Hydrodynamics in heavy-ion collisions

Dirk H. Rischke

Institut für Theoretische Physik

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

thanks to:

Barbara Betz, Ioannis Bouras, Gabriel S. Denicol, Carsten Greiner, Pasi Huovinen, Etele Molnár, Harri Niemi, Jorge Noronha, Zhe Xu

Central questions

1. **Is there collective flow of hot and dense strong-interaction matter created in heavy-ion collisions?**

2. **Can hydrodynamics describe this collective flow?**

Central questions and answers

1. Is there collective flow of hot and dense strong-interaction matter created in heavy-ion collisions?

$\Longrightarrow$ **yes!**

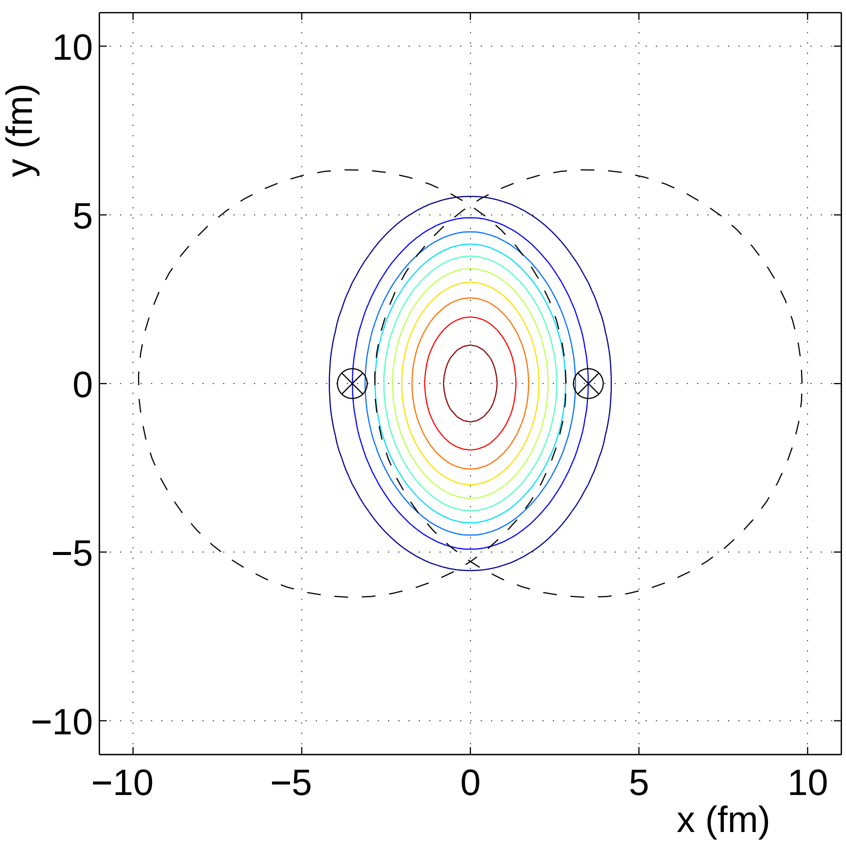
2. Can hydrodynamics describe this collective flow ?

$\Longrightarrow$ **Qualitatively: yes!**

Quantitatively: only under favorable circumstances!

Origin of collective flow

non-central heavy-ion collision
in transverse $(x - y)$ plane $(z = 0)$
(event-averaged):



If particles do not interact with each other, they stream freely towards the detector

$\Longrightarrow$ single-inclusive particle spectrum:

$$E\,\frac{dN}{d^3\vec{p}} \equiv E\,\frac{dN}{dp_z\,d^2\vec{p}_\perp}\,, \quad p_\perp = \sqrt{p_x^2 + p_y^2}$$

transverse momentum

$$\equiv E\,\frac{dN}{dp_z\,p_\perp dp_\perp\,d\varphi} \equiv \frac{dN}{dy\,p_\perp dp_\perp d\varphi}\,,$$

$\tanh y \equiv \frac{p_z}{E}$, y : longitudinal rapidity

is independent of azimuthal angle φ

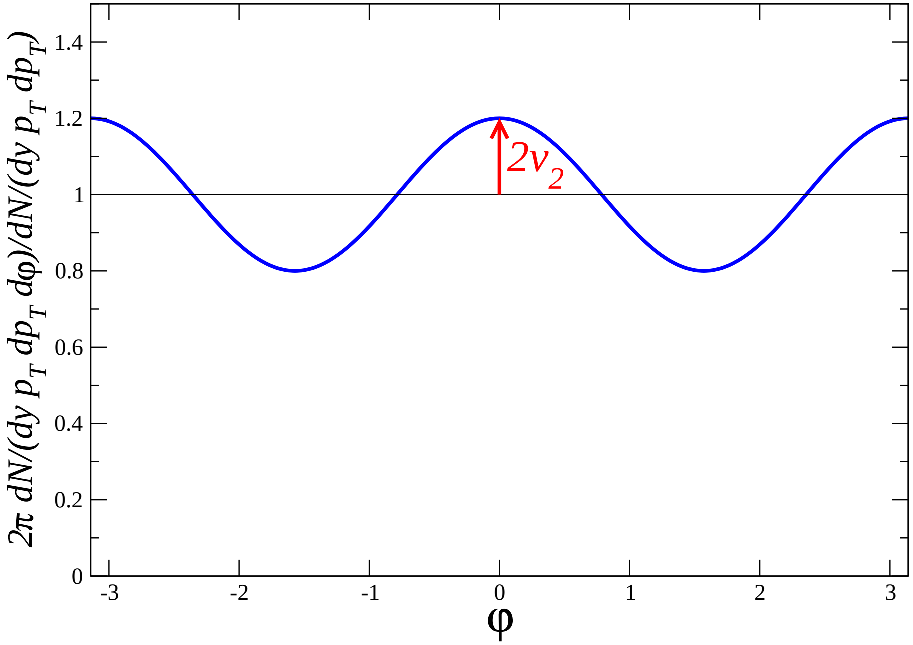
$\Longrightarrow$ information on initial geometry is lost

But: If particles interact strongly (like in a fluid), collective flow develops

$\Longrightarrow$ initial spatial asymmetry is, by difference in pressure gradients, converted to final momentum anisotropy

Characterization of collective flow

Event-averaged single-inclusive particle spectrum at $y = 0$ as function of φ:



$\Longrightarrow$ **preferential emission of particles in the reaction $(x - z)$ plane**

$\Longrightarrow$ **Fourier decomposition of single-inclusive particle spectrum:**

$$E\,\frac{dN}{d^3\vec{p}} \equiv \frac{dN}{dy\, p_\perp dp_\perp d\varphi} \equiv \frac{1}{2\pi}\,\frac{dN}{dy\, p_\perp dp_\perp}\left(1 + 2\sum_{n=1}^{\infty} v_n(y, p_\perp)\,\cos\left\{n\left[\varphi - \Psi_n(y, p_\perp)\right]\right\}\right)$$

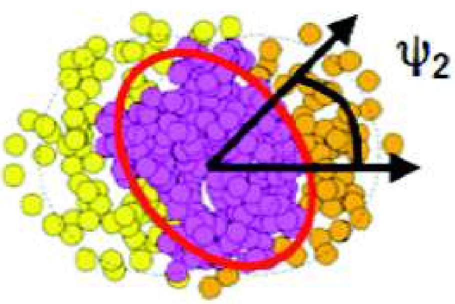
v_1 : directed flow, v_2 : elliptic flow v_3 : triangular flow , etc.

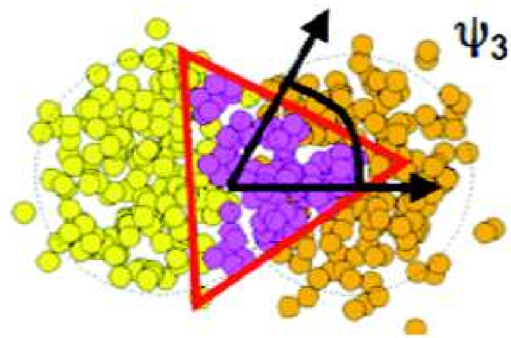
Event-by-event collective flow (I)

event by event: fluctuations of initial geometry

$\Longrightarrow$ **rotate participant plane vs. reaction plane** $\boldsymbol{\Psi_n \neq 0}$

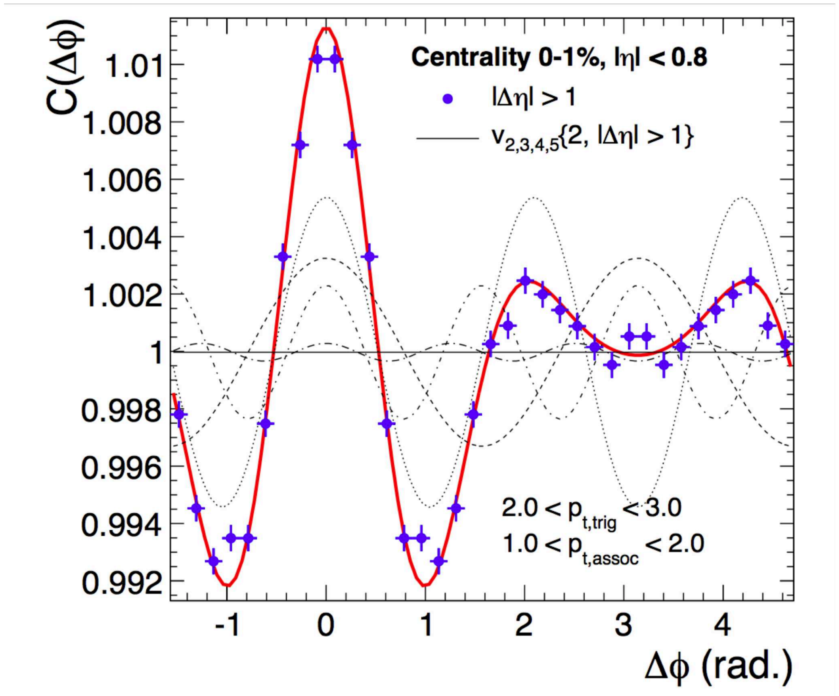
$\Longrightarrow$ **induce higher flow harmonics** $\boldsymbol{v_n \neq 0\,,\ n = 3, 4, \ldots}$



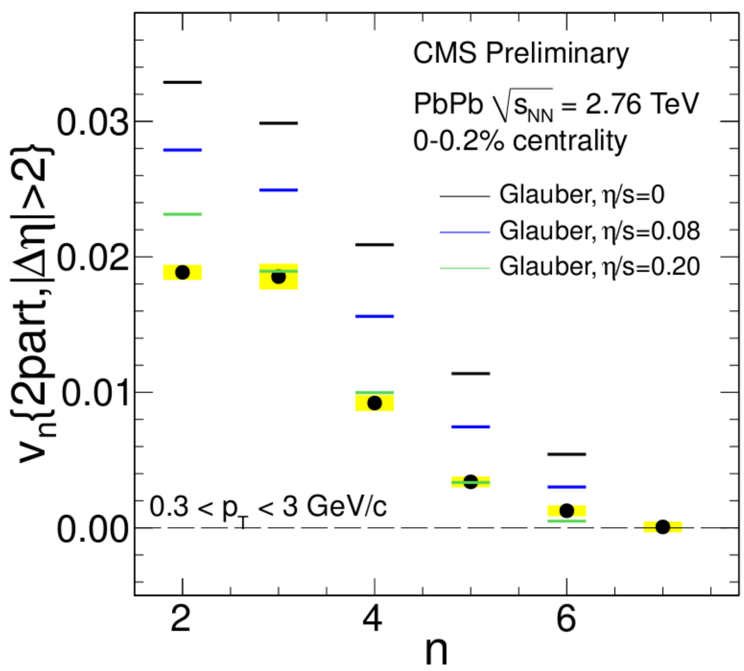


Event-by-event collective flow (II)

⟹ **two-particle correlation functions can be explained as superposition of Fourier components** B. Alver, G. Roland, PRC 81 (2010) 054905



ALICE Collaboration, PRL 107 (2011) 032301



G. Roland for the CMS collaboration, presentation at QM 2012

⟹ **1. Is there collective flow of hot and dense strong-interaction matter created in heavy-ion collisions?** ⟹ yes! (and even event by event!)

Theoretical description of collective flow

Quintessential theory of collective flow: hydrodynamics

$\Longrightarrow$ **2. Can hydrodynamics describe collective flow of hot and dense strong-interaction matter created in heavy-ion collisions?**

Hydrodynamics: degrees of freedom

1. Net charge (e.g., baryon number, strangeness, etc.) current: $\boxed{N^\mu = n\, u^\mu + n^\mu}$

   u^μ fluid 4-velocity, $u^\mu u_\mu = u^\mu g_{\mu\nu} u^\nu = 1$

   $g_{\mu\nu} \equiv \mathrm{diag}(+,-,-,-)$ (West coast!!) metric tensor

   $n \equiv u^\mu N_\mu$ net charge density in fluid rest frame

   $n^\mu \equiv \Delta^{\mu\nu} N_\nu \equiv N^{<\mu>}$ diffusion current (flow of net charge relative to u^μ), $n^\mu u_\mu = 0$

   $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ projector onto 3-space orthogonal to u^μ, $\Delta^{\mu\nu} u_\nu = 0$

2. Energy-momentum tensor: $\boxed{T^{\mu\nu} = \epsilon\, u^\mu u^\nu - (p + \Pi)\, \Delta^{\mu\nu} + 2\, q^{(\mu} u^{\nu)} + \pi^{\mu\nu}}$

   $\epsilon \equiv u^\mu T_{\mu\nu} u^\nu$ energy density in fluid rest frame

   p pressure in fluid rest frame

   Π bulk viscous pressure, $p + \Pi \equiv -\frac{1}{3}\, \Delta^{\mu\nu} T_{\mu\nu}$

   $q^\mu \equiv \Delta^{\mu\nu} T_{\nu\lambda} u^\lambda$ heat flux current (flow of energy relative to u^μ), $q^\mu u_\mu = 0$

   $\pi^{\mu\nu} \equiv T^{<\mu\nu>}$ shear stress tensor, $\pi^{\mu\nu} u_\mu = \pi^{\mu\nu} u_\nu = 0\,,\ \pi^\mu{}_\mu = 0$

   $a^{(\mu\nu)} \equiv \frac{1}{2}\,(a^{\mu\nu} + a^{\nu\mu})$ symmetrized tensor

   $a^{<\mu\nu>} \equiv \left(\Delta_\alpha{}^{(\mu} \Delta^{\nu)}{}_\beta - \frac{1}{3}\, \Delta^{\mu\nu} \Delta_{\alpha\beta}\right) a^{\alpha\beta}$ symmetrized, traceless spatial projection

Hydrodynamics: equations of motion

1. **Net charge conservation:**

$$\boxed{\partial_\mu N^\mu = 0} \Longleftrightarrow \boxed{\dot{n} + n\,\theta + \partial \cdot n = 0}$$

$\dot{a} \equiv u^\mu \partial_\mu a$ convective (comoving) derivative
(time derivative in fluid rest frame, $\dot{a}_{\mathrm{RF}} \equiv \partial_t a$)

$\theta \equiv \partial_\mu u^\mu$ expansion scalar

2. **Energy-momentum conservation:**

$\boxed{\partial_\mu T^{\mu\nu} = 0} \Longleftrightarrow$ energy conservation:

$$\boxed{u_\nu\, \partial_\mu T^{\mu\nu} = \dot{\epsilon} + (\epsilon + p + \Pi)\,\theta + \partial \cdot q - q \cdot \dot{u} - \pi^{\mu\nu}\,\partial_\mu u_\nu = 0}$$

acceleration equation:

$$\Delta^{\mu\nu}\,\partial^\lambda T_{\nu\lambda} = 0 \quad \Longleftrightarrow$$

$$(\epsilon + p)\dot{u}^\mu = \nabla^\mu(p + \Pi) - \Pi\dot{u}^\mu - \Delta^{\mu\nu}\dot{q}_\nu - q^\mu\theta - q \cdot \partial u^\mu - \Delta^{\mu\nu}\,\partial^\lambda \pi_{\nu\lambda}$$

$\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ 3-gradient,
(spatial gradient in fluid rest frame, $u^\mu_{\mathrm{RF}} \equiv (1,0,0,0)$)

Solvability

Problem:

5 equations, but 15 unknowns (for given u^μ): ϵ , p , n , Π , n^μ (3) , q^μ (3) , $\pi^{\mu\nu}$ (5)

Solution:

1. **clever choice of frame (Eckart, Landau,...): eliminate n^μ or q^μ**

   $\Longrightarrow$ **does not help! Promotes u^μ to dynamical variable!**

2. **ideal fluid limit: all dissipative terms vanish, $\Pi = n^\mu = q^\mu = \pi^{\mu\nu} = 0$**

   $\Longrightarrow$ **6 unknowns: ϵ , p , n , u^μ (3)   (not quite there yet...)**

   $\Longrightarrow$ **fluid is in local thermodynamical equilibrium**

   $\Longrightarrow$ **provide equation of state (EoS) $p(\epsilon, n)$ to close system of equations**

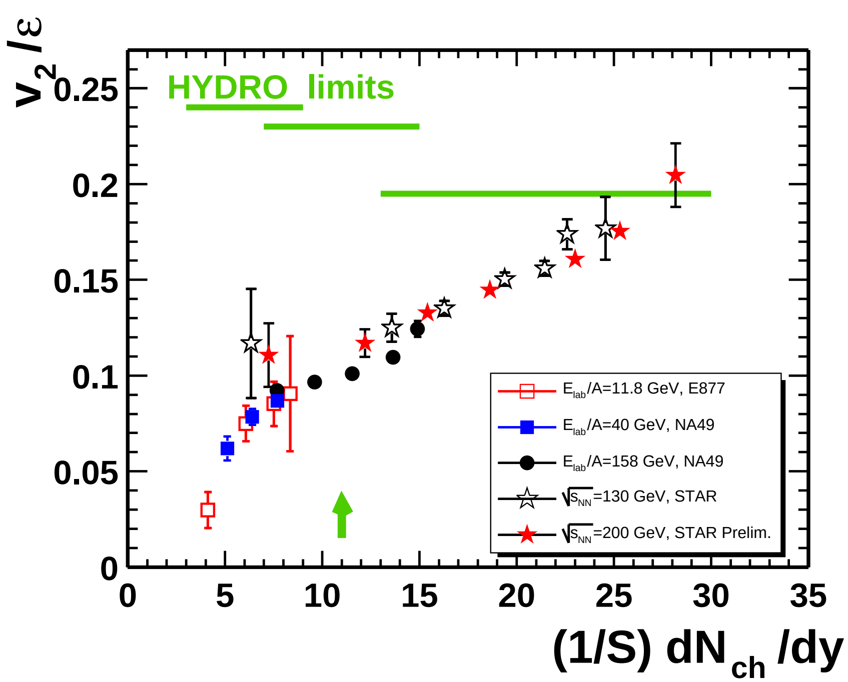
3. **provide additional equations for dissipative quantities**

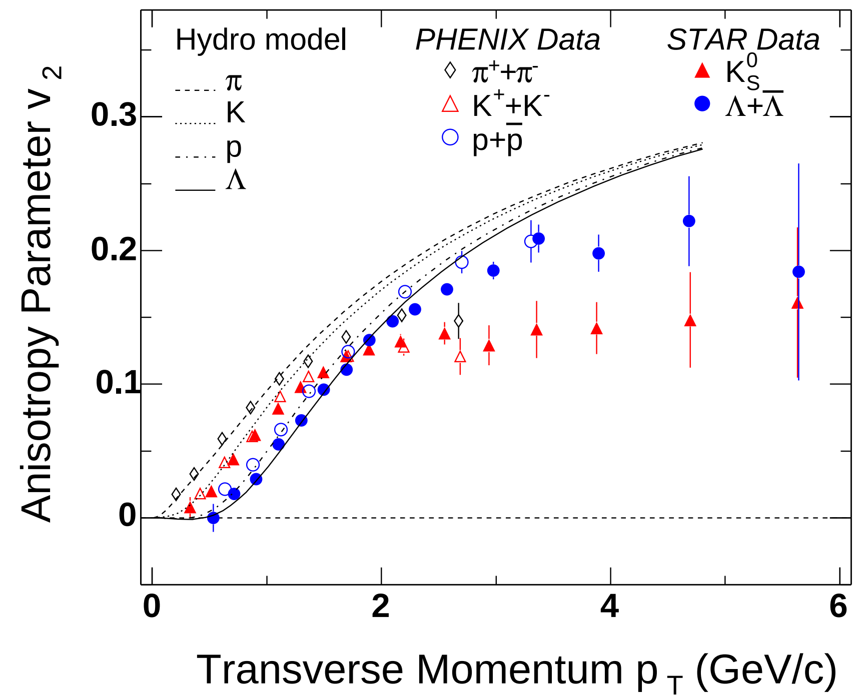
   $\Longrightarrow$ **dissipative relativistic hydrodynamics**

   (a) **First-order theories: e.g. generalization of Navier-Stokes (NS) equations to the relativistic case (Landau, Lifshitz)**

   (b) **Second-order theories: e.g. Israel-Stewart (IS) equations**

Ideal hydrodynamics confronts data





$\Longrightarrow$ approach to ideal fluid limit with increasing centrality and beam energy

$\Longrightarrow$ quantitative description of elliptic flow at RHIC within ideal hydrodynamics

$\Longrightarrow$ no dissipative effects! $\Longrightarrow$ "RHIC physicists serve up the perfect fluid"

$\Longrightarrow$ answer to question 2.: yes!
at least qualitatively at lower beam energies and centralities,
and even quantitatively at RHIC energies (case closed! ...?)

Two problems (I)

1. **There is no real ideal fluid!**

shear viscosity $\eta \sim \dfrac{T}{\langle\sigma\rangle} \to 0 \quad \Longleftrightarrow \quad$ **average scattering cross section** $\langle\sigma\rangle \to \infty$

minimal value for shear viscosity to entropy density ratio:

(i) from uncertainty principle ("quantum limit"): $\dfrac{\eta}{s} \simeq \dfrac{1}{12}$

P. Danielewicz, M. Gyulassy, PRD 31 (1985) 53

(ii) from AdS/CFT correspondence: conjectured lower bound $\dfrac{\eta}{s} = \dfrac{1}{4\pi}$

P. Kovtun, D.T. Son, A. Starinets, PRL 94 (2005) 111601

$\Longrightarrow$ **What is** $\dfrac{\eta}{s}$ **of hot and dense hadronic matter?**

If $\dfrac{\eta}{s} \ll 1 \quad \Longrightarrow$ **matter is strongly interacting!**

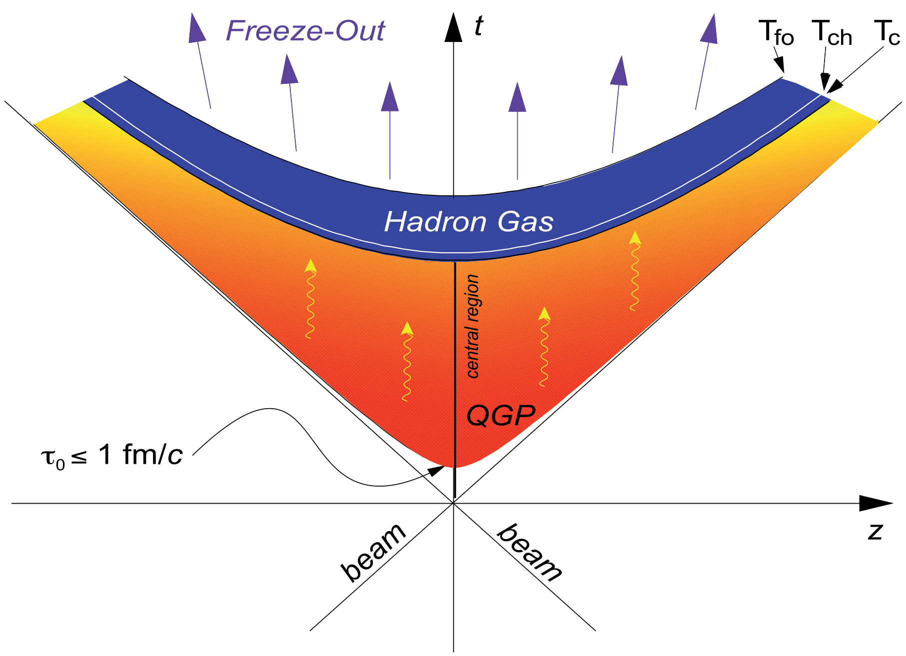
$\Longrightarrow$ **"strongly coupled quark-gluon plasma" (sQGP)**

Two problems (II)

2. Hydrodynamical equations of motion: $\partial_\mu T^{\mu\nu} = 0$

$\Longrightarrow$ **partial differential equations**

$\Longrightarrow$ **require initial conditions on a space-time hypersurface**



energy-momentum tensor $T^{\mu\nu}(\tau_0, \vec{x})$
on initial space-time hypersurface
$\tau \equiv \sqrt{t^2 - z^2} \equiv \tau_0 = const.$

$\Longrightarrow$ **continuum of parameters to fit to experimental data**

$\Longrightarrow$ **experimental data may allow for non-zero viscosity!**

$\Longrightarrow$ **need calculations within dissipative hydrodynamics and with realistic initial conditions**

Navier-Stokes equations

Navier-Stokes (NS) equations: first-order dissipative relativistic hydrodynamics

1. bulk viscous pressure: $\Pi_{\rm NS} = -\zeta\,\theta$

   ζ bulk viscosity

2. diffusion current: $n^\mu_{\rm NS} = \kappa_n\,\nabla^\mu\alpha$

   $\beta \equiv 1/T$ inverse temperature,

   $\alpha \equiv \beta\,\mu$, μ chemical potential,

   κ_n net-charge diffusion coefficient

3. shear stress tensor: $\pi^{\mu\nu}_{\rm NS} = 2\,\eta\,\sigma^{\mu\nu}$

   η shear viscosity,

   $\sigma^{\mu\nu} = \nabla^{<\mu}u^{\nu>}$ shear tensor

$\Longrightarrow$ algebraic expressions in terms of thermodynamic and fluid variables

$\Longrightarrow$ simple... but: unstable and acausal equations of motion!!

W.A. Hiscock, L. Lindblom, PRD 31 (1985) 725

Israel-Stewart equations

Israel-Stewart (IS) equations: second-order dissipative relativistic hydrodynamics

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

"Simplified" version:

$$\tau_\Pi \, \dot{\Pi} + \Pi = \Pi_{\rm NS}$$
$$\tau_n \, \dot{n}^{<\mu>} + n^\mu = n^\mu_{\rm NS}$$
$$\tau_\pi \, \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\rm NS}$$

cf. also T. Koide, G.S. Denicol, Ph. Mota, T. Kodama, PRC 75 (2007) 034909

$\Longrightarrow$ **dynamical (instead of algebraic) equations for dissipative terms!**

solution: e.g. bulk viscous pressure $\Pi(t) = \Pi_{\rm NS}\left(1 - e^{-t/\tau_\Pi}\right) + \Pi(0)\, e^{-t/\tau_\Pi}$

$\Longrightarrow$ **dissipative quantities** $\Pi\,,\ n^\mu\,,\ \pi^{\mu\nu}$ **relax to their respective NS values** $\Pi_{\rm NS}\,,\ n^\mu_{\rm NS}\,,\ \pi^{\mu\nu}_{\rm NS}$ **on time scales** $\tau_\Pi\,,\ \tau_n\,,\ \tau_\pi$

$\Longrightarrow$ **stable and causal hydrodynamical equations of motion!**

see, e.g., S. Pu, T. Koide, DHR, PRD 81 (2010) 114039

Power counting (I)

3 length scales: 2 microscopic, 1 macroscopic

- **thermal wavelength** $\lambda_{\text{th}} \sim \beta \equiv 1/T$
- **mean free path** $\ell_{\text{mfp}} \sim (\langle\sigma\rangle n)^{-1}$

  $\langle\sigma\rangle$ **averaged cross section,** $n \sim T^3 = \beta^{-3} \sim \lambda_{\text{th}}^{-3}$
- **length scale over which macroscopic fluid fields vary** L_{hydro} , $\partial_\mu \sim L_{\text{hydro}}^{-1}$

Note: **since** $\eta \sim (\langle\sigma\rangle\lambda_{\text{th}})^{-1} \Longrightarrow$

$$\frac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \frac{1}{\langle\sigma\rangle n}\,\frac{1}{\lambda_{\text{th}}} \sim \frac{\lambda_{\text{th}}^3}{\langle\sigma\rangle\lambda_{\text{th}}} \sim \frac{\lambda_{\text{th}}^3}{\langle\sigma\rangle\lambda_{\text{th}}} \sim \frac{\eta}{s}$$

s **entropy density,** $s \sim n \sim T^3 = \beta^{-3} \sim \lambda_{\text{th}}^{-3}$

$\Longrightarrow \dfrac{\eta}{s}$ **solely determined by the 2 microscopic length scales!**

Note: similar argument holds for $\dfrac{\zeta}{s}\,,\ \dfrac{\kappa_n}{\beta\, s}$

$\Longrightarrow$ **transport coefficients are material properties (like the EoS)**

Power counting (II)

$\Longrightarrow$ **What is the ratio of microscopic to macroscopic length scale?**

$\Longrightarrow$ **Knudsen number:**

$$K \equiv \frac{\ell_{\mathrm{mfp}}}{L_{\mathrm{hydro}}} \sim \ell_{\mathrm{mfp}}\, \partial_\mu$$

$\Longrightarrow$ **If $K \ll 1$: separation of macroscopic fluid dynamics (large scale $\sim L_{\mathrm{hydro}}$) from microscopic particle dynamics (small scale $\sim \ell_{\mathrm{mfp}}$)**

$\Longrightarrow$ **well-defined gradient (derivative) expansion!**

$\Longrightarrow$ **NS terms (divided by p or n) are of first order in K:**

$$\text{e.g. } \frac{\Pi_{\mathrm{NS}}}{p} \sim -\frac{\zeta}{Ts}\,\partial_\mu u^\mu \sim -\frac{\zeta}{s}\,\lambda_{\mathrm{th}}\,\partial_\mu u^\mu \sim \frac{\ell_{\mathrm{mfp}}}{\lambda_{\mathrm{th}}}\,\lambda_{\mathrm{th}}\,\partial_\mu u^\mu \sim \ell_{\mathrm{mfp}}\partial_\mu u^\mu \sim K$$

But: in IS equations, $\Pi\,,\; n^\mu\,,\; \pi^{\mu\nu}$ are independent dynamical variables!

$\Longrightarrow$ **inverse Reynolds numbers**

$$R_\Pi^{-1} \equiv \frac{|\Pi|}{p}\,,\quad R_n^{-1} \equiv \frac{|n^\mu|}{n}\,,\quad R_\pi^{-1} \equiv \frac{|\pi^{\mu\nu}|}{p}$$

Note: If $\Pi \sim \Pi_{\mathrm{NS}}\,,\; n^\mu \sim n^\mu_{\mathrm{NS}}\,,\; \pi^{\mu\nu} \sim \pi^{\mu\nu}_{\mathrm{NS}} \quad \Longrightarrow \quad R_i^{-1} \sim K$

Israel-Stewart equations revisited (I)

additional relaxation terms in IS equations are of second order in K and R_i^{-1}:

e.g. $$\frac{1}{p}\tau_\Pi\,\dot{\Pi} \sim \frac{1}{p}u^\mu\,\ell_{\rm mfp}\,\partial_\mu\,\Pi \sim K\,\frac{\Pi}{p} \sim K\,R_\Pi^{-1}$$

$\Longrightarrow$ **to be consistent, have to include other second-order terms as well!**

to $O(K^2)$:

$$\tau_\Pi\,\dot{\Pi} + \Pi = \Pi_{\rm NS} + \mathcal{K}$$
$$\tau_n\,\dot{n}^{<\mu>} + n^\mu = n^\mu_{\rm NS} + \mathcal{K}^\mu$$
$$\tau_\pi\,\dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\rm NS} + \mathcal{K}^{\mu\nu}$$

$$\mathcal{K} = \zeta_1\,\omega_{\mu\nu}\,\omega^{\mu\nu} + \zeta_2\,\sigma^{\mu\nu}\,\sigma_{\mu\nu} + \zeta_3\,\theta^2 + \zeta_4\,(\nabla\alpha)^2 + \zeta_5\,(\nabla p)^2 + \zeta_6\,\nabla\alpha\cdot\nabla p + \zeta_7\,\nabla^2\alpha + \zeta_8\,\nabla^2 p\,,$$

$$\mathcal{K}^\mu = \kappa_1\,\sigma^{\mu\nu}\,\nabla_\nu\alpha + \kappa_2\,\sigma^{\mu\nu}\,\nabla_\nu p + \kappa_3\,\theta\,\nabla^\mu\alpha + \kappa_4\,\theta\,\nabla^\mu p + \kappa_5\,\omega^{\mu\nu}\,\nabla_\nu\alpha + \kappa_6\,\Delta^{\mu\lambda}\partial^\nu\sigma_{\lambda\nu} + \kappa_7\,\nabla^\mu\theta\,,$$

$$\mathcal{K}^{\mu\nu} = \eta_1\,\omega_\lambda{}^{<\mu}\,\omega^{\nu>\lambda} + \eta_2\,\theta\,\sigma^{\mu\nu} + \eta_3\,\sigma_\lambda{}^{<\mu}\,\sigma^{\nu>\lambda} + \eta_4\,\sigma_\lambda{}^{<\mu}\,\omega^{\nu>\lambda} + \eta_5\,\nabla^{<\mu}\alpha\,\nabla^{\nu>}\alpha$$
$$+\,\eta_6\,\nabla^{<\mu}p\,\nabla^{\nu>}p + \eta_7\,\nabla^{<\mu}\alpha\,\nabla^{\nu>}p + \eta_8\,\nabla^{<\mu}\nabla^{\nu>}\alpha + \eta_9\,\nabla^{<\mu}\nabla^{\nu>}p$$

where $\omega^{\mu\nu} \equiv \nabla^{<\mu}u^{\nu>}$ vorticity

cf. R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100
P. Romatschke, Class. Quant. Grav. 27 (2010) 025006

Israel-Stewart equations revisited (II)

unfortunately, including second-order gradient terms renders eqs. of motion parabolic

$\Longrightarrow$ acausal, unstable $\Longrightarrow$ in general, $\mathcal{K}$, $\mathcal{K}^\mu$, $\mathcal{K}^{\mu\nu}$ have to be omitted!

... but there are more terms of second order: to $O(K R_i^{-1})$, $O(R_i^{-2})$

$$\tau_\Pi \dot{\Pi} + \Pi = \Pi_{\rm NS} + \mathcal{K} + \mathcal{J} + \mathcal{R}$$

$$\tau_n \dot{n}^{<\mu>} + n^\mu = n^\mu_{\rm NS} + \mathcal{K}^\mu + \mathcal{J}^\mu + \mathcal{R}^\mu$$

$$\tau_\pi \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\rm NS} + \mathcal{K}^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu}$$

$$\mathcal{J} = -\ell_{\Pi n}\, \nabla \cdot n - \tau_{\Pi n}\, n \cdot \nabla p - \delta_{\Pi\Pi}\, \theta\, \Pi - \lambda_{\Pi n}\, n \cdot \nabla \alpha + \lambda_{\Pi\pi}\, \pi^{\mu\nu} \sigma_{\mu\nu}$$

$$\mathcal{J}^\mu = \omega^{\mu\nu}\, n_\nu - \delta_{nn}\, \theta\, n^\mu - \ell_{n\Pi}\, \nabla^\mu \Pi + \ell_{n\pi} \Delta^{\mu\nu}\, \nabla^\lambda \pi_{\nu\lambda} + \tau_{n\Pi}\, \Pi\, \nabla^\mu p - \tau_{n\pi}\, \pi^{\mu\nu}\, \nabla_\nu p - \lambda_{nn}\, \sigma^{\mu\nu}\, n_\nu + \lambda_{n\Pi}\, \Pi\, \nabla^\mu \alpha - \lambda_{n\pi}\, \pi^{\mu\nu}\, \nabla_\nu \alpha$$

$$\mathcal{J}^{\mu\nu} = 2\, \pi_\lambda{}^{<\mu}\, \omega^{\nu>\lambda} - \delta_{\pi\pi}\, \theta\, \pi^{\mu\nu} - \tau_{\pi\pi}\, \pi_\lambda{}^{<\mu}\, \sigma^{\nu>\lambda} + \lambda_{\pi\Pi}\, \Pi\, \sigma^{\mu\nu} - \tau_{\pi n}\, n^{<\mu}\, \nabla^{\nu>} p + \ell_{\pi n}\, \nabla^{<\mu} n^{\nu>} + \lambda_{\pi n}\, n^{<\mu} \nabla^{\nu>} \alpha$$

$$\mathcal{R} = \varphi_1\, \Pi^2 + \varphi_2\, n \cdot n + \varphi_3\, \pi^{\mu\nu} \pi_{\mu\nu}$$

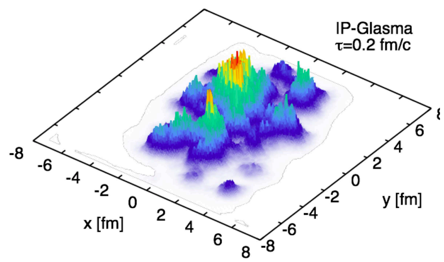
$$\mathcal{R}^\mu = \varphi_4\, \pi^{\mu\nu}\, n_\nu + \varphi_5\, \Pi\, n^\mu$$

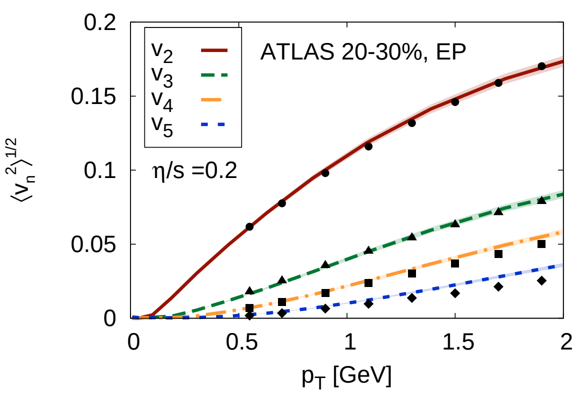
$$\mathcal{R}^{\mu\nu} = \varphi_6\, \Pi\, \pi^{\mu\nu} + \varphi_7\, \pi_\lambda{}^{<\mu}\, \pi^{\nu>\lambda} + \varphi_8\, n^{<\mu}\, n^{\nu>}$$

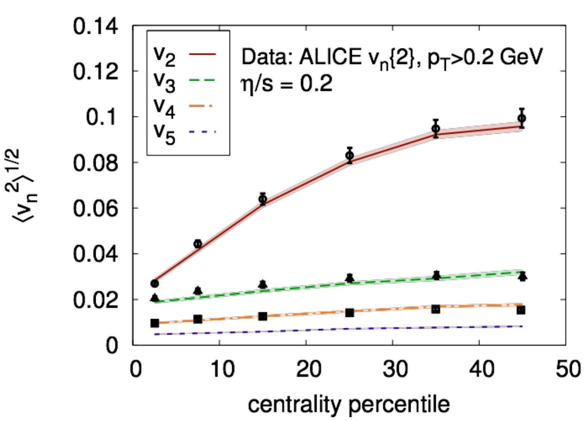
transport coefficients can be determined by matching to the underlying theory, e.g. kinetic theory G.S. Denicol, H. Niemi, E. Molnár, DHR, PRD 85 (2012) 114047

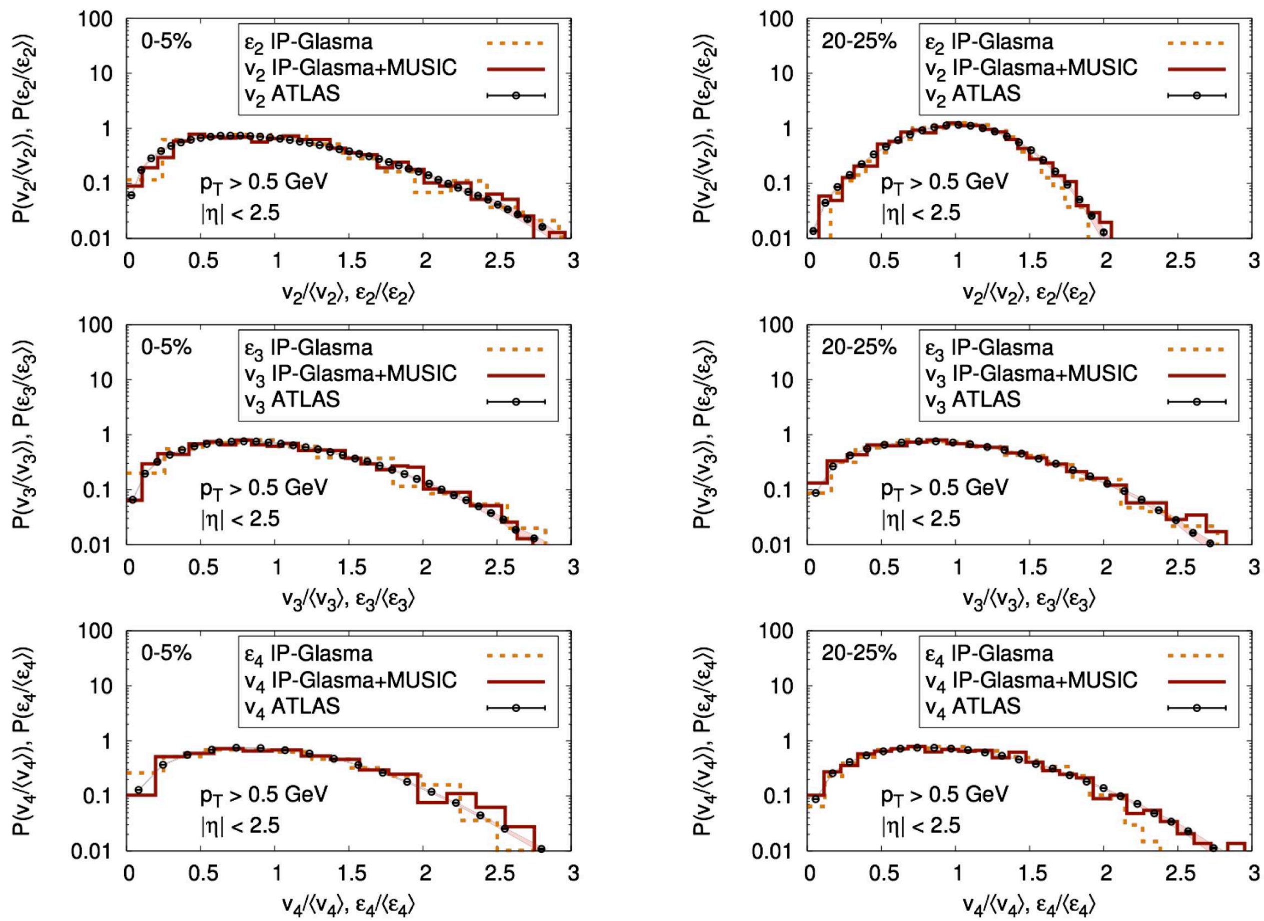
Dissipative hydrodynamics confronts data

C. Gale, S. Jeon, B. Schenke, IJMPA 28 (2013) 1340011









$\Longrightarrow$ Quantitative description of collective flow by dissipative hydrodynamics, for all centralities and event by event, with η/s close to AdS/CFT bound! (case closed! ...?)

Consistency check (I)

Could a conspiration of

1. **the particular choice of initial conditions**
2. **the value of η/s and the other transport coefficients**
3. **the particular form of the hydrodynamical equations**

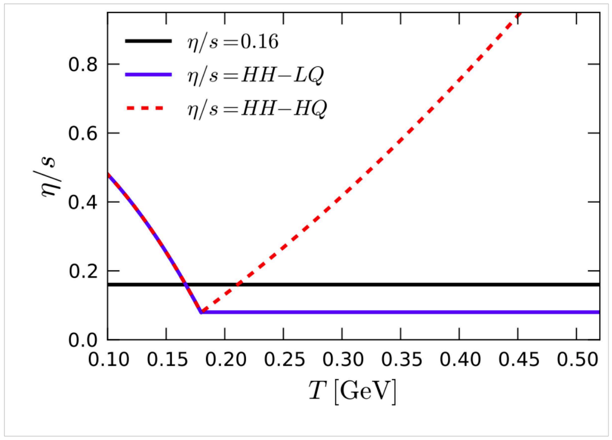
produce correct results for the v_n's, although some (or all) of them are incorrect?

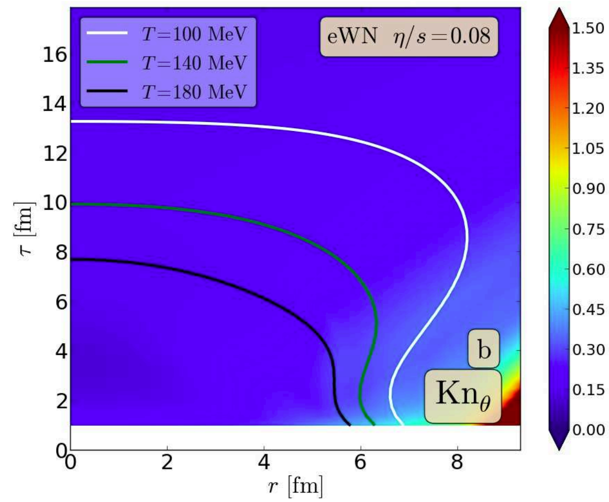
To exclude

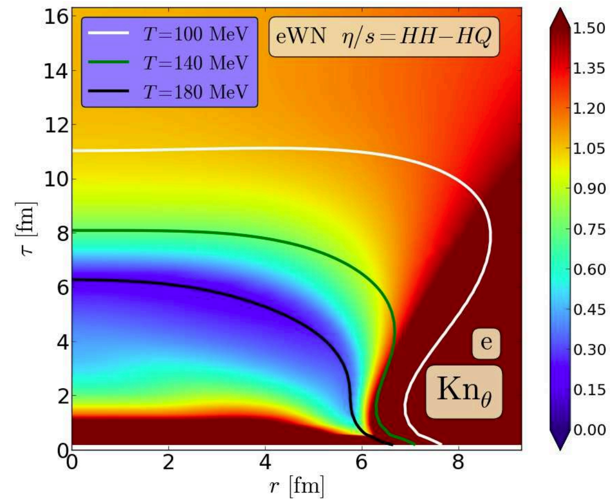
1. **, one needs better understanding of the initial stages**
2. **, one needs independent input (e.g. from LQCD, FRG, etc.), similar to EoS**
3. **, one can perform consistency check:**
   are K , R_i^{-1} sufficiently small, so that the equations of motion of second-order dissipative relativistic hydrodynamics are applicable?

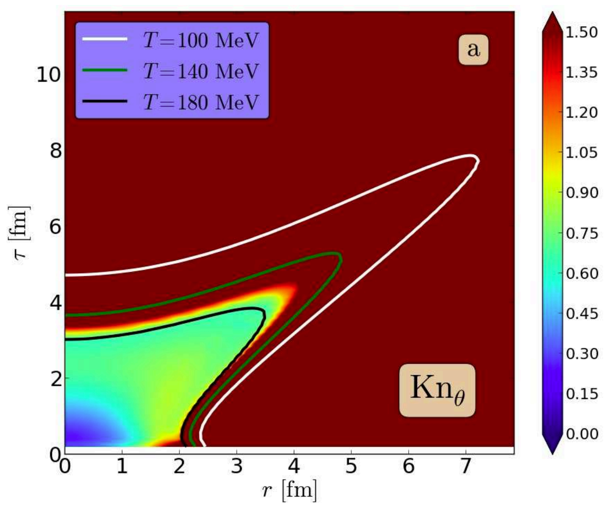
Consistency check (II)

H. Niemi, G.S. Denicol, arXiv:1404.7327 [nucl-th]









- **AA collisions, event-averaged initial conditions: If η/s is sufficiently small and τ_0 sufficiently large, second-order hydrodynamics is applicable**
- **pA collisions: Even for $HH - LQ$ parametrization of η/s, second-order hydrodynamics is barely applicable**

Conclusions

1. Is there collective flow of hot and dense strong-interaction matter created in heavy-ion collisions?

$\Longrightarrow$ **yes!**

2. Can hydrodynamics describe this collective flow ?

$\Longrightarrow$ **Qualitatively: yes!**

Quantitatively: only under favorable circumstances!

hydrodynamics: effective theory for the long-wavelength, small-frequency limit

$\Longrightarrow$ **a quantitatively correct description may require an extension of hydrodynamics to higher orders in the Knudsen number!**

Details (I)

1. Boltzmann equation $k \cdot \partial f_k = C[f]$ for single-particle distribution function f_k
2. introduce irreducible tensors of rank ℓ: $k_{<\mu_1} \cdots k_{\mu_\ell>} \equiv \Delta^{\nu_1 \cdots \nu_\ell}_{\mu_1 \cdots \mu_\ell} k_{\nu_1} \cdots k_{\nu_\ell}$

   $\Delta^{\nu_1 \cdots \nu_\ell}_{\mu_1 \cdots \mu_\ell}$ are projectors onto subspaces orthogonal to u^μ, symmetric in μ_i, ν_i, and traceless
3. f_k can be expanded in terms of $k_{<\mu_1} \cdots k_{\mu_\ell>}$

$$f_k = f_{0k} + f_{0k}\, \tilde{f}_{0k} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}^{(\ell)}_{kn}\, \rho_n^{\mu_1 \cdots \mu_\ell}\, k_{<\mu_1} \cdots k_{\mu_\ell>}$$

where

(a) $f_{0k} = [\exp(\beta\, u \cdot k - \alpha) + a]^{-1}$ single-particle distribution function in local equilibrium, $a = \pm 1/0$ for Fermi/Bose/Boltzmann statistics

(b) $\tilde{f}_{0k} = 1 - a\, f_{0k}$

(c) $\mathcal{H}^{(\ell)}_{kn} = \dfrac{W^{(\ell)}}{\ell!} \sum_{m=n}^{N_\ell} a^{(\ell)}_{mn}\, P^{(\ell)}_{km}$, where

$P^{(\ell)}_{kn} = \sum_{r=0}^{n} a^{(\ell)}_{nr}\, E_k^r$ are orthogonal polynomials of order n in energy $E_k \equiv u \cdot k$

$\Longrightarrow$ $\mathcal{H}^{(\ell)}_{kn}$ are polynomials of order N_ℓ in energy E_k

(d) irreducible moments of $\delta f_k \equiv f_k - f_{0k}$: $\rho_n^{\mu_1 \cdots \mu_\ell} = \int \mathrm{d}K\, \delta f_k\, E_k^n\, k^{<\mu_1} \cdots k^{\mu_\ell>}$

Details (II)

4. rewrite Boltzmann equation in the form $\delta \dot{f}_k = -\dot{f}_{0k} - \frac{1}{E_k}\left\{ k \cdot \nabla \left(f_{0k} + \delta f_k \right) - C[f] \right\}$

5. derive equations of motion for irreducible moments, e.g. up to $\ell = 2$:

$$\begin{aligned}
\dot{\rho}_r &= C_{r-1} + \alpha_r^{(0)}\,\theta - \tfrac{G_{2r}}{D_{20}}\,\theta\,\Pi + \tfrac{G_{2r}}{D_{20}}\,\sigma^{\mu\nu}\,\pi_{\mu\nu} + \tfrac{G_{3r}}{D_{20}}\,\partial\cdot n + (r-1)\sigma_{\mu\nu}\,\rho_{r-2}^{\mu\nu} + r\dot{u}_\mu\rho_{r-1}^{\mu} \\
&\quad - \nabla_\mu\rho_{r-1}^{\mu} - \tfrac{1}{3}\left[(r+2)\rho_r - (r-1)m^2\rho_{r-2}\right]\theta \\
\dot{\rho}_r^{<\mu>} &= C_{r-1}^{<\mu>} + \alpha_r^{(1)}\,\nabla^\mu\alpha + \omega^{\mu}{}_{\nu}\rho_r^{\nu} - \tfrac{1}{3}\left[(r+3)\rho_r^{\mu} - (r-1)m^2\rho_{r-2}^{\mu}\right]\theta - \Delta_\lambda^\mu\nabla_\nu\rho_{r-1}^{\lambda\nu} \\
&\quad - \tfrac{1}{5}\left[(2r+3)\rho_r^{\nu} - 2(r-1)m^2\rho_{r-2}^{\nu}\right]\sigma_\nu^\mu - \tfrac{1}{3}\left[(r+3)\rho_{r+1} - rm^2\rho_{r-1}\right]\dot{u}^\mu \\
&\quad + \tfrac{\beta J_{r+2,1}}{\epsilon+p}\left(\Pi\dot{u}^\mu - \nabla^\mu\Pi + \Delta^{\mu\nu}\partial^\lambda\pi_{\lambda\nu}\right) + \tfrac{1}{3}\nabla^\mu\left(\rho_{r+1} - m^2\rho_{r-1}\right) \\
&\quad + (r-1)\rho_{r-2}^{\mu\nu\lambda}\sigma_{\lambda\nu} + r\dot{u}_\nu\rho_{r-1}^{\mu\nu} \\
\dot{\rho}_r^{<\mu\nu>} &= C_{r-1}^{<\mu\nu>} + 2\,\alpha_r^{(2)}\sigma^{\mu\nu} - \tfrac{2}{7}\left[(2r+5)\rho_r^{\lambda<\mu} - 2(r-1)m^2\rho_{r-2}^{\lambda<\mu}\right]\sigma_\lambda^{\nu>} + 2\rho_r^{\lambda<\mu}\omega_\lambda{}^{\nu>} \\
&\quad + \tfrac{2}{15}\left[(r+4)\rho_{r+2} - (2r+3)m^2\rho_r + (r-1)m^4\rho_{r-2}\right]\sigma^{\mu\nu} \\
&\quad + \tfrac{2}{5}\nabla^{<\mu}\left(\rho_{r+1}^{\nu>} - m^2\rho_{r-2}^{\nu>}\right) - \tfrac{2}{5}\left[(r+5)\rho_{r+1}^{<\mu} - rm^2\rho_{r-1}^{<\mu}\right]\dot{u}^{\nu>} \\
&\quad - \tfrac{1}{3}\left[(r+4)\rho_r^{\mu\nu} - (r-1)m^2\rho_{r-2}^{\mu\nu}\right]\theta + (r-1)\rho_{r-2}^{\mu\nu\lambda\rho}\sigma_{\lambda\rho} - \Delta_{\alpha\beta}^{\mu\nu}\nabla_\lambda\rho_{r-1}^{\alpha\beta\lambda} + r\rho_{r-1}^{\mu\nu\lambda}\dot{u}_\lambda
\end{aligned}$$

$\alpha_r^{(\ell)}$, G_{nm}, D_{nq}, J_{nq} thermodynamic functions

$C_r^{<\mu_1\cdots\mu_\ell>} = \int \mathrm{d}K\, E_k^r\, k^{<\mu_1}\cdots k^{\mu_\ell>}\, C[f]$ irreducible moment of collision integral

Details (III)

Remarks:

(a) system of infinitely many coupled equations for irreducible moments $\rho_r^{\mu_1 \cdots \mu_\ell}$

(b) system completely equivalent to Boltzmann equation

(c) by definition $\rho_0 = -\frac{3}{m^2}\Pi\,,\ \rho_0^\mu = n^\mu\,,\ \rho_0^{\mu\nu} = \pi^{\mu\nu}$

(d) matching conditions in Landau frame imply $\rho_1 = \rho_2 = \rho_1^\mu = 0$

6. fluid dynamics comprises tensors up to rank 2 $\Longrightarrow$ neglect $\rho_r^{\mu_1 \cdots \mu_\ell}$ with $\ell > 2$

7. linearize collision integral: $C_{r-1}^{<\mu_1 \cdots \mu_\ell>} = -\sum_{n=0}^{N_\ell} \mathcal{A}_{rn}^{(\ell)}\, \rho_n^{\mu_1 \cdots \mu_\ell} + O(\delta f_k^2)$

$\Longrightarrow$ linearized equation of motion for irreducible moments:

$$\dot{\vec{\rho}} + \mathcal{A}^{(0)}\,\vec{\rho} \simeq \vec{\alpha}^{(0)}\theta + \ldots$$
$$\dot{\vec{\rho}}^{\,\mu} + \mathcal{A}^{(1)}\,\vec{\rho}^{\,\mu} \simeq \vec{\alpha}^{(1)}\nabla^\mu\alpha + \ldots$$
$$\dot{\vec{\rho}}^{\,\mu\nu} + \mathcal{A}^{(2)}\,\vec{\rho}^{\,\mu\nu} \simeq 2\,\vec{\alpha}^{(2)}\sigma^{\mu\nu} + \ldots$$

8. diagonalize collision matrix: $(\Omega^{-1})^{(\ell)}\mathcal{A}^{(\ell)}\Omega^{(\ell)} = \mathrm{diag}(\chi_0^{(\ell)}, \ldots, \chi_j^{(\ell)}, \ldots)$

for later purposes: $\tau^{(\ell)} \equiv \left(\mathcal{A}^{-1}\right)^{(\ell)} = \Omega^{(\ell)}\ \mathrm{diag}\left(1/\chi_0^{(\ell)}\,,\ \ldots\,,\ 1/\chi_j^{(\ell)}\,,\ \ldots\right)\ \left(\Omega^{-1}\right)^{(\ell)}$

$\Longrightarrow$

$$\tau^{(0)}\,\dot{\vec{\rho}} + \vec{\rho} \simeq \tau^{(0)}\,\vec{\alpha}^{(0)}\theta + \ldots$$
$$\tau^{(1)}\,\dot{\vec{\rho}}^{\,\mu} + \vec{\rho}^{\,\mu} \simeq \tau^{(1)}\,\vec{\alpha}^{(1)}\nabla^\mu\alpha + \ldots$$
$$\tau^{(2)}\,\dot{\vec{\rho}}^{\,\mu\nu} + \vec{\rho}^{\,\mu\nu} \simeq 2\,\tau^{(2)}\,\vec{\alpha}^{(2)}\sigma^{\mu\nu} + \ldots$$

Details (IV)

9. eigenmodes of linearized equations of motion: $X_i^{\mu_1\cdots\mu_\ell} = \sum_{j=0}^{N_\ell} (\Omega^{-1})_{ij}^{(\ell)} \rho_j^{\mu_1\cdots\mu_\ell}$

$\Longrightarrow$ equations of motion for eigenmodes decouple:

$$\dot{X}_i + \chi_i^{(0)} X_i = \beta_i^{(0)} \theta + \ldots$$
$$\dot{X}_i^{<\mu>} + \chi_i^{(1)} X_i^\mu = \beta_i^{(1)} \nabla^\mu \alpha + \ldots$$
$$\dot{X}_i^{<\mu\nu>} + \chi_i^{(2)} X_i^{\mu\nu} = \beta_i^{(2)} \sigma^{\mu\nu} + \ldots$$

where $\beta_i^{(0)} = \sum_{j=0,\neq 1,2}^{N_0} \left(\Omega^{-1}\right)_{ij}^{(0)} \alpha_j^{(0)}\,, \quad \beta_i^{(1)} = \sum_{j=0,\neq 1}^{N_1} \left(\Omega^{-1}\right)_{ij}^{(1)} \alpha_j^{(1)}\,, \quad \beta_i^{(2)} = 2\sum_{j=0}^{N_2} \left(\Omega^{-1}\right)_{ij}^{(2)} \alpha_j^{(2)}$

10. slowest eigenmodes (w/o r.o.g. $i=0$) remain dynamical, all faster ones ($i \neq 0$) are replaced by their asymptotic (NS) values:

$$X_i \simeq \frac{\beta_i^{(0)}}{\chi_i^{(0)}}\,\theta\,, \quad X_i^\mu \simeq \frac{\beta_i^{(1)}}{\chi_i^{(1)}}\,\nabla^\mu \alpha\,, \quad X_i^{\mu\nu} \simeq \frac{\beta_i^{(2)}}{\chi_i^{(2)}}\,\sigma^{\mu\nu}$$

11. Since $\rho_i^{\mu_1\cdots\mu_\ell} = \sum_{j=0}^{N_\ell} \Omega_{ij}^{(\ell)} X_j^{\mu_1\cdots\mu_\ell}$:

$$\rho_i \simeq \Omega_{i0}^{(0)} X_0 + \sum_{j=3}^{N_0} \Omega_{ij}^{(0)} \frac{\beta_j^{(0)}}{\chi_j^{(0)}}\,\theta$$
$$\rho_i^\mu \simeq \Omega_{i0}^{(1)} X_0^\mu + \sum_{j=2}^{N_1} \Omega_{ij}^{(1)} \frac{\beta_j^{(1)}}{\chi_j^{(1)}}\,\nabla^\mu \alpha$$
$$\rho_i^{\mu\nu} \simeq \Omega_{i0}^{(2)} X_0^{\mu\nu} + \sum_{j=1}^{N_2} \Omega_{ij}^{(2)} \frac{\beta_i^{(2)}}{\chi_j^{(2)}}\,\sigma^{\mu\nu}$$

Details (V)

$\Longrightarrow$ express X_0 , X_0^μ , $X_0^{\mu\nu}$ in terms of Π , n^μ , $\pi^{\mu\nu}$ as well as θ , $\nabla^\mu\alpha$, $\sigma^{\mu\nu}$ (w/o r.o.g. $\Omega_{00}^{(\ell)} \equiv 1$):

$$X_0 \simeq -\frac{3}{m^2}\Pi - \sum_{j=3}^{N_0} \Omega_{0j}^{(0)} \frac{\beta_j^{(0)}}{\chi_j^{(0)}} \theta$$

$$X_0^\mu \simeq n^\mu - \sum_{j=2}^{N_1} \Omega_{0j}^{(1)} \frac{\beta_j^{(1)}}{\chi_j^{(1)}} \nabla^\mu\alpha$$

$$X_0^{\mu\nu} \simeq \pi^{\mu\nu} - \sum_{j=1}^{N_2} \Omega_{0j}^{(2)} \frac{\beta_i^{(2)}}{\chi_j^{(2)}} \sigma^{\mu\nu}$$

$\Longrightarrow$ express ρ_i , ρ_i^μ , $\rho_i^{\mu\nu}$ in terms of Π , n^μ , $\pi^{\mu\nu}$ as well as θ , $\nabla^\mu\alpha$, $\sigma^{\mu\nu}$:

$$\boxed{\begin{aligned} \frac{m^2}{3}\rho_i &\simeq -\Omega_{i0}^{(0)}\Pi + \left(\zeta_i - \Omega_{i0}^{(0)}\zeta_0\right)\theta \\ \rho_i^\mu &\simeq \Omega_{i0}^{(1)} n^\mu + \left(\kappa_{n\,i} - \Omega_{i0}^{(1)}\kappa_{n\,0}\right)\nabla^\mu\alpha \\ \rho_i^{\mu\nu} &\simeq \Omega_{i0}^{(2)}\pi^{\mu\nu} + 2\left(\eta_i - \Omega_{i0}^{(2)}\eta_0\right)\sigma^{\mu\nu} \end{aligned}}$$

where $\zeta_i = \frac{m^2}{3} \sum_{r=0,\neq 1,2}^{N_0} \tau_{ir}^{(0)} \alpha_r^{(0)}$, $\kappa_{n\,i} = \sum_{r=0,\neq 1}^{N_1} \tau_{ir}^{(1)} \alpha_r^{(1)}$, $\eta_i = \sum_{r=0}^{N_2} \tau_{ir}^{(2)} \alpha_r^{(2)}$

$\Longrightarrow$ equations of motion for irreducible moments become identical with equations of motion for dissipative quantities Π , n^μ , $\pi^{\mu\nu}$

$\Longrightarrow$ identify transport coefficients

Discussion (I)

1. **Basis of expansion for δf_k is orthogonal in irreducible subspaces**
   $\Longrightarrow$ **truncation at any order in ℓ and N_ℓ possible!**
2. **14-moment approximation corresponds to choice $N_0 = 2\,,\ N_1 = 1\,,\ N_2 = 0$ and leads to IS equations**
3. **approximation can be systematically improved by increasing N_ℓ**
4. **transport coefficients approach Chapman-Enskog values already for $N_0 = 5\,,\ N_1 = 4\,,\ N_2 = 3$ (41-moment approximation)**
   Example: classical massless gas with constant cross section σ, $\ell_{\text{mfp}} = (\sigma n)^{-1}$

# of moments	η	$\tau_\pi[\ell_{\text{mfp}}]$	$\tau_{\pi\pi}[\tau_\pi]$	$\lambda_{\pi n}[\tau_\pi]$	$\delta_{\pi\pi}[\tau_\pi]$	$\ell_{\pi n}[\tau_\pi]$	$\tau_{\pi n}[\tau_\pi]$
14	$4/(3\sigma\beta)$	$5/3$	$10/7$	0	$4/3$	0	0
23	$14/(11\sigma\beta)$	2	$134/77$	$0.344/\beta$	$4/3$	$-0.689/\beta$	$-0.689/n$
32	$1.268/(\sigma\beta)$	2	1.69	$0.254/\beta$	$4/3$	$-0.687/\beta$	$-0.687/n$
41	$1.267/(\sigma\beta)$	2	1.69	$0.244/\beta$	$4/3$	$-0.685/\beta$	$-0.685/n$

# of moments	κ_n	$\tau_n[\ell_{\text{mfp}}]$	$\delta_{nn}[\tau_n]$	$\lambda_{nn}[\tau_n]$	$\lambda_{n\pi}[\tau_n]$	$\ell_{n\pi}[\tau_n]$	$\tau_{n\pi}[\tau_n]$
14	$3/(16\sigma)$	$9/4$	1	$3/5$	$\beta/20$	$\beta/20$	$0.0125\beta/p$
23	$21/(128\sigma)$	2.59	1.0	0.96	0.054β	0.118β	$0.0295\beta/p$
32	$0.1605/\sigma$	2.57	1.0	0.93	0.052β	0.119β	$0.0297\beta/p$
41	$0.1596/\sigma$	2.57	1.0	0.92	0.052β	0.119β	$0.0297\beta/p$

Discussion (II)

5. **approach can be further systematically improved:**
   - (a) **consider also faster eigenmodes $X_i\,,\ X_i^{\mu}\,,\ X_i^{\mu\nu}\,,\ i>0$, to be dynamical**
   - (b) **take into account irreducible moments of tensor rank $\ell > 2$**
   - (c) **take into account second-order corrections in the collision integral (compute coefficients $\varphi_1, \ldots, \varphi_8$)**

     E. Molnar, H. Niemi, G.S. Denicol, DHR, PRD 89 (2014) 074010

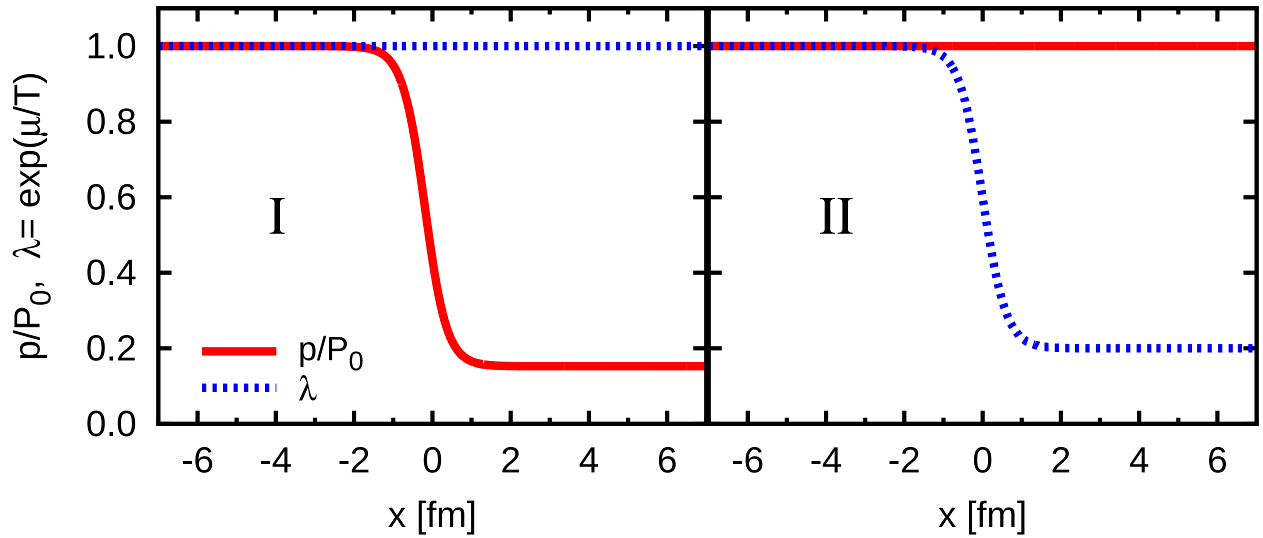
     e.g. massless Boltzmann gas: $\varphi_4 = \dfrac{1}{25\,p}\,,\ \varphi_7 = \dfrac{9}{70\,p}\,,\ \varphi_8 = \dfrac{8}{5\,\beta^2 p}$

Application: heat-flow problem (I)

G.S. Denicol, H. Niemi, I. Bouras, E. Molnár, Z. Xu, DHR, C. Greiner, arXiv:1207.6811[nucl-th]

Initial conditions: discontinuity in

(I) pressure **(II) fugacity**



$\Longrightarrow$ **first-order (NS) terms can be vanishingly small:**

$$\text{(I)}: \nabla^\mu \alpha \simeq 0 \qquad \text{(II)}: \nabla^\mu p \simeq \dot{u}^\mu \simeq 0 \implies \sigma^{\mu\nu} \simeq 0$$

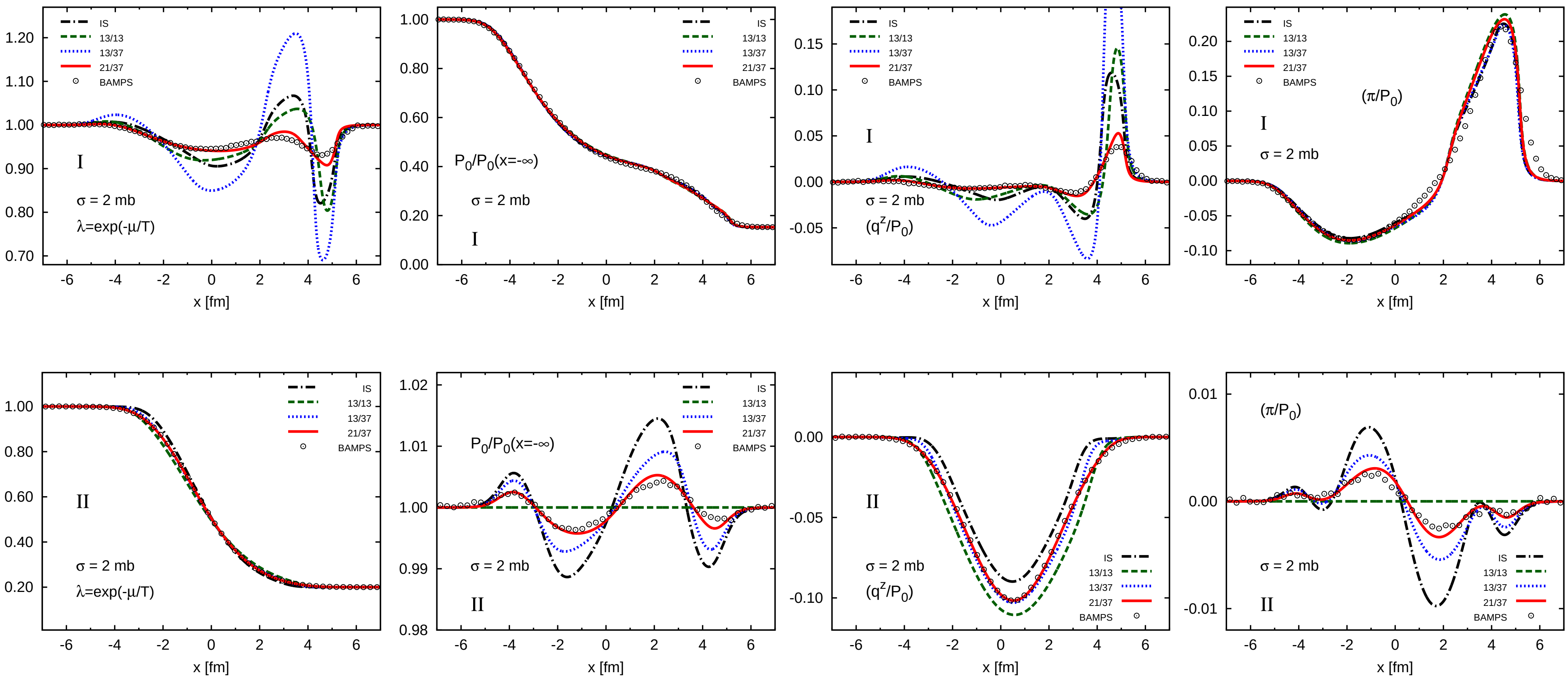
$\Longrightarrow$ **second-order terms can become larger than first-order terms!**

$\Longrightarrow$ **power-counting scheme in terms of Knudsen number is invalidated!**

Application: heat-flow problem (II)

Solution: consider ρ_2^{μ} , $\rho_1^{\mu\nu}$ as additional dynamical variables!

$e^{-\alpha}$ $\quad$ p $\quad$ q^{μ} $\quad$ $\pi^{\mu\nu}$



Conclusions

1. **Second-order fluid dynamics has been systematically derived as long-wavelength, small-frequency limit of kinetic theory**
2. **Transport coefficients agree with values from Chapman-Enskog expansion**
3. **Heat-flow problem can be solved by taking higher irreducible moments to be dynamical variables**
4. **Further systematic improvements are possible and should be explored**