INTERPLAY OF $\gamma$-RIGID AND $\gamma$-STABLE COLLECTIVE MOTION IN THE PHASE TRANSITION FROM SPHERICAL TO DEFORMED NUCLEAR SHAPES

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The classical Hamiltonian function of the liquid drop model has 5 degrees of freedom, namely the two shape variables $\beta$ and $\gamma$ and the three Euler angles.

$$\mathcal{H} = \frac{B}{2} \left( \dot{\beta}^2 + \dot{\gamma}^2 \right) + \frac{1}{2} \sum_{k=1}^{3} \omega_k^2 I_k + V(\beta, \gamma).$$

Bohr-Mottelson Hamiltonian after quantization

Imposing a certain value for the $\gamma$ shape variable, one reaches the $\gamma$-rigid version of the collective model which is interesting by itself due to its description of the basic rotation-vibration coupling.

- $\gamma \neq 0^\circ \Rightarrow$ 4 degrees of freedom $(\beta, \theta_1, \theta_2, \theta_3) \Rightarrow$ Davydov-Chaban Hamiltonian Davydov & Chaban NP 20 (1960) 499
- $\gamma = 0^\circ \Rightarrow$ 3 degrees of freedom $(\beta, \theta_1, \theta_2) \Rightarrow$ X(3)-type Hamiltonian Bonatsos et. al. PLB 632 (2006) 238

Although the $\gamma$-rigidity hypothesis is somewhat crude it provides simple approaches to the successful reproduction of the relevant experimental data.


The similarity between the $\beta$ excited bands of the X(5) and X(3) solutions addresses the question about the importance of rigidity in explaining the critical collective phenomena.
The kinetic energy operator $\hat{T}_{vib} + \hat{T}_{rot}$ in the five-dimensional shape phase space

$$T_s = -\frac{\hbar^2}{2B} \left[ \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} - \frac{1}{4\beta^2} \sum_{k=1}^{3} \frac{Q_k^2}{\sin^2 (\gamma - \frac{2}{3}\pi k)} \right]$$

In the prolate $\gamma$-rigid regime defined only by three degrees of freedom, the same operator gets a simpler form

$$T_r = -\frac{\hbar^2}{2B} \left[ \frac{1}{\beta^2} \frac{\partial}{\partial \beta} \beta^2 \frac{\partial}{\partial \beta} - \frac{Q^2}{3\beta^2} \right]$$

The interplay between $\gamma$-stable and $\gamma$-rigid collective motion is achieved by considering the Hamiltonian:

$$H = \chi T_r + (1 - \chi) T_s + V(\beta, \gamma), \quad 0 \leq \chi < 1$$

$\beta$ variable is separated from the $\gamma$-angular ones if the potential have the structure

$$v(\beta, \gamma) = \frac{2B}{\hbar^2} V(\beta, \gamma) = u(\beta) + (1 - \chi) \frac{u(\gamma)}{\beta^2}$$
Factorizing the total wave function as $\Psi(\beta, \gamma, \Omega) = \xi(\beta)\varphi(\gamma, \Omega)$, the associated Schrödinger equation is separated in two parts:

$$
\begin{bmatrix}
(1 - \chi) \left( -\frac{1}{\sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} + \sum_{k=1}^{3} \frac{Q_k^2}{4 \sin^2 \left( \gamma - \frac{2}{3} \pi k \right)} + u(\gamma) \right) + \frac{\chi}{3} Q^2 \\
\end{bmatrix} \varphi(\gamma, \Omega) = W \varphi(\gamma, \Omega)
$$

Small angle approximation $\Rightarrow$ $u(\gamma) = \left(3a\right)^2 \gamma^2$  

\begin{align*}
W &= 3a(1 - \chi)(n_\gamma + 1) + \frac{L(L + 1) - (1 - \chi)K^2}{3}, \quad \varphi(\gamma, \Omega) = \eta(\gamma)D_{MK}^{L}(\Omega) \\
\eta_{n_\gamma, |K|}(\gamma) &= N_{n, |K|}\gamma^{\frac{|K|}{2}} \exp \left(-3a \frac{\gamma^2}{2}\right) L_n^{|K|/2}(3a\gamma^2), \quad n = \frac{1}{2} \left(n_\gamma - \frac{|K|}{2}\right)
\end{align*}

$$
\begin{bmatrix}
-\frac{\partial^2}{\partial \beta^2} - \frac{2(2 - \chi)}{\beta} \frac{\partial}{\partial \beta} + \frac{W}{\beta^2} + u(\beta) \\
\end{bmatrix} \xi(\beta) = \epsilon \xi(\beta), \quad \text{Flat potential} \quad u(\beta) = \left\{ \begin{array}{ll}
0, \quad \beta \leq \beta_W, \\
\infty, \quad \beta > \beta_W.
\end{array} \right.
$$

$$
\Rightarrow \epsilon_{L,K,s,n_\gamma}(\beta_W) = \left( \frac{x_{s,\nu}}{\beta_W} \right)^2, \quad \xi_{L,K,s,n_\gamma}(\beta) = N_{s,\nu,\beta} \chi^{-\frac{3}{2}} J_\nu \left( \frac{x_{s,\nu,\beta}}{\beta_W} \right)
$$

$x_{s,\nu}$ is $s$-th zero of the Bessel function $J_\nu(x_{s,\nu,\beta}/\beta_W)$ and $n_\beta = s - 1.$

$$
\nu = \left[ \frac{L(L + 1) - (1 - \chi)K^2}{3} + (\frac{3}{2} - \chi)^2 + (1 - \chi)3a(n_\gamma + 1) \right]^{\frac{1}{2}}.
$$
The excitation energy of the whole system in respect to the ground state

\[ E_{K,n,\beta,n,\gamma}(L) = \frac{\hbar^2}{2B} \left[ \epsilon_{L,K,n,\beta+1,n,\gamma}(\beta_W) - \epsilon_{0,0,1,0}(\beta_W) \right] \]

The full solution after proper normalization and symmetrization reads:

\[ \Psi_{LMKn,\beta,n,\gamma}(\beta,\gamma,\Omega) = \xi_{L,K,n,\beta,n,\gamma}(\beta)\eta_{n,\gamma} |K| \langle \gamma \rangle \sqrt{\frac{2L+1}{16\pi^2(1+\delta_{0,0})}} \left[ D_{LK}(\Omega) + (-)^L D_{LM-K}(\Omega) \right] \]

The transition rates are calculated by employing the general expression for the quadrupole transition operator:

\[ T_{\mu}^{(E2)} = t\beta \left[ D_{\mu 0}^2(\Omega) \cos \gamma + \frac{1}{\sqrt{2}} \left( D_{\mu 2}^2(\Omega) + D_{\mu -2}^2(\Omega) \right) \sin \gamma \right] \]

The final result for the \( E2 \) transition probability is given in a factorized form:

\[ B(E2; LKn,\beta,n,\gamma \to L'K'n',\beta,n',\gamma) = \frac{5t^2}{16\pi} \left( C_{KK'-KK'}^{L,L'} B_{L'K'n',\beta,n',\gamma}^{LK} G_{K'n',\gamma}^{Kn,\gamma} \right)^2 \]
The evolution as function of $\chi$ and $a$ of theoretically evaluated spectral observables such as $R_{4/2} = E(4^+_g)/E(2^+_g)$ (a) ratio and the $\beta$ (b) and $\gamma$ (c) band heads normalized to the energy of the first excited state.

Bonatsos et. al. PLB 649 (2007) 394

$\chi = 0 \xrightarrow{x} X(3)$

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The ground state $\beta$ probability density in respect to the $d\beta$ integration measure.
NUMERICAL APPLICATION

- **Experimental ground, $\gamma$ and $\beta$ band states**
- **Experimental ground, $\gamma$ and $\beta$ band states with uncertain assignment**
- **Theoretical ground, $\gamma$ and $\beta$ band predictions**

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>0.948</th>
<th>0.269</th>
<th>0.848</th>
</tr>
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<tbody>
<tr>
<td>$a$</td>
<td>168.899</td>
<td>10.309</td>
<td>41.191</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.567</td>
<td>0.845</td>
<td>1.359</td>
</tr>
<tr>
<td>Nr. of states</td>
<td>19</td>
<td>32</td>
<td>24</td>
</tr>
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</table>
Comparison of theoretical results with experiment and rigid rotor (R. R.) predictions for several $E2$ transition probabilities.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$\frac{4^+_g \rightarrow 2^+_g}{2^+_g \rightarrow 0^+_g}$</th>
<th>$\frac{6^+_g \rightarrow 4^+_g}{2^+_g \rightarrow 0^+_g}$</th>
<th>$\frac{8^+_g \rightarrow 6^+_g}{2^+_g \rightarrow 0^+_g}$</th>
<th>$\frac{10^+_g \rightarrow 8^+_g}{2^+_g \rightarrow 0^+_g}$</th>
<th>$\frac{12^+_g \rightarrow 10^+_g}{2^+_g \rightarrow 0^+_g}$</th>
<th>$\frac{14^+_g \rightarrow 12^+_g}{2^+_g \rightarrow 0^+_g}$</th>
<th>$\frac{2^+_\gamma \rightarrow 2^+<em>g}{2^+</em>\gamma \rightarrow 0^+_g}$</th>
<th>$\frac{2^+_\gamma \rightarrow 4^+<em>g}{2^+</em>\gamma \rightarrow 0^+_g}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{160}\text{Gd}$</td>
<td>1.45</td>
<td>1.62</td>
<td>1.74</td>
<td>1.83</td>
<td>1.90</td>
<td>1.97</td>
<td>1.441</td>
<td>0.073</td>
</tr>
<tr>
<td>$^{162}\text{Dy}$</td>
<td>1.42(6)</td>
<td>1.48(9)</td>
<td>1.70(9)</td>
<td>1.72(11)</td>
<td>1.62(20)</td>
<td>1.62(20)</td>
<td>1.78(16)</td>
<td>0.137(12)</td>
</tr>
<tr>
<td>$^{166}\text{Er}$</td>
<td>1.44(6)</td>
<td>1.71(10)</td>
<td>1.72(8)</td>
<td>1.80(9)</td>
<td>1.71(10)</td>
<td>1.84(23)</td>
<td>1.86(14)</td>
<td>0.151(10)</td>
</tr>
<tr>
<td>R. R.</td>
<td>1.43</td>
<td>1.57</td>
<td>1.65</td>
<td>1.69</td>
<td>1.72</td>
<td>1.74</td>
<td>1.429</td>
<td>0.071</td>
</tr>
</tbody>
</table>

- Very good agreement with experiment for $\Delta K = 0$ transitions.
- The $\Delta K = 2$ experimental transitions rates are slightly underestimated.
- Very weak dependence on the two parameters.
- $\Delta K = 0$ theoretical predictions are overestimated in respect to the R. R., while the $\Delta K = 2$ ones are almost identical.

The last result is related to the quasi-rigid description of the $\gamma$ degree of freedom.
A simple exactly separable model was constructed by taking the kinetic energy of the Bohr Hamiltonian as a combination of prolate $\gamma$-rigid and $\gamma$-stable rotation-vibration kinetic operators.

The relative weight of these two components is managed through a so-called rigidity parameter which bridges the $X(3)$ $\gamma$-rigid solution to its $\gamma$-stable counterpart represented by ES-$X(5)$ model when an infinite square well potential in $\beta$ is adopted.

The best experimental realization of the model is found in few rare earth nuclei around $N = 96$, namely $^{160}$Gd and $^{162}$Dy, while $^{166}$Er is another suitable candidate.

The proposed hybrid formalism unveils alternative features of the collective motion in the vicinity of the critical point of a spherical to deformed shape phase transition.