# Bound-electron $g$-factor and fundamental constants

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# Outline

# Free electron $g$ factor

$$g_{\mathrm{free}} = 2\left(1 + \frac{\alpha}{\pi}A^{(2)} + \left(\frac{\alpha}{\pi}\right)^2 A^{(4)} + \left(\frac{\alpha}{\pi}\right)^3 A^{(6)} + \cdots + \Delta\right)$$

| | $A^{(2i)}$ | |
|---|---|---|
| | | 2.000 000 000 000 |
| $\alpha/\pi$ | 0.5 | 0.002 322 819 466 |
| $(\alpha/\pi)^2$ | $-0.328\ldots$ | $-0.000$ 003 544 610 |
| $(\alpha/\pi)^3$ | $1.181\ldots$ | 0.000 000 029 608 |
| $(\alpha/\pi)^4$ | $-1.911(2)$ | $-0.000$ 000 000 111 |
| $(\alpha/\pi)^5$ | 9.16(58) | 0.000 000 000 001 |
| $\Delta$ | | 0.000 000 000 002 |
| $g_{\mathrm{free}}$ | | 2.002 319 304 361 |

# Free electron $g$ factor

$$g_{\mathrm{free}} = 2\left(1 + \frac{\alpha}{\pi}A^{(2)} + \left(\frac{\alpha}{\pi}\right)^2 A^{(4)} + \left(\frac{\alpha}{\pi}\right)^3 A^{(6)} + \cdots + \Delta\right)$$

Experiment: $g_{\mathrm{free}} = 2.002\,319\,304\,361\,46(56)$

Hanneke, Fogwell, Gabrielse, PRL (2008)

Theory: $A^{(i)}, \Delta$

Aoyama, Hayakawa, Kinoshita, Nio, PRL (2012)

$\rightarrow \alpha = 1/137.035\,999\,173\,(35)$

vs $\alpha = 1/137.035\,999\,044\,(90)$

Bouchendira, Cladé, Guellati-Khélifa, Nez, Biraben, PRL (2011)

# Highly charged ions

Nucleus + several electrons: $N_e \ll Z$

Nuclear potential: $V_{\rm nuc}(r) = -\dfrac{\alpha Z}{r}$

Electron velocity: $v/c \sim \alpha Z$

Low-$Z$ systems: $\alpha Z \ll 1 \rightarrow \alpha Z$-expansion

High-$Z$ systems: $\alpha Z \sim 1 \rightarrow$ all orders in $\alpha Z$

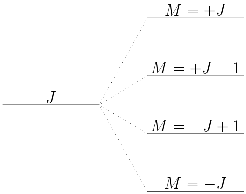
# Highly charged ions

Simple systems: (closed shell(s)) + 1 electron

- H-like: $1s$
- Li-like: $(1s)^2\,2s$
- B-like: $(1s)^2\,(2s)^2\,2p$

Zeeman splitting:

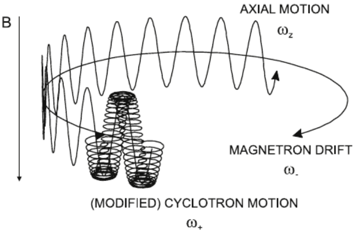
$$\Delta E_{\mathrm{mag}} = \langle -\boldsymbol{\mu}\mathbf{B}\rangle = g\,M\,\mu_0\,B$$



# Experiments: past and future

| Z | H-like: $1s$ | Li-like: $(1s)^2 2s$ | B-like: $(1s)^2(2s)^2 2p$ |
|---|---|---|---|
| 6 | 2.001 041 596 (5)<br>2.001 041 590 2 (4) | | |
| 8 | 2.000 047 025 (2) | | |
| 14 | 1.995 348 952 (1) | 2.000 889 890 (2) | |
| 18 | | | *in progress*<br>GSI |
| 20 | *planned*<br>Uni Mainz | *in progress*<br>Uni Mainz | |
| ...<br>up to 92 | MPIK, FAIR | | |

# Measurement principle
## University of Mainz




$$\begin{aligned}
\omega_c^2 &= \omega_+^2 + \omega_-^2 + \omega_z^2 \\
\omega_c &= \frac{qe}{m_{\rm ion}} B \\
\omega_L &= \frac{g}{2} \frac{e}{m_e} B \\
\frac{\omega_L}{\omega_c} &= \frac{g}{2q} \frac{m_{\rm ion}}{m_e}
\end{aligned}$$

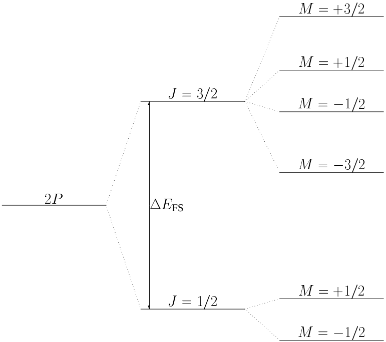
$$m_e = 0.000\,548\,579\,909\,46\,(22)\ \mathrm{u}$$

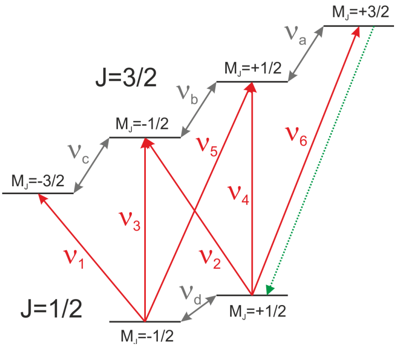
Häffner et al., PRL (2000); Beier et al., PRL (2002); Verdú et al., PRL (2004)

$$m_e = 0.000\,548\,579\,909\,067\,(17)\ \mathrm{u}$$

Sturm et al., Nature (2014)

# Measurement principle
## ARTEMIS experiment at GSI



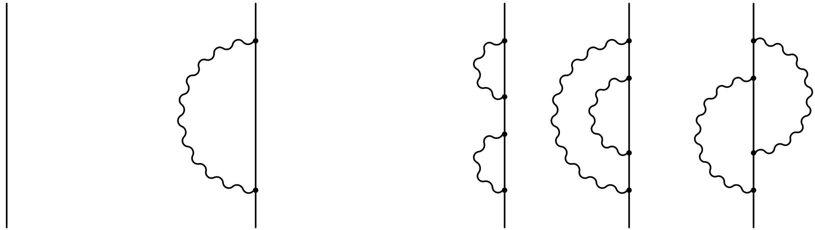




Laser-microwave double-resonance spectroscopy

von Lindenfels et al., PRA (2013)

# QED for bound states

$\alpha \approx 1/137 \ll 1$ — good expansion parameter

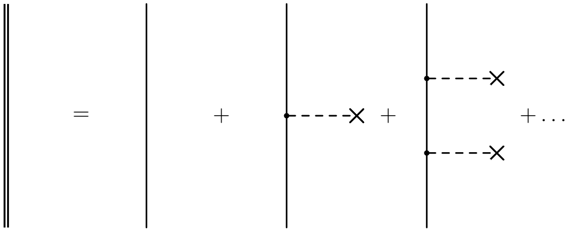


# QED for bound states

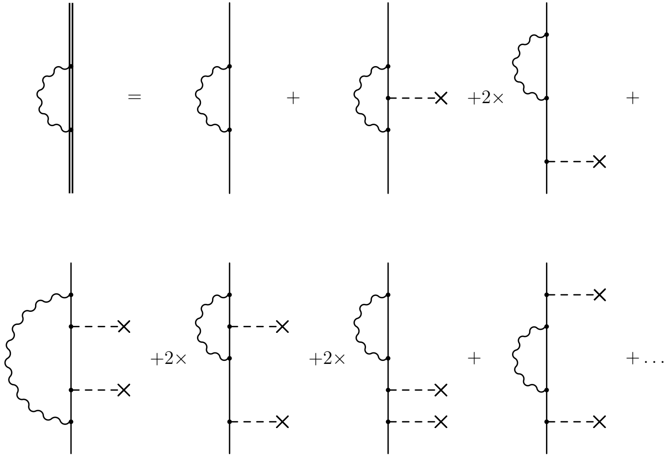
$\alpha \approx 1/137 \ll 1$ — good expansion parameter

$1/137 \leq \alpha Z \lesssim 1$ — not really

*Furry picture* of QED:



# QED for bound states

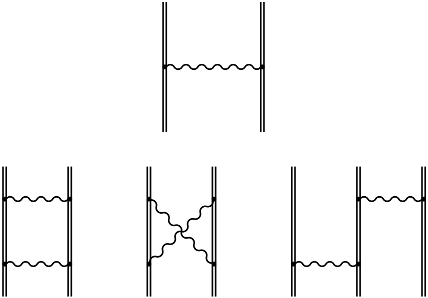


# QED for bound states

$\alpha \approx 1/137 \ll 1$ — good expansion parameter

$1/137 \leq \alpha Z \lesssim 1$ — not really

$1/Z = \alpha/\alpha Z$ — it depends



# Bound electron $g$ factor

$$g_{\text{bound}} = g_{\text{L}} \left( A^{(0)}(\alpha Z) + \frac{\alpha}{\pi} A^{(2)}(\alpha Z) + \left(\frac{\alpha}{\pi}\right)^2 A^{(4)}(\alpha Z) + \ldots \right)$$

Non-relativistic limit:

$$g_{\text{L}} = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)} = \begin{cases} 2 & \text{for } ns \\ 2/3 & \text{for } np_{1/2} \\ 4/3 & \text{for } np_{3/2} \\ \ldots & \end{cases}$$

Binding effect:

$$B^{(i)}(\alpha Z) = A^{(i)}(\alpha Z) - A^{(i)}(0) \sim (\alpha Z)^2$$

# Bound electron $g$ factor

## Determination of $\alpha$

$$g_{\mathrm{bound}} = g_{\mathrm{L}} \left( A^{(0)}(\alpha Z) + \frac{\alpha}{\pi} A^{(2)}(\alpha Z) + \left(\frac{\alpha}{\pi}\right)^2 A^{(4)}(\alpha Z) + \dots \right)$$

$$1s: \qquad A^{(0)}(\alpha Z) = \frac{1}{3}\left(1 + 2\sqrt{1-(\alpha Z)^2}\right)$$

$$\frac{\mathrm{d}g_{1s}}{\mathrm{d}\alpha} \approx -\frac{4\alpha Z^2}{3\sqrt{1-(\alpha Z)^2}} \approx \begin{cases} 0.01 & \text{for } Z = 1 \\ 111 & \text{for } Z = 92 \end{cases}$$

$$\frac{\mathrm{d}g_{\mathrm{free}}}{\mathrm{d}\alpha} \approx \frac{1}{\pi} \approx 0.32$$

# Bound electron $g$ factor

## Determination of $\alpha$

free electron

$$\frac{\delta\alpha}{\alpha} = \frac{2\pi}{\alpha}\frac{\delta g_{\mathrm{free}}}{g_{\mathrm{free}}}$$

$$\delta g_{free}^{\mathrm{exp}} = 3 \times 10^{-13}$$

$$\rightarrow \frac{\delta\alpha}{\alpha} = 3 \times 10^{-10}$$

bound electron

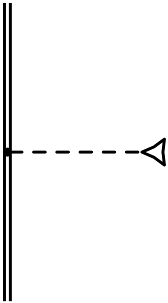
$$\frac{\delta\alpha}{\alpha} \sim \frac{1}{(\alpha Z)^2}\frac{\delta g_{1s}}{g_{1s}}$$

$$\delta g_{1s}^{\mathrm{exp}} = 1.5 \times 10^{-10}$$

$$\rightarrow \frac{\delta\alpha}{\alpha} = 3 \times 10^{-10}$$

(for Pb, $Z = 82$)

# $g$ factor: leading order



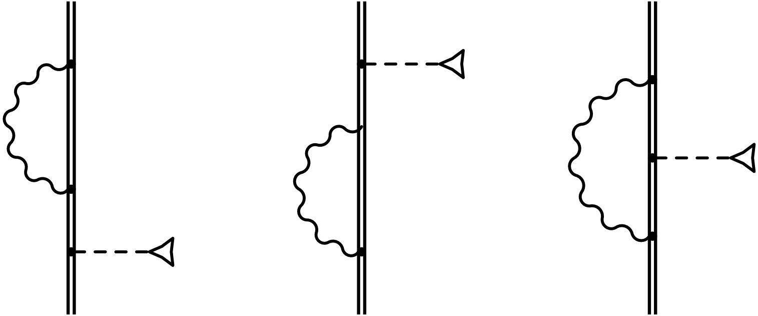
$$g_{\rm D}(\alpha Z) = \frac{\kappa(2\kappa\varepsilon - 1)}{2j(j+1)}$$

$$g_{\rm D}[1s](\alpha Z) = \frac{2}{3}\left(1 + 2\sqrt{1-(\alpha Z)^2}\right)$$

$$g_{\rm D}(\alpha Z) \to g_{\rm D}(0) = g_{\rm L}$$

$$g_{\rm L} = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)}$$

# Self-energy

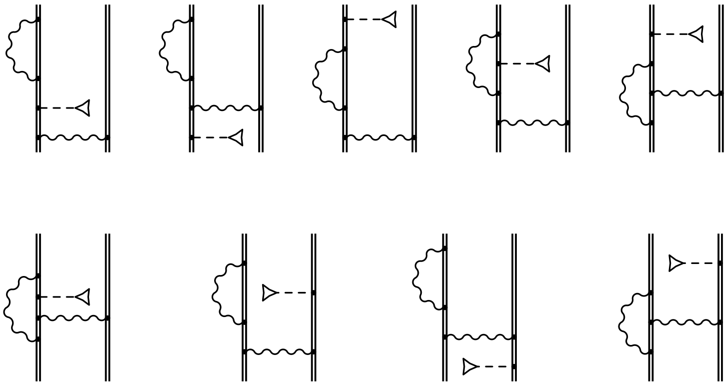


Yerokhin, Indelicato, Shabaev, PRL(2002); PRA (2004)

Yerokhin, Jentschura, PRA (2010)

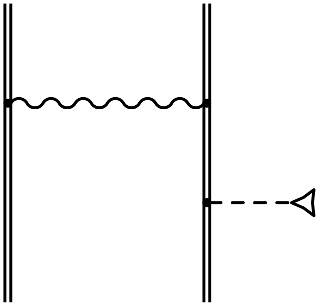
Glazov et al., PLA (2006)

# Two-electron self-energy



Volotka et al., PRL (2009); Glazov et al., PRA (2010)

# One-photon exchange

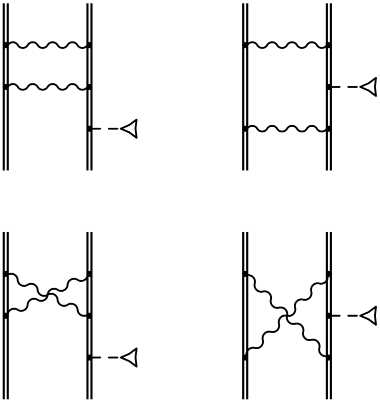


$$\Delta g_{\rm int} = \Delta g_{\rm int}^{(1)} + \Delta g_{\rm int}^{(2)} + \dots$$

$$\Delta g_{\rm int}^{(1)} = \frac{1}{Z} (\alpha Z)^2 B(\alpha Z)$$

Shabaev et al., PRA (2002)

# Two-photon exchange: 2-electron diagrams

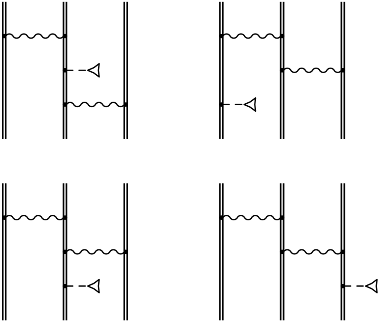


$$\Delta g_{\mathrm{int}} = \Delta g_{\mathrm{int}}^{(1)} + \Delta g_{\mathrm{int}}^{(2)} + \dots$$

$$\Delta g_{\mathrm{int}}^{(2)} = \frac{1}{Z^2} (\alpha Z)^2 \, C(\alpha Z)$$

Wagner et al., PRL (2013); Volotka et al., PRL (2014)

# Two-photon exchange: 3-electron diagrams



$$\Delta g_{\rm int} = \Delta g_{\rm int}^{(1)} + \Delta g_{\rm int}^{(2)} + \ldots$$

$$\Delta g_{\rm int}^{(2)} = \frac{1}{Z^2}(\alpha Z)^2 C(\alpha Z)$$

Wagner et al., PRL (2013); Volotka et al., PRL (2014)

# Higher-order contributions: beyond $1/Z$

$$\Delta g_{\rm int} = \Delta g_{\rm int}^{(1)} + \Delta g_{\rm int}^{(2+)}$$

$\Delta g_{\rm int}^{(2+)}$: large-scale CI Dirac-Fock-Sturm

Li-like silicon $^{28}\mathrm{Si}^{11+}$

| | |
|---|---|
| $\Delta g_{\rm int}^{(1)}$ | 0.000 321 592 |
| $\Delta g_{\rm int}^{(2+)}$ | $-$0.000 006 689 (74) |
| $\Delta g_{\rm int}$ | 0.000 314 903 (74) |

Glazov et al., PRA (2004); Glazov et al., PLA (2006)

# Higher-order contributions: beyond $1/Z^2$

$$\Delta g_{\rm int} = \Delta g_{\rm int}^{(1)} + \Delta g_{\rm int}^{(2)} + \Delta g_{\rm int}^{(3+)}$$

$\Delta g_{\rm int}^{(3+)}$: large-scale CI Dirac-Fock-Sturm

Li-like silicon $^{28}\mathrm{Si}^{11+}$

| | |
|---|---|
| $\Delta g_{\rm int}^{(1)}$ | 0.000 321 592 |
| $\Delta g_{\rm int}^{(2)}$ | $-$0.000 006 876 (1) |
| $\Delta g_{\rm int}^{(3+)}$ | 0.000 000 085 (22) |
| $\Delta g_{\rm int}$ | 0.000 314 801 (22) |

Wagner et al., PRL (2013)

# Higher-order contributions: screening potential

$$\Delta g_{\rm int} = \Delta g_{\rm int}^{(0)} + \Delta g_{\rm int}^{(1)} + \Delta g_{\rm int}^{(2)} + \Delta g_{\rm int}^{(3+)}$$

$\Delta g_{\rm int}^{(3+)}$: large-scale CI Dirac-Fock-Sturm

Li-like silicon $^{28}\mathrm{Si}^{11+}$

| | |
|---|---|
| $\Delta g_{\rm int}^{(0)}$ | 0.000 349 636 |
| $\Delta g_{\rm int}^{(1)}$ | −0.000 033 846 |
| $\Delta g_{\rm int}^{(2)}$ | −0.000 000 976 |
| $\Delta g_{\rm int}^{(3+)}$ | −0.000 000 005 (6) |
| $\Delta g_{\rm int}$ | 0.000 314 808 (6) |

Volotka et al., PRL (2014)

# $g$ factor of Li-like silicon (Z=14)

| | | |
|---|---|---|
| Free electron | Dirac + QED | 2.002 319 304 |
| Binding | Dirac | −0.001 745 249 |
| | QED $\sim \alpha$ | 0.000 001 224 (3) |
| | QED $\sim \alpha^2$ | −0.000 000 001 |
| $e^-$–$e^-$ interaction | $\sim 1/Z$ | 0.000 321 592 |
| | $\sim 1/Z^2$ | −0.000 006 876 (1) |
| | $\sim 1/Z^{3+}$ | 0.000 000 085 (22) |
| | $\sim \alpha/Z^+$ | −0.000 000 212 (46) |
| Nuclear effects | Recoil | 0.000 000 039 (1) |
| | Finite size | 0.000 000 003 |
| | Total theory | 2.000 889 909 (51) |
| | Experiment | 2.000 889 890 (2) |

Wagner et al., PRL (2013)

# $g$ factor of Li-like silicon (Z=14)

| | | |
|---|---|---|
| Free electron | Dirac + QED | 2.002 319 304 |
| Binding | Dirac | −0.001 395 613 |
| | QED $\sim \alpha$ | 0.000 001 224 (3) |
| | QED $\sim \alpha^2$ | −0.000 000 001 |
| $e^-$–$e^-$ interaction | $\sim 1/Z$ | −0.000 033 846 |
| | $\sim 1/Z^2$ | −0.000 000 976 |
| | $\sim 1/Z^{3+}$ | −0.000 000 005 (6) |
| | $\sim \alpha/Z^+$ | −0.000 000 236 (5) |
| Nuclear effects | Recoil | 0.000 000 039 (1) |
| | Finite size | 0.000 000 003 |
| | Total theory | 2.000 889 892 (8) |
| | Experiment | 2.000 889 890 (2) |

Wagner et al., PRL (2013); Volotka et al., PRL (2014)

# Nuclear effects

$$g_{\text{bound}} = g_{\text{L}} \left( A^{(0)}(\alpha Z) + \frac{\alpha}{\pi} A^{(2)}(\alpha Z) + \left(\frac{\alpha}{\pi}\right)^2 A^{(4)}(\alpha Z) + \dots \right) + \Delta g_{\text{nuc}}$$

Problem: $\Delta g_{\text{nuc}} = \text{nuclear} \begin{cases} \text{recoil} \\ \text{finite size} \\ \text{polarization} \end{cases}$

Solutions:

- H-like + Li-like ions: $g' = g_{(1s)^2 2s} - \xi g_{1s}$

  Shabaev et al., PRA (2002)
- H-like + B-like ions (high Z): $g' = g_{(1s)^2 (2s)^2 2p} - \xi g_{1s}$

  Shabaev et al., PRL (2006); Volotka and Plunien, PRL (2014)

# Nuclear magnetic moment

Total angular momentum: $\mathbf{F} = \mathbf{j} + \mathbf{I}$

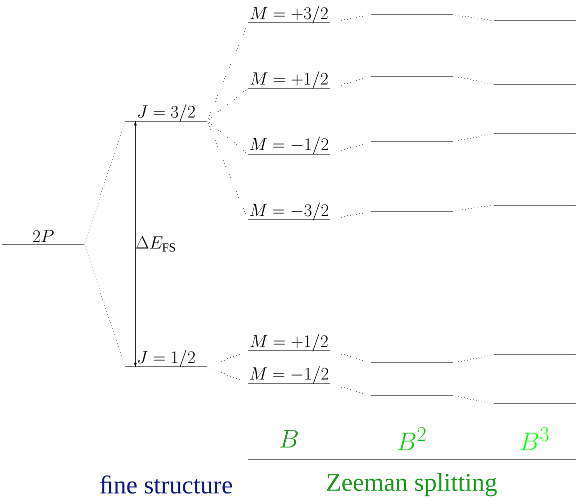
$$F = |j - I|, \ldots j + I \,, \quad M_F = -F, \cdots + F$$

Total magnetic moment: $\boldsymbol{\mu} = \boldsymbol{\mu}_j + \boldsymbol{\mu}_I$

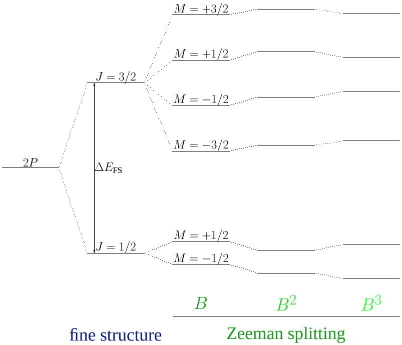
$$\Delta E_{\rm mag} = g_F \mu_0 B M_F \quad \text{if} \quad \Delta E_{\rm mag} \ll \Delta E_{\rm HFS}$$

$$g_F = g_j \frac{F(F+1) + j(j+1) - I(I+1)}{2F(F+1)} - \frac{m_e}{m_p} g_I \frac{F(F+1) - j(j+1) + I(I+1)}{2F(F+1)}$$

# Zeeman splitting in B-like ion

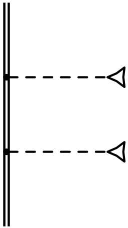


# Non-linear contributions in $B$




$$
\begin{aligned}
E(B) &= E^{(0)} \\
&+ \left[\Delta E^{(1)} = g\, M\, \mu_0\, B\right] \\
&+ \left[\Delta E^{(2)} = g^{(2)}(M)\,(\mu_0 B)^2\right] \\
&+ \left[\Delta E^{(3)} = g^{(3)}(M)\,(\mu_0 B)^3\right] \\
&+ \ldots
\end{aligned}
$$

# Second-order Zeeman effect

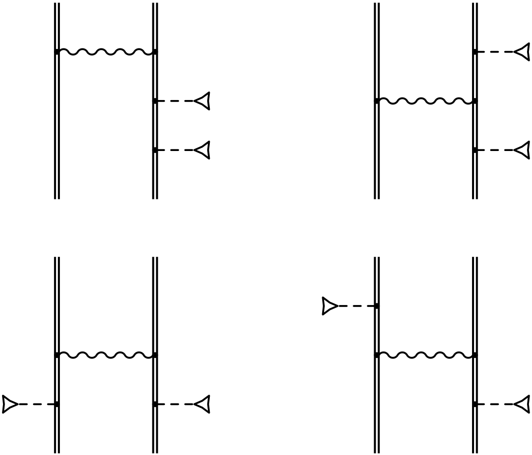


$$\Delta E^{(2)}(B) = g^{(2)}(\mu_0 B)^2 \approx \frac{\langle a|\hat{V}_{\mathrm{m}}|b\rangle\langle b|\hat{V}_{\mathrm{m}}|a\rangle}{\Delta E_{\mathrm{FS}}}$$

$$\approx 0.92 \cdot 10^{-4} \times \Delta E^{(1)}(B)$$

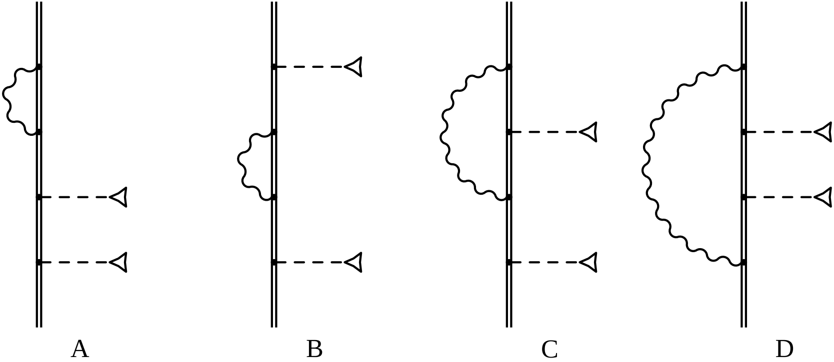
$$a = 2p_{1/2}, \quad b = 2p_{3/2} \qquad \text{or} \qquad a = 2p_{3/2}, \quad b = 2p_{1/2}$$

# One-photon exchange



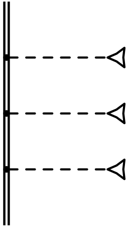
to be published

# QED correction. Self-energy diagrams




*in progress*

# Third-order Zeeman effect



$$\Delta E^{(3)}(B) = g^{(3)}(\mu_0 B)^3 \approx \frac{\langle a|\hat{V}_\mathrm{m}|b\rangle\langle b|\hat{V}_\mathrm{m}|a\rangle}{\Delta E_\mathrm{FS}^2}\left(\langle b|\hat{V}_\mathrm{m}|b\rangle - \langle a|\hat{V}_\mathrm{m}|a\rangle\right)$$

$$\approx 4.3\cdot 10^{-9} \times \Delta E^{(1)}(B)$$

$a = 2p_{1/2}, \quad b = 2p_{3/2}$ or $a = 2p_{3/2}, \quad b = 2p_{1/2}$

# Summary

$g$ factor of HCI is available for both

- high-precision measurements
- accurate calculations

$\rightarrow$ Test of theory: bound-state QED

$\rightarrow$ Determination of

- $\alpha$ — fine structure constant
- $m_e$ — electron mass (in a.u.)
- $\mu_N$ — nuclear magnetic moment
- $R_N$ — nuclear charge radius