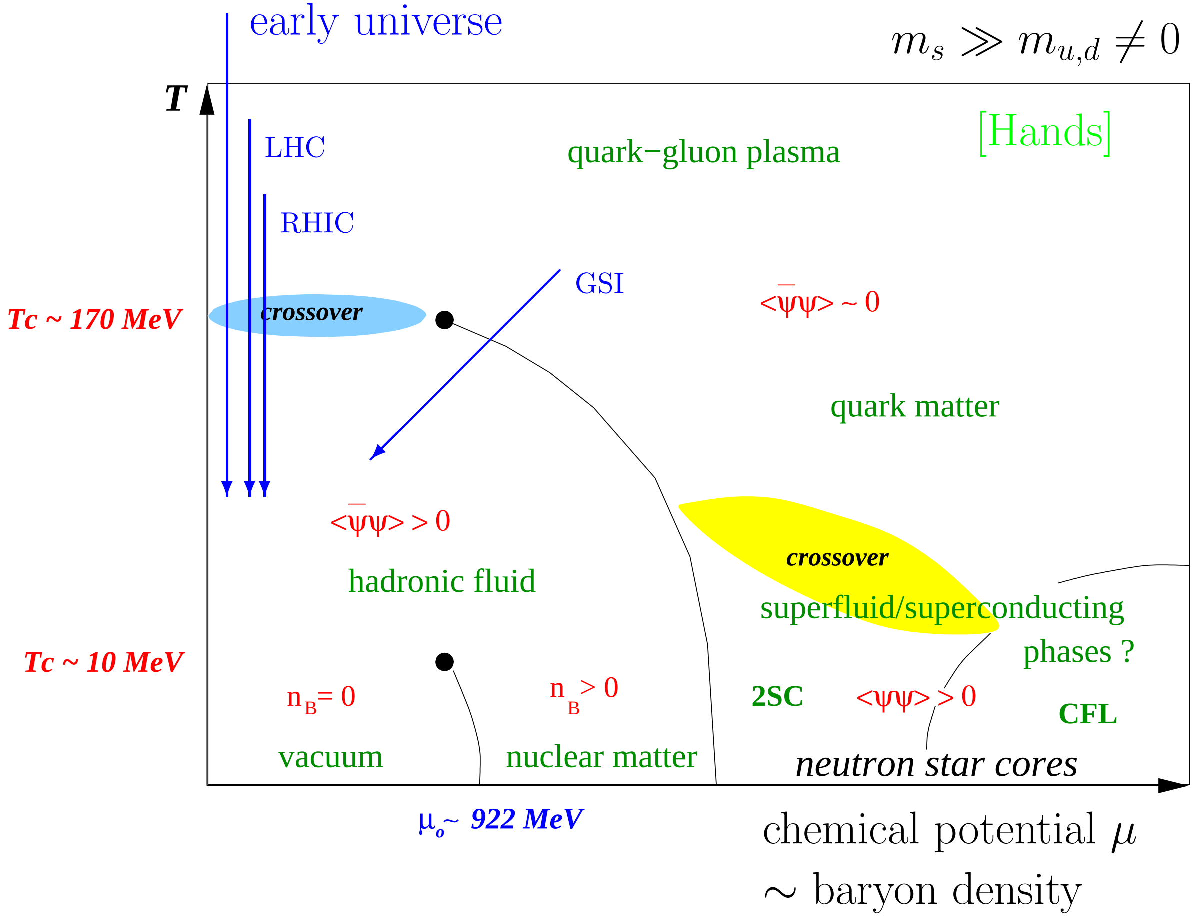
- ▷ results closer to continuum limit and at small quark masses are coming in
- ▷ thanks to dedicated TFlops machines available now !



$$T_c(m_{PS}, a) = T_c(m_\pi, 0) + c_1 m_{PS} + c_2 a^2 \qquad \Rightarrow \qquad T_c(m_\pi, 0) = 180 MeV \pm ?$$

Phase diagram

expected properties :



in detail dependent on
masses of light flavors

$m_{u,d} \ll m_s \quad N_F = 2$

$m_{u,d} < m_s \quad N_F = 2 + 1$

$m_{u,d} \simeq m_s \quad N_F = 3$

see e.g. Rajagopal, Wilczek, hep-ph/0011333