Poincaré invariance in low-energy EFTs for QCD

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based on [Berwein, Brambilla, Hwang, Vairo, TUM-EFT 74/15]

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- QCD contains both perturbative (asymptotic freedom) [Wilczek, Gross, PRL (1973), Politzer, PRL (1973)] and non-perturbative aspects (confinement) [Wilson, PRD (1974)]

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- Frameworks of EFT enable to do so
- Variety of EFTs for QCD depending on physical processes (HQET, NRQCD, pNRQCD, SCET)
- Focus on a pair of heavy quark and antiquark bound state (e.g, bottomonium, charmonium)

Heavy quark-antiquark pair

Non-relativistic QCD (NRQCD)

[Caswell, Lepage, Phys. Lett. B (1986), Bodowin, Braaten, Lepage PRD (1995)]

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- Limitation: not the most useful theory for bound states

From NRQCD to pNRQCD

Potential NRQCD [Pineda, Soto, PRD (1998), Brambilla, Pineda, Soto, Vairo, Nucl. Phys. B (2000)]: a nice theoretical tool for describing the mass spectra of heavy quark-antiquark bound system

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- Operators with Wilson coefficients

Wilson coefficients in EFT

Wilson coefficients c_n: undetermined scalar functions in front of the series of operators in EFT (e.g., HQET/NRQCD):

$$\mathcal{L}_{NRQCD} = \sum_n c_n \frac{\mathcal{O}_n}{M^{d_n-4}}, \quad \text{where} \quad [\mathcal{O}_n] = d_n$$

- $\mathcal{O}_n$ consist of effective DOF

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- Benefits from symmetry: reduce the matching calculation
- Poincaré invariance from its UV theory is the symmetry used

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- Lagrangian structure of pNRQCD and field transformations
- Field redefinitions and constraints in pNRQCD
- Outlook (Effective String Theory)

Relativistic field under spatial boost

Field transformation: $\psi_a \to \mathcal{M}(\Lambda)_{ab}\psi_b(\Lambda^{-1}x)$ with $\mathcal{M}(\Lambda)$ a representation of the Poincaré group

- Generic field under the spatial boost $\mathcal{B}$:

$$\psi_a(x) \to (e^{\mp\boldsymbol{\eta}\cdot\boldsymbol{\Sigma}})_{ab}\psi_b(\mathcal{B}^{-1}x)$$

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- Massive field [Heinonen, Hill, Solon, PRD (2012)]:

$$\psi_a(x) \to \exp\left[\mp\boldsymbol{\eta}\cdot\left(\frac{\boldsymbol{\Sigma}\times\boldsymbol{\partial}}{M+\sqrt{M^2-\boldsymbol{\partial}^2}}\right)\right]_{ab}\psi_b(\mathcal{B}^{-1}x),$$

  with reference frame chosen $v=(1,0,0,0)$ (e.g., rest frame of heavy particle)

NR field under spatial boost

- Non-relativistic expansion [Heinonen, Hill, Solon, PRD (2012)]:

$$\psi_a(x) \to \Big\{1 + iM\boldsymbol{\eta}\cdot\mathbf{x} - \frac{i\boldsymbol{\eta}\cdot\boldsymbol{\partial}}{2M} - \frac{i\boldsymbol{\eta}\cdot\boldsymbol{\partial}\boldsymbol{\partial}^2}{4M^3} + \frac{(\boldsymbol{\Sigma}\times\boldsymbol{\eta})\cdot\boldsymbol{\partial}}{2M}\left[1 + \frac{\boldsymbol{\partial}^2}{4M^2}\right] + \mathcal{O}(1/M^4)\Big\}\psi_a(\mathcal{B}^{-1}x)$$

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- Boost of the interacting field with symmetries under C, P, T, and rotation [Berwein, Brambilla, Hwang, Vairo, TUM-EFT 74/15]:

$$\psi_a(x) \to \Big\{1 + iM\boldsymbol{\eta}\cdot\mathbf{x} - \frac{ik_1\boldsymbol{\eta}\cdot\boldsymbol{D}}{2M} + \frac{k_2(\boldsymbol{\Sigma}\times\boldsymbol{\eta})\cdot\boldsymbol{D}}{2M} - \frac{ik_3\boldsymbol{\eta}\cdot\boldsymbol{D}\boldsymbol{D}^2}{4M^3} + \frac{k_4(\boldsymbol{\Sigma}\times\boldsymbol{\eta})\cdot\boldsymbol{D}}{2M}\frac{\boldsymbol{D}^2}{4M^2} + \mathcal{O}(g, 1/M^4)\Big\}\psi_a(\mathcal{B}^{-1}x)$$

NRQCD Lagrangian [Caswell, Lepage, Phys. Lett. B (1986)]

Bilinear sector of the NRQCD Lagrangian up to $1/M^2$:

$$
\begin{aligned}
\mathcal{L}_{NRQCD} \ni \; & \psi^\dagger \Big\{ iD_0 + c_1 \frac{\mathbf{D}^2}{2M} + c_2 \frac{\mathbf{D}^4}{8M^3} + c_F g \frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{2M} \\
& + c_D g \frac{[\mathbf{D}\cdot, \mathbf{E}]}{8M^2} + i c_s g \frac{\boldsymbol{\sigma}\cdot[\mathbf{D}\times, \mathbf{E}]}{8M^2} \Big\} \psi \\
& + \chi^\dagger \Big\{ iD_0 - c_1 \frac{\mathbf{D}^2}{2M} - c_2 \frac{\mathbf{D}^4}{8M^3} - c_F g \frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{2M} \\
& + c_D g \frac{[\mathbf{D}\cdot, \mathbf{E}]}{8M^2} + i c_s g \frac{\boldsymbol{\sigma}\cdot[\mathbf{D}\times, \mathbf{E}]}{8M^2} \Big\} \chi
\end{aligned}
$$

- ψ annihilates heavy quark
- χ creates heavy antiquark

Heavy quark field transformation

- Quark and antiquark field transformations ($x' = \mathcal{B}^{-1}x$):

$$
\begin{aligned}
\psi(x) &\rightarrow \left\{1 + iM\boldsymbol{\eta}\cdot\mathbf{x} - \frac{ik_1}{2M}\boldsymbol{\eta}\cdot\mathbf{D} + \frac{k_2}{4M}\boldsymbol{\eta}\cdot(\mathbf{D}\times\boldsymbol{\sigma}) \right. \\
&\quad \left. +\mathcal{O}(g, M^{-2})\right\}\psi(x') \\
\chi(x) &\rightarrow \left\{1 - iM\boldsymbol{\eta}\cdot\mathbf{x} + \frac{ik_1}{2M}\boldsymbol{\eta}\cdot\mathbf{D} - \frac{k_2}{4M}\boldsymbol{\eta}\cdot(\mathbf{D}\times\boldsymbol{\sigma}) \right. \\
&\quad \left. +\mathcal{O}(g, M^{-2})\right\}\chi(x')
\end{aligned}
$$

- Extra terms in the Lagrangian up to 1/M :

$$
\begin{aligned}
\delta\mathcal{L}_{2\psi} &= \psi^\dagger\left[i(1-c_1)\boldsymbol{\eta}\cdot\mathbf{D} - \frac{1}{2M}(k_1 - c_1)\{D_0, \boldsymbol{\eta}\cdot\mathbf{D}\} \right. \\
&\quad \left. +\frac{1}{4M}(1 - 2c_F + c_s)\boldsymbol{\eta}\cdot(g\mathbf{E}\times\boldsymbol{\sigma})\right]\psi
\end{aligned}
$$

Constraints in NRQCD

Invariance of the Lagrangian, $\delta\mathcal{L}_{2\psi} = 0$ up to $1/M$

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- Relations valid at higher loop matching calculations
- Similar calculations performed also for the 4-quark operators, and matches to the known results. [Brambilla, Vairo, Mereghetti, PRD (2009)] Full results to be submitted [Berwein, Brambilla, Hwang, Vairo, TUM-EFT 74/15]

pNRQCD: power counting and field contents

Potential non-relativistic QCD (pNRQCD): by integrating out the relative momentum, $Mv \sim 1/\mathrm{r}$, from NRQCD

- Coordinates: $\mathbf{R} \equiv (\mathbf{x}_1 + \mathbf{x}_2)/2$, $\mathbf{r} \equiv \mathbf{x}_1 - \mathbf{x}_2$, t

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\begin{aligned}
\boldsymbol{\nabla}_r, \frac{1}{r} &\sim Mv \\
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- Quark-antiquark color singlet $S(t, \mathbf{R}, \mathbf{r})$ and octet $O^a(t, \mathbf{R}, \mathbf{r})$
- Multipole expanded ultra-soft gluon $A^a_\mu(t, \mathbf{R})$

pNRQCD: Lagrangian

Schematic form of the Lagrangian bilinear in singlet and octet:

$$\mathcal{L}_{pNRQCD} \ni \int d^3r \mathrm{Tr}\left\{S^\dagger \mathcal{K}_{SS} S + O^\dagger \mathcal{K}_{OO} O + [S^\dagger \mathcal{K}_{SO} O + \mathrm{H.C.}]\right\}$$
$$- \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu}(\mathbf{R}, t)$$

- Integration over relative distance r

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- Focus on $S^\dagger \mathcal{K}_{SS} S$ for the simplicity here

Bilinear in singlet up to $1/M^2$ [Brambilla, Gromes, Vairo, Phys. Lett. B (2003)]

$$
\begin{aligned}
S^\dagger \mathcal{K}_{SS} S = S^\dagger \Big(& i\partial_0 + \frac{1}{2M}\left\{ c_s^{(1,-2)}, \boldsymbol{\nabla}_r^2 \right\} + \frac{c_s^{(1,0)}}{4M}\boldsymbol{\nabla}_R^2 - V_S^{(0)} \\
& - \frac{V_S^{(1)}}{M} + \frac{V_{p^2 Sa}}{8M^2}\boldsymbol{\nabla}_R^2 + \frac{1}{2M^2}\left\{ \boldsymbol{\nabla}_r^2, V_{p^2 Sb} \right\} + \frac{V_{L^2 Sa}}{4M^2 r^2}(\mathbf{r}\times\boldsymbol{\nabla}_R)^2 \\
& + \frac{V_{L^2 Sb}}{4M^2 r^2}(\mathbf{r}\times\boldsymbol{\nabla}_r)^2 - \frac{V_{S_{12} S}}{M^2 r^2}\Big(3(\mathbf{r}\cdot\boldsymbol{\sigma}^{(1)})(\mathbf{r}\cdot\boldsymbol{\sigma}^{(2)}) - \mathbf{r}^2 \boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)} \Big) \\
& - \frac{V_{S^2 S}}{4M^2}\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)} + \frac{i V_{LSSa}}{4M^2}(\mathbf{r}\times\boldsymbol{\nabla}_R)\cdot(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}) \\
& + \frac{V_{LSSb}}{4M^2}(\mathbf{r}\times\boldsymbol{\nabla}_r)\cdot(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}) \Big) S
\end{aligned}
$$

- (a, b): a denotes order of $1/M$, b denotes order of r
- $\boldsymbol{\sigma}^{(1/2)}$: spin matrix for quark/antiquark
- c, V: Wilson coefficients.

Minimal requirements of symmetry

Construct the most generalised form of the spatial boost $\mathbf{k}$ for the singlet field with following criteria

- Transformations under C, P, T

$$\mathbf{k} \xrightarrow{P} -\mathbf{k}, \quad \mathbf{k} \xrightarrow{C} -\sigma_2 \mathbf{k}^* \sigma_2, \qquad \mathbf{k} \xrightarrow{T} \sigma_2 \mathbf{k} \sigma_2$$

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- Proper order of truncation: up to $1/M$ and r
- Same criteria apply to octet field

Generalised boost

$$
\begin{aligned}
S'(t,\boldsymbol{R},\boldsymbol{r}) = \Bigg(& 1 - 2iM\boldsymbol{\eta}\cdot\boldsymbol{R} + \frac{ik_D^{(1,0)}}{4M}\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_R + \frac{i}{4M}\left\{k_{a'}^{(1,0)}\boldsymbol{\eta}\cdot\boldsymbol{r}, \boldsymbol{\nabla}_R\cdot\boldsymbol{\nabla}_r\right\} \\
&+ \frac{i}{4M}\left\{k_{a''}^{(1,0)}\boldsymbol{r}\cdot\boldsymbol{\nabla}_R, \boldsymbol{\eta}\cdot\boldsymbol{\nabla}_r\right\} + \frac{i}{4M}\left\{k_{a'''}^{(1,0)}\boldsymbol{r}\cdot, \boldsymbol{\nabla}_r(\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_R)\right\} + \frac{i}{4M}\left\{\frac{k_b^{(1,0)}}{r^2}(\boldsymbol{\eta}\cdot\boldsymbol{r})(\boldsymbol{r}\cdot\boldsymbol{\nabla}_R)\boldsymbol{r}\cdot, \boldsymbol{\nabla}_r\right\} \\
&- \frac{k_c^{(1,0)}}{8M}\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_R\times\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right) + \frac{k_{d'}^{(1,0)}}{8Mr^2}(\boldsymbol{\eta}\cdot\boldsymbol{r}\times\boldsymbol{\nabla}_R)\left(\boldsymbol{r}\cdot\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right) \\
&- \frac{k_{d''}^{(1,0)}}{8Mr^2}\left(\boldsymbol{\eta}\cdot\boldsymbol{r}\times\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)(\boldsymbol{r}\cdot\boldsymbol{\nabla}_R) - \frac{1}{8M}\left\{k_a^{(1,-1)}, \boldsymbol{\eta}\cdot\boldsymbol{\nabla}_r\times\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right\} \\
&+ \frac{1}{8M}\left\{\frac{k_{b'}^{(1,-1)}}{r^2}\left(\boldsymbol{r}\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)(\boldsymbol{\eta}\times\boldsymbol{r})\cdot, \boldsymbol{\nabla}_r\right\} \\
&- \frac{1}{8M}\left\{\frac{k_{b''}^{(1,-1)}}{r^2}\left(\boldsymbol{\eta}\cdot\boldsymbol{r}\times\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)\boldsymbol{r}\cdot, \boldsymbol{\nabla}_r\right\}\Bigg) S(t',\boldsymbol{R}',\boldsymbol{r}')
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- More undetermined coefficients, k's $\rightarrow$ more problems?

Generalised boost

$$
\begin{aligned}
S'(t,\boldsymbol{R},\boldsymbol{r}) = \Bigg(& 1 - 2iM\boldsymbol{\eta}\cdot\boldsymbol{R} + \frac{ik_D^{(1,0)}}{4M}\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_R + \frac{i}{4M}\left\{k_{a'}^{(1,0)}\boldsymbol{\eta}\cdot\boldsymbol{r},\boldsymbol{\nabla}_R\cdot\boldsymbol{\nabla}_r\right\} \\
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- More undetermined coefficients, k's $\to$ more problems?
- Coefficients removed by field redefinition
- Field redefinition by unitary transformation

Field redefinition

Define new singlet via unitary transformation: $S = \mathcal{U}_S \widetilde{S}$

- $\mathcal{U}_S$: unitary operator for singlet

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- Order of $1/M$ expansion on the exponents?

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- Boost coefficients are shifted:
  $k_{a'}^{(1,0)} \to k_{a'}^{(1,0)} - 2q_{a'}^{(1,0)} - 2q_{a''}^{(1,0)}$,
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  $\dots$
- Eliminate as many terms as possible to simplify the generalised expression of the boost

Intermezzo

Dropping tilde notation on singlet

$$
\begin{aligned}
S' = &\Bigg(1 - 2iM\boldsymbol{\eta}\cdot\boldsymbol{R} + \frac{ik_D^{(1,0)}}{4M}\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_R + \frac{i}{4M}\left\{k_{a'}^{(1,0)}\boldsymbol{\eta}\cdot\boldsymbol{r}, \boldsymbol{\nabla}_r\cdot\boldsymbol{\nabla}_R\right\} \\
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\equiv&(1 - i\boldsymbol{\eta}\cdot\boldsymbol{k})S
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$$

- 5 coefficients to be determined

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- 5 coefficients to be determined
- Any other constraints to impose?

1st constraint: commutation relation

- Commutation relation between boost generators $\boldsymbol{k}$

$$[1 - i\boldsymbol{\xi}\cdot\boldsymbol{k}, 1 - i\boldsymbol{\eta}\cdot\boldsymbol{k}]S \stackrel{!}{=} i(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{j}S$$

fixes coefficients: $k_{a'}^{(1,0)} = k_{c}^{(1,0)} = 1$ and $k_{d'}^{(1,0)} = k_{d''}^{(1,0)} = 0$

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- $k_D^{(1,0)}$ remains to be constrained

2nd constraint: Lorentz invariance

- Boost transformation of the Lagrangian

$$
\begin{aligned}
\delta\mathcal{L}_{2S} \quad = \quad & S^\dagger\Bigg(i\left(1-c_S^{(1,0)}\right)\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_R-\frac{1}{2M}\left(k_D^{(1,0)}-c_S^{(1,0)}\right)\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_R\partial_0 \\
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& +\frac{i}{Mr^2}\left(V_{L^2Sa}+\frac{r}{2}\partial_r V_S^{(0)}\right)(\boldsymbol{\eta}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\nabla}_R) \\
& +\frac{1}{2M}\left(V_{LSSa}+\frac{1}{2r}\partial_r V_S^{(0)}\right)\boldsymbol{\eta}\cdot(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)})\times\mathbf{r}\Bigg)S
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- Constraints on the boost/Wilson coefficients:

$$
\begin{aligned}
k_D^{(1,0)} &= c_S^{(1,0)}=1, \quad V_{p^2Sa}+V_{L^2Sa}+\frac{1}{2}V_S^{(0)}=0, \\
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$$

Coda [Berwein, Brambilla, Hwang, Vairo, TUM-EFT 74/15]

Final version of the boost after parameters fixed

$$S'(t,\mathbf{R},\mathbf{r}) = \Big(1 - 2iM\boldsymbol{\eta}\cdot\mathbf{R} + \frac{i}{4M}\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_R + \frac{i}{4M}\{\boldsymbol{\eta}\cdot\mathbf{r},\boldsymbol{\nabla}_R\cdot\boldsymbol{\nabla}_r\}$$
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$$+\mathcal{O}(1/M^2)\Big)S(t',\mathbf{R}',\mathbf{r}')$$

- Agrees with the expression from *induced representation* of Wigner's *little group formalism*

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- Similar procedures in the octet-octet sector

Outlook: non-perturbative regime

Effective String Theory (EST) [Nambu, Phys. Lett. B (1979)]

- Heavy quark-antiquark bound system with long distance separation $r\Lambda_{QCD} \gg 1$ [Kogut, Parisi, Phys. Rev. Lett. (1981)]

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- Poincaré invariance in pNRQCD reduces parameters of EST [Brambilla, Groher, Martinez, Vairo PRD (2014)]
- Work in progress: employ EFT systematics in EST to include higher order terms

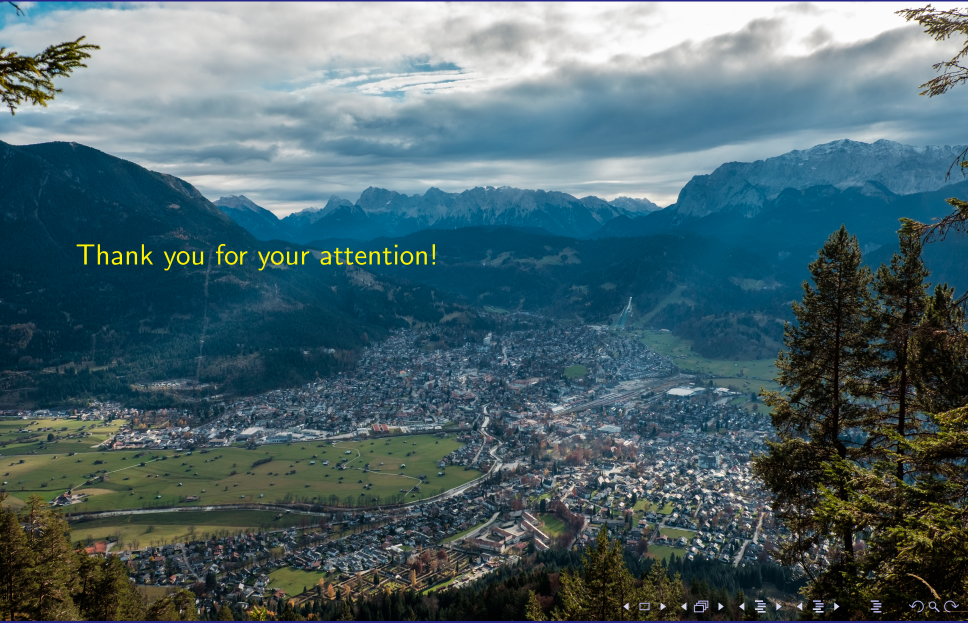
Summary and Outlook

Summary

- Implementation of boost in NRQCD
- Lagrangian structure of pNRQCD
- Boost transformations in pNRQCD and field redefinition
- Constraints in Wilson coefficients of pNRQCD

Outlook

- Effective String Theory
- Dark matter direct detection (bottom-up approach in EFT)



Thank you for your attention!

Little group element

- $L(p)$ a rotation in the plane of $k/M \equiv w$ and $p/M \equiv v$:

$$L(w,v)^{\mu}_{\nu} \;=\; g^{\mu}_{\nu} - \frac{1}{1+v\cdot w}(w^{\mu}w_{\nu} + v^{\mu}v_{\nu}) + w^{\mu}v_{\nu} - v^{\mu}w_{\nu}$$

- For $\Lambda = \mathcal{B}(\eta)$, $\mathcal{B}(\eta)v = v + \eta$, and explicit form of the boost:

$$\mathcal{B}(\eta)^{\mu}_{\nu} = g^{\mu}_{\nu} - (v^{\mu}\eta_{\nu} - \eta^{\mu}v_{\nu}) + \mathcal{O}(\eta^2)$$

- Little group element for the infinitesimal boost:

$$W(\mathcal{B}(\eta),p) \;=\; 1 + \frac{i}{2}\left[\frac{1}{M+v\cdot p}(\eta^{\alpha}p^{\beta}_{\perp} - p^{\alpha}_{\perp}\eta^{\beta})\mathcal{J}_{\alpha\beta}\right] + \mathcal{O}(\eta^2)$$

where $p^{\beta}_{\perp} \equiv p^{\beta} - (v\cdot p)v^{\beta}$ and $(\mathcal{J}^{\alpha\beta})_{\mu\nu} = i(g^{\alpha}_{\mu}g^{\beta}_{\nu} - g^{\beta}_{\mu}g^{\alpha}_{\nu})$

Unitary operator: octet-octet

- Unitary operator for the octet - octet sector

$$\mathcal{U}_o = \exp\left[\frac{i}{2M}\left(-i\mathbf{D}_R\cdot(\tilde{\mathbf{k}}_{oo}^{(0,2)}+\tilde{\mathbf{k}}_{oo}^{(1,-1)}+\tilde{\mathbf{k}}_{oo}^{(1,0)}\right)+h.c.\right]$$

- Boost on the octet after the field redefinition:

$$\begin{aligned}\widetilde{O}' = & \Big(1 - i\boldsymbol{\eta}\cdot\mathbf{k} - \frac{i}{8}\tilde{k}_a^{(0,2)}(\mathbf{r}\cdot g\mathbf{E})(\mathbf{r}\cdot\boldsymbol{\eta}) - \frac{i}{8}\tilde{k}_b^{(0,2)}\mathbf{r}^2(\boldsymbol{\eta}\cdot g\mathbf{E}) \\ & -\frac{i}{2M}\{\tilde{k}_{a'}^{(1,0)}\boldsymbol{\eta}\cdot\mathbf{r},\boldsymbol{\nabla}_r\cdot\mathbf{D}_R\} - \frac{i}{2M}\{\tilde{k}_{a''}^{(1,0)}\mathbf{r}\cdot\boldsymbol{D}_R,\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_r\} \\ & +\cdots+\mathcal{O}\left(\frac{1}{M^2}\right)\Big)\widetilde{O}\end{aligned}$$

- Similar shift in the parameters

Constraints: octet-octet sector (1/2)

Octet - octet sector includes:

$$\begin{aligned}
\mathcal{L}_{2O} \ni\ & O^\dagger \Big(\frac{i}{8M} V^{(1,0)}_{OOa}(r) \{ \boldsymbol{\nabla}_r \cdot, \mathbf{r} \times g\mathbf{B} \} \\
& + \frac{1}{16M^2} \{ (\boldsymbol{\nabla}_r \cdot \mathbf{D}_R), V^{(2,0)}_{OOc'}(r)(\mathbf{r} \cdot g\mathbf{E}) \} \\
& + \frac{1}{16M^2} \{ \boldsymbol{\nabla}^i_r \mathbf{D}^j_R, V^{(2,0)}_{OOc''}(r) \mathbf{r}^j g\mathbf{E}^i \} \\
& + \frac{1}{16M^2} \{ \boldsymbol{\nabla}^i_r \mathbf{D}^j_R, V^{(2,0)}_{OOc'''}(r) \mathbf{r}^i g\mathbf{E}^j \} \\
& + \frac{1}{16M^2} \{ \boldsymbol{\nabla}^i_r \mathbf{D}^j_R, \frac{V^{(2,0)}_{OOd}(r)}{r^2} \mathbf{r}^i \mathbf{r}^j (\mathbf{r} \cdot \mathbf{E}) \} \Big) O + C.C.
\end{aligned}$$

Constraints: octet-octet sector (2/2)

Constraints under the little group:

$$
\begin{aligned}
V^{(1,0)}_{OOa} + V^{(2,0)}_{OOc'} &= 0 \\
V^{(1,0)}_{OOa} - V^{(2,0)}_{OOc''} &= 2 \\
rV^{(2,0)}_{OOc'''} &= 0 \\
V^{(2,0)}_{OOd} &= 0 \qquad (1)
\end{aligned}
$$

References

D. J. Gross and F. Wilczek (1973)
Ultraviolet Behavior of Non-Abelian Gauge Theories
Phys. Rev. Lett. 30, 1343

H. D. Politzer (1973)
Reliable Perturbative Results for Strong Interactions?
Phys. Rev. Lett. 30, 1346

K. G. Wilson (1974)
Confinement of quarks
Phys. Rev. D 10, 2445

W.E. Caswell and G.P. Lepage (1986)
Effective lagrangians for bound state problems in QED, QCD, and other field theories
Phys. Lett. B 167, 437 - 442

G. T. Bodwin, E. Braaten, and G. P. Lepage (1995)
Rigorous QCD analysis of inclusive annihilation and production of heavy quarkonium
Phys. Rev. D 51, 1125

References

N. Brambilla, D. Gromes, and A. Vairo (2003)
Poincaré invariance constrains on NRQCD and potential NRQCD
Phys. Lett. B 576, 314 - 327

J. Heinonen, R. Hill, and M. Solon (2012)
Lorentz invariance in heavy particle effective theories
Phys. Rev. D 86, 094020 (2012)

M. Luke and A. Manohar (1992)
Reparametrisation invariance constraints on heavy particle effective field theories
Phys. Lett. B 286, 348 - 354

S. Weinberg (1995)
Quantum Theory of Fields I
Cambridge Univ. Press

References

N. Brambilla, A. Pineda, J. Soto, and A. Vairo (2000)
Potential NRQCD: an effective theory for heavy quarkonium
Nucl. Phys. B 566 (2000), 275 - 310

A. Pineda, J. Soto (1998)
Matching at one loop for the four-quark operators in NRQCD
Phys. Rev. D 58 (1998), 114011

N. Brambilla, A. Vairo, and E. Mereghetti (2009)
Hadronic quarkonium decays at order v7
Phys. Rev. D 83 (2009), 079904

Y. Nambu (1979)
QCD and the string model
Phys. Lett. B 80 (1979), 372

J. Kogut, G. Parisi (1981)
Long-Range Spin-Spin Forces in Gauge Theories
Phys. Rev. Lett. 47 (1981), 16

References

N. Brambilla, A. Pineda, J. Soto, A. Vairo (2000)

QCD potential at O(1/m)

PRD 63 (2000), 014023

G. Perez-Nadal, J. Soto (2009)

Effective-string theory constraints on the long-distance behavior of the subleading potentials.

PRD 79 (2009), 114002

N. Brambilla, M. Groher, H.E. Martinez, A. Vairo (2014)

Effective string theory and the long-range relativistic corrections to the quark-antiquark potential

PRD 90 (2014), 114032

M. Berwein, N. Brambilla, S. Hwang, A. Vairo (2016)

Poincare invaraince in NRQCD and pNRQCD

To be submitted to *PRD* (2016), TUM-EFT 74/15

E.P.Wigner (1939)

On unitary representations of the inhomogeneous Lorentz group

Annals Math. 40, 149 (1939)